APPLICATION 1: Fraunhofer Diffraction

To illustrate the relationship consider the example of a finite window of width $\epsilon$, and having a periodic refractive index so that the phase imparted to a transmitted wave is given by

$$F_{\epsilon}(\omega) = e^{\frac{2\pi i \omega \epsilon}{2\pi}}$$

(corresponding to a transmission grating with $e$ exponential rulings on its transparent opening)

Fraunhofer diffraction patterns from same window, but having different transparencies corresponding to $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$, exponential rulings respectively on the interval $[\epsilon, \frac{1}{(1+\epsilon)}]\).
The relation between the transmission function of a screen and its Fraunhofer diffraction pattern is the same as the relation between the Fourier transform of a function and the function itself. The set of orthonormal wave packets is the direct mathematical formulation of this fact.

As an example consider a transmission grating of length $L$. For notational reasons we let the coordinate along the length of this grating be $x$ and the coordinate along the diffraction pattern be $t$.

Let the transmission function of the grating be

$$F(x) = F_0(x) + \frac{a}{2\lambda} (F_0(x) + F_0(-x))$$

$$F(x) = \begin{cases} \frac{1}{\lambda} \left[ 1 + a \cos \frac{2\pi x}{\lambda} \right] & -\frac{\lambda}{2} < x < 0 \frac{\lambda}{2} \\ 0 & \frac{\lambda}{2} < |x| \end{cases}$$

Transmission function

$$P_0(t)$$

$$\frac{a}{2} P_0(t)$$

$$\frac{a}{2} P_0(t)$$

$$\frac{a}{2} P_0(t)$$

Diffraction grating with red transmission ("transfer")

Fraunhofer diffraction pattern

$$\frac{1}{\lambda} \left[ 1 + a \cos \frac{2\pi x}{\lambda} \right]$$

$$\frac{a}{2} e^{-i\omega t} + \frac{a}{2} e^{i\omega t}$$

$$\text{Fourier transform of } P_0(t) + \frac{a}{2} P_0(t) + \frac{a}{2} P_0(t)$$

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Comments:

One can make the following observations about the orthonormal wave packets associated with transmission function in the Fourier domain:

(i) A wave packet is the diffraction pattern of a slit of width $E$.

(ii) If the slit contains no grating ($a=0$) then there is only the central ($L=0$) unshifted diffraction pattern.

(iii) If a slit of width $E$ has $L$ grating periods then the first wave packet with non-zero amplitude is found at $t = \pm 2\pi L$.

(iv) The fact that the transmission function is real implies that the first pair of wave-packets is symmetrically placed at $t = \pm 2\pi L$.

APPLICATION 2: Phased Antennas Array

(i) If the transmission function of a grating does not vary harmonically along its aperture, then higher order pairs of wave packets at $t = \pm 2\pi L, \pm 4\pi L, 2, \pm 2\pi 3, \ldots$ are found. The transmission function will be found in the diffraction pattern. The higher order wave packets (i.e., diffraction patterns) will have less amplitude the more closely the transmission function varies along the aperture trigonometrically.

PHASED ARRAY ANTENNAS

An important issue is whether a diffraction pattern must always be symmetrically placed around the zero order ($L=0$) pattern. In other words, can one construct a transmission...
The factor $F_{eo} = C_{eo}$ is approximated as

$$\frac{1}{N} - N \approx 0.$$

By controlling the relative phase delay, the corresponding emitted wave packet $F(t)$ generates a wave that is reflected at the surface. Some wave energy is sent into the air, and it is possible to use microwave horns, dipole antennas, etc.

The planar array patterns are one-sided directional patterns. By adding a $\pi$ phase shift, we can construct a super-fast Fourier transformer. This phase shift can be done using a function of $x$.

$$F(x) = \int P(x) e^{-i2\pi x} dx.$$

For complex configuration, the relation between a function and its Fourier transform is described by the Fourier transform function.

**Phased Antennas Array**

Application: Super Fast Fourier Transformer

18-8
Notation: Signal Analysis (ω,t - coordinates)

Fourier Optics (x₁, x₀ - coordinates)

In Fourier optics, the transmission function and the screen where its diffraction pattern appears are labelled by x₁ and x₀, not by ω and t.

\[ \frac{1}{\text{length}} \quad \omega \quad \text{(length)} \quad x_1 \quad (\text{length}) \]

\[ e^{i} \quad \{ \quad \text{L = aperture dimension} \quad \delta t = \frac{2\pi}{\lambda} \quad \delta \alpha \} \quad \text{aperture} \quad \frac{2\pi}{\delta} \]

Transmission Screen

Reception Screen

Consequently, the wave packet

\[ P_2(t) = \int_{-\infty}^{\infty} P_1(\omega) e^{i \omega t} \frac{e^{i \omega t}}{\sqrt{2\pi}} d\omega \]

becomes

\[ P_2(x_0) = \int_{-\infty}^{\infty} \frac{e^{-i \frac{2\pi}{\lambda \delta} x_1}}{\sqrt{1 - \frac{2\pi}{\lambda \delta}}} \int_{-\infty}^{\infty} e^{-i \frac{2\pi}{\lambda \delta} x_1 x_0} \frac{2\pi}{\lambda \delta} dx_0 \]

The indeterminacy was the window in the transmission screen, and at, the width of the wave packet, become respectively

\[ \Delta x_1 = \omega \Delta \frac{x}{2\pi} = e \frac{\lambda}{2\pi} \]

\[ \Delta x_0 = \delta t = \frac{2\pi}{\epsilon} \]

Their product gives an indeterminacy relation:

\[ \Delta x_1 \Delta x_0 = \lambda \frac{\pi}{2} \]

Thus, the narrower the transmission window \( \Delta x_1 \), the wider the diffraction pattern \( \Delta x_0 \).

This is related to the quantum mechanical indeterminacy relation.

\[ \Delta x_1 \Delta p = \hbar \]

if one introduces

\[ \Delta p = \hbar \times \Delta (\text{transverse wave number}) = \hbar \left( \frac{\Delta x_1}{\lambda \delta} \right) \]
Fig. 9-7. Raster display of 48 channels versus time illustrating delayed arrival of pulsar pulses due to dispersion of interstellar medium. On a single frequency channel, the pulsar pulse is not noteworthy but in a raster display of 48 channels the pulses connect as a very distinctive diagonal line. (by Martin Ewing).

Time delay $= t_2 - t_1 = \frac{81L\tilde{N}}{2c} (\nu_2^{-2} - \nu_1^{-2})$  \hspace{1cm} (9-12)