# Monday, April 9

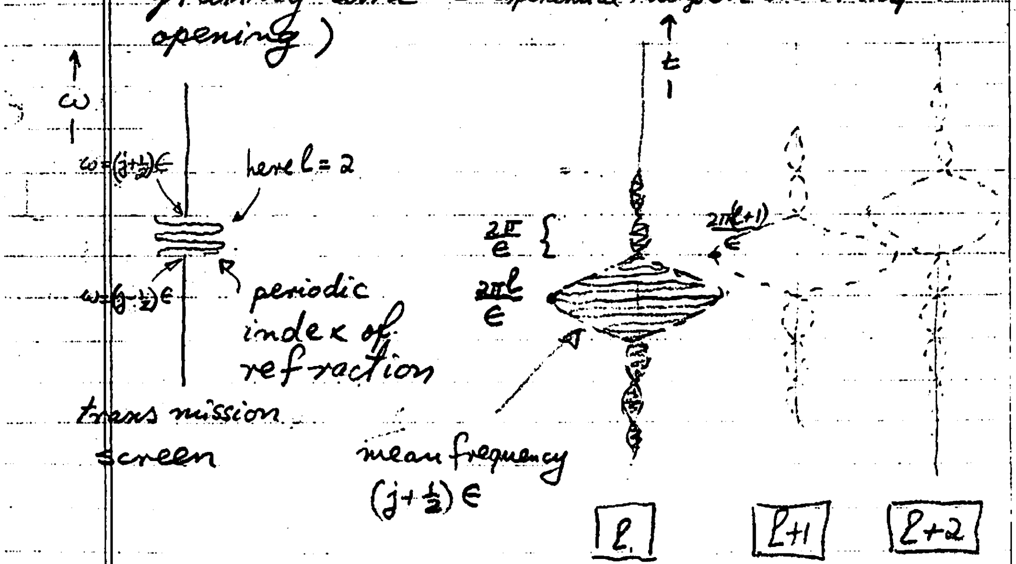
# 4 Applications of O.N. wavepackets.

## APPLICATION 1: Fraunhofer Diffraction 

To illustrate the relationship consider the example of a finite window of width $\epsilon$ and having a periodic refractive index so that the phase imparted to a transmitted wave is given by

$$F_{j\ell}(\omega) = \frac{e^{2\pi i \ell\omega/\epsilon}}{\sqrt{\epsilon}}$$

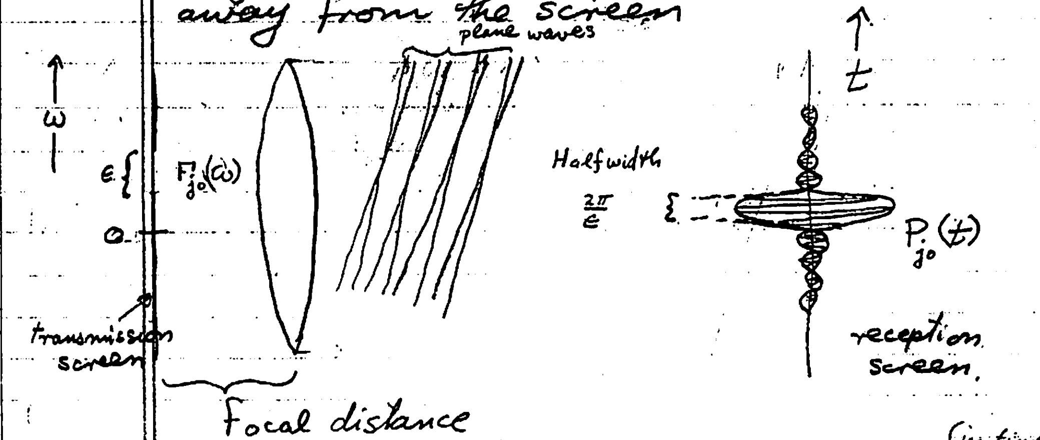
(corresponding to a transmission grating with $\ell$ "exponential rulings" on its transparent opening)




Fraunhofer diffraction patterns from same window, but having different transparencies corresponding to $\ell$, $\ell+1$, $\ell+2$ "exponential rulings" respectively in the interval $[j\epsilon, (j+1)\epsilon]$.



A optical way of obtaining the Fraunhoffer diffraction pattern of a transmission screen is by means of a lens placed 1 focal distance away from the screen




An O.N. wave packet with mean frequency $j\epsilon$ and mean position (in time) $\frac{2\pi\ell}{\epsilon}$

$$P_{j\ell}(t) = \int_{j\epsilon}^{(j+1)\epsilon} \frac{e^{2\pi i \ell\omega/\epsilon}}{\sqrt{\epsilon}} \frac{e^{-i\omega t}}{\sqrt{2\pi}}\, d\omega$$

$$= \frac{e^{i\left(\frac{2\pi\ell}{\epsilon}-t\right)(j+1)\epsilon} - e^{i\left(\frac{2\pi\ell}{\epsilon}-t\right)j\epsilon}}{\sqrt{2\pi\epsilon}\; i\left(\frac{2\pi\ell}{\epsilon}-t\right)}$$

$$= e^{i\left(\frac{2\pi\ell}{\epsilon}-t\right)(j+\frac{1}{2})\epsilon} \underbrace{\frac{2}{\sqrt{2\pi\epsilon}} \frac{\sin\left(\frac{2\pi\ell}{\epsilon}-t\right)\frac{\epsilon}{2}}{\frac{2\pi\ell}{\epsilon}-t}}_{\text{modulus of wave packet}}$$

OBSERVATION: SHIFTING the transmission screen ("frequency" = $j\epsilon$) window does not alter the position of the wave packet

The relation between the transmission function of a screen and its Fraunhofer diffraction pattern is the same as the relation between the Fourier Transform of a function and the function itself.
The set of orthonormal wave packets is the direct mathematical formulation of this fact.

As an example consider a transmission grating of length $\epsilon$. For notational reasons we let the coordinate along the length of this grating be $\omega$ and the coordinate along the diffraction pattern be $t$.
Let the transmission ("transfer") function of the grating be

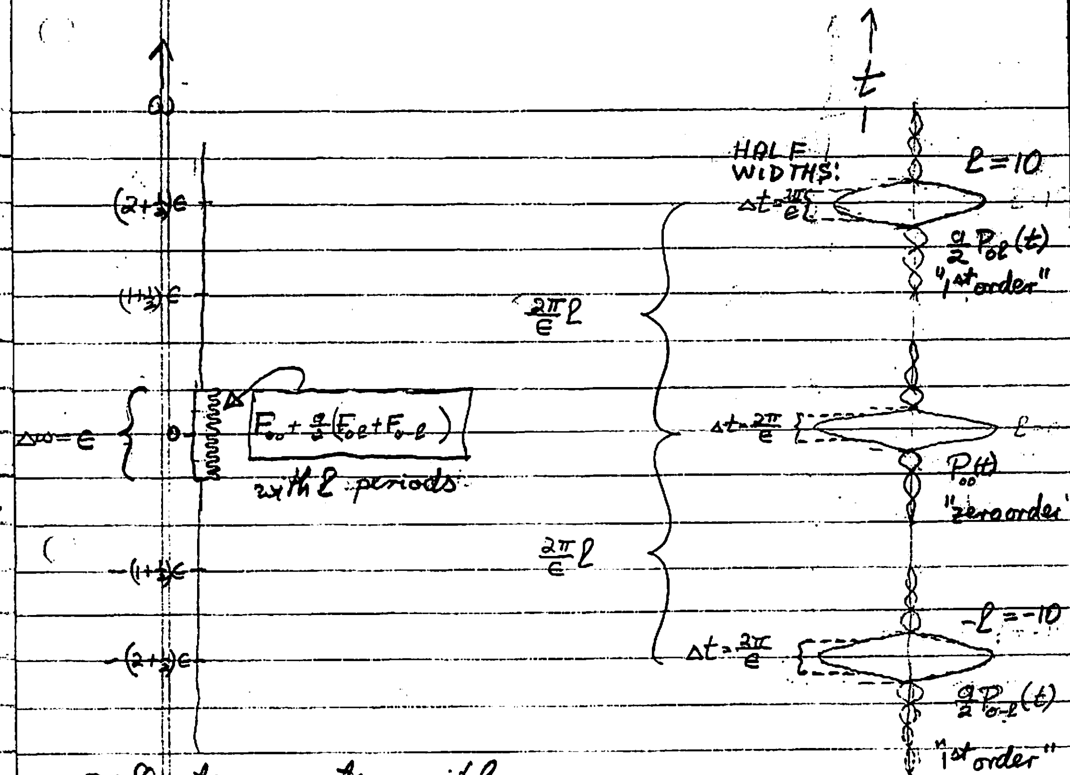
$$F(\omega) = F_{00}(\omega) + \frac{a}{2}\left(F_{0\ell}(\omega) + F_{0-\ell}(\omega)\right)$$

$$= \begin{cases} \frac{1}{\sqrt{\epsilon}}\left[1 + a\cos\frac{2\pi\ell\omega}{\epsilon}\right] & -\frac{\epsilon}{2} < \omega < \frac{\epsilon}{2} \\ 0 & \frac{\epsilon}{2} < |\omega| \end{cases}$$



Transmission function

Its diffraction pattern




Diffraction grating with real transmission ("transfer") function

$$\frac{1}{\sqrt{\epsilon}}\left[1 + a\cos\frac{2\pi\ell\omega}{\epsilon}\right]$$

$$\frac{1}{\sqrt{\epsilon}}\left[1 + \frac{a}{2}e^{2\pi i\ell\omega/\epsilon} + \frac{a}{2}e^{-2\pi i\ell\omega/\epsilon}\right] \xrightarrow{\text{Fourier x form}} P_{00}(t) + \frac{a}{2}P_{0\ell}(t) + \frac{a}{2}P_{0-\ell}(t)$$

$$-\frac{\epsilon}{2} < \omega < \frac{\epsilon}{2}$$

The corresponding Fraunhofer diffraction pattern is

Comments:

One can make the following observations about the orthonormal wave packets associated with transmission function in the Fourier domain:

(i) A wave packet is the diffraction pattern of a slit of width $\epsilon$.

(ii) If the slit contains no grating ($a=0$), then there is only the central ($\ell=0$) unshifted diffraction pattern.

(iii) If a slit of width $\epsilon$ has $\ell$ grating periods then the first wave packet with non-zero amplitude is found at $t=\frac{2\pi}{\epsilon}\ell$

(iv) The fact that the transmission function is real,

$$F(\omega) = F_{00}(\omega) + \frac{a}{2}\left(F_{0\ell} + c.c.\right)$$

implies that the first pair of wave-packets is symmetrically placed at $t = \pm\frac{2\pi}{\epsilon}\ell$.



(v) If the transmission function (=grating) does not vary harmonically along its aperture, then "higher order" pairs of wave packets

$$\pm\frac{2\pi}{\epsilon}\ell,\ \pm\frac{2\pi}{\epsilon}\ell\cdot 2,\ \pm\frac{2\pi}{\epsilon}3, \ldots$$

$$P_{0\pm\ell},\ P_{0\pm 2\ell},\ P_{0\pm 3\ell}, \ldots$$

1st order, 2nd order, 3rd order ... diffraction

will be found in the diffraction pattern. These higher order wave packets (i.e diffraction patterns) will have less amplitude the more closely the transmission function varies along the aperture trigonometrically.

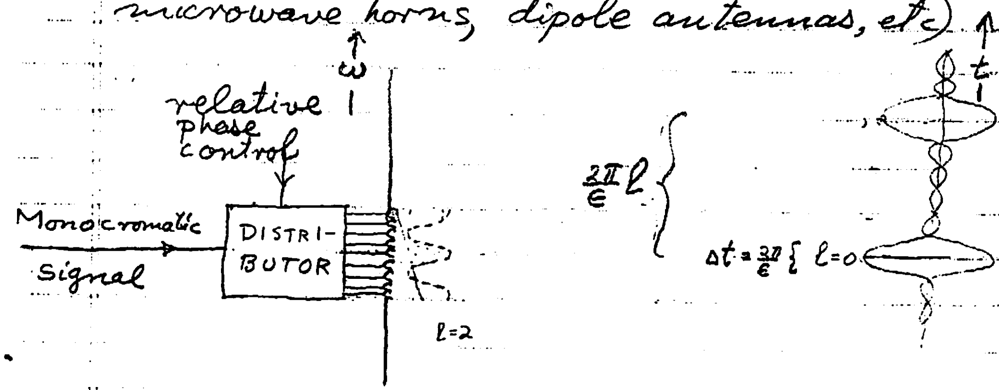
PHASED ARRAY ANTENNAS

An important issue is whether a diffraction pattern must always be symmetrically placed around the zero order ($\ell=0$) pattern. In other words, can one construct a transmission

## PHASED ANTENNAS ARRAY

screen which yields only one sided diffraction patterns. The answer to this is yes, and it is the principle behind a phased antenna array.

One replaces the $\ell$-period diffraction grating with an array of 2N wave emitters (e.g. loudspeakers, microwave horns, dipole antennas, etc)




By controlling the relative phase delay among the N wave emitters one can generate emitted wave packets $P_{j\ell}(t)$ corresponding to

$$j=0, \quad -N < \ell < N.$$

The factor $F_{0\ell}(\omega) = e^{2\pi i\omega\ell/\epsilon}$ is approximated discretely.

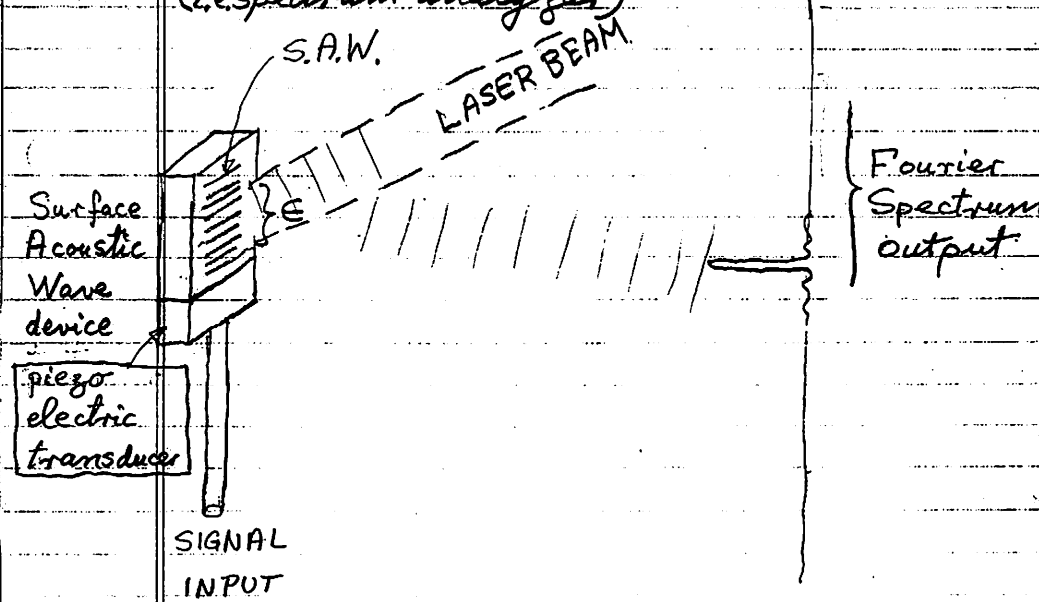


## APPLICATION 3: Super Fast Fourier Transformer

Except for complex conjugation, the relation between a function and its Fourier transform is reciprocal

$$\bar{F}_{j\ell}(\omega) = \int_{-\infty}^{\infty} \bar{P}_{j\ell}(t)\, \frac{e^{-i\omega t}}{\sqrt{2\pi}}\, dt$$

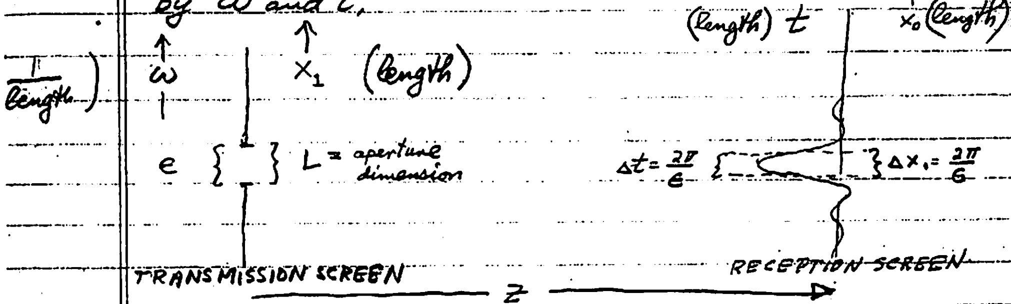
Thus one can construct a Super Fast Fourier Transformer (i.e. spectrum analyzer)



NOTATION: SIGNAL ANALYSIS ($\omega, t$ - coordinates)

vs FOURIER OPTICS ($x_1, x_0$ - coordinates)

In Fourier optics the transmission function screen and the screen where its diffraction pattern appears are labelled by $x_1$ and $x_0$, not by $\omega$ and $t$.




Let $z$ = distance from transmission screen to screen where diffraction pattern appears.

$\lambda$ = wave length of monocromatic wave

$x_1$ = coordinate along transmission screen

$x_0$ = coordinate along screen where diffraction pattern apears.

Then from the theory of Fourier optics we have

$$\omega = \frac{2\pi}{\lambda z} x_1 \qquad \epsilon = \frac{2\pi}{\lambda z} L$$

$$t = x_0 \qquad \frac{2\pi \omega l}{\epsilon} = \frac{2\pi l}{L} x_1$$



Consequently the wave packet

$$P_{jl}(t) = \int_{j\epsilon}^{(j+1)\epsilon} \frac{e^{\frac{2\pi i \omega l}{\epsilon}}}{\sqrt{\epsilon}} \frac{e^{-i\omega t}}{\sqrt{2\pi}} d\omega$$

becomes

$$P_{jl}(x_0) = \int_{jL}^{(j+1)L} \frac{e^{\frac{2\pi l}{L} x_1}}{\sqrt{L}\sqrt{\frac{2\pi}{\lambda z}}} e^{-i\frac{2\pi}{\lambda z} x_1 x_0} \left(\frac{2\pi}{\lambda z}\right) dx_0$$

4 The indeterminacy $\Delta\omega = \epsilon$, the window in the transmission screen, and $\Delta t$, the width of the wave packet, become respectively

$$\Delta X_1 = \Delta\omega \frac{\lambda z}{2\pi} = \epsilon \frac{\lambda z}{2\pi}$$

$$\Delta X_0 = \Delta t = \frac{2\pi}{e}$$

Their product gives an indeterminacy relation.

$$\boxed{\Delta X_1 \, \Delta X_0 = \lambda z}$$

Thus, the narrower the transmission window $\Delta x_1$, the wider the diffraction pattern $\Delta x_0$.

This is related to the quantum mechanical indeterminacy relation

$$\Delta x_1 \Delta p = \hbar$$

if one introduces

$$\Delta p = \hbar \times \Delta(\text{transverse wave number}) = \hbar\left(\frac{1}{\lambda} \frac{\Delta x_0}{z}\right)$$

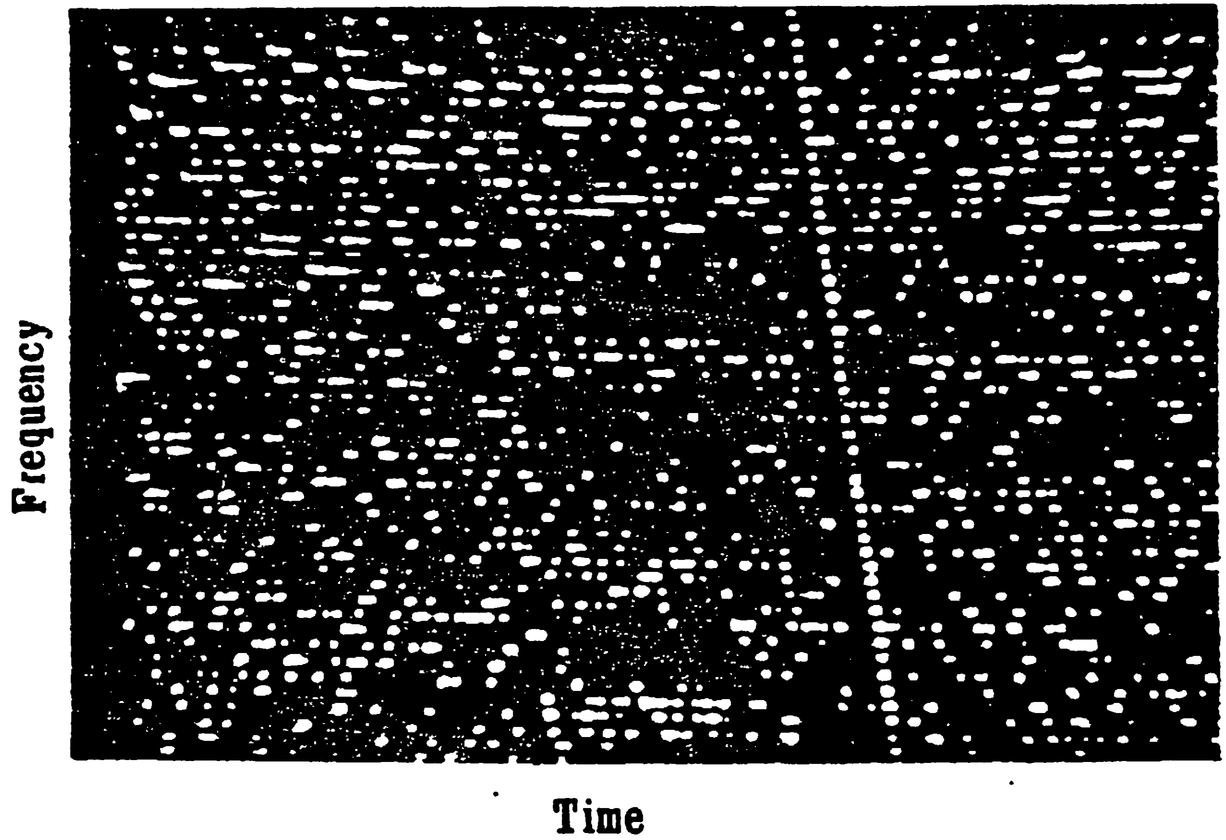



*Fig. 9-7.* Raster display of 48 channels versus time illustrating delayed arrival of pulsar pulses due to dispersion of interstellar medium. On a single frequency channel, the pulsar pulse is not noteworthy but in a raster display of 48 channels the pulses connect as a very distinctive diagonal line. (by Martin Ewing).

$$\text{Time delay} = t_2 - t_1 = \frac{81 L\bar{N}}{2c}\left(\nu_2^{-2} - \nu_1^{-2}\right) \tag{9-12}$$

Application 4: "Water fall display" (9-12