When do Fourier series converge
(point-wise)?

Recall that $f$ is called differentiable at $x$ if
$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ exists}; \therefore f'(x)$$
and $f'(x)$ = slope of the tan line to the graph of $f$.
at the point $(x, f(x))$.

\[ \tan \theta = f'(x) \]

**Def.** $f$ is said to have a right-hand derivative
if
$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ exists}; \quad f'_+(x)$$

slope of the tan line
to the part of the graph
to the right of $x$.

Similarly
**Def.** $f$ has a left-hand derivative
if
$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ exists}; \quad f'_-(x)$$
Example of a function which has both left and right derivatives, but it is not differentiable:

Ex.: \( f(x) = |x| \) has right and left derivatives at \( x = 0 \)

\[ f'_+(0) = 1 \quad \text{and} \quad f'_-(0) = -1 \]

**Theorem**

Suppose \( f \) is piecewise continuous on \((-\pi, \pi)\) and periodic on \( \mathbb{R} \) with period \( 2\pi \).

Then at each point \( x \) where \( f'_+(x), f'_-(x) \) exist

\[ \lim_{N \to \infty} \frac{S_N(f)(x) = f(x+ \frac{2\pi}{N}) + f(x- \frac{2\pi}{N})}{2} \]

In particular, if \( f \) is continuous at \( x \) then

\[ \lim_{N \to \infty} S_N(f)(x) = f(x) \]

**Remark**

be careful about \( x = \pm \pi \)

the function

is periodic, period \( 2\pi \)

but the points \( x = -\pi \) and \( x = \pi \) are points of
discontinuity ( \( f \) is periodic and continuous from
\(-\pi, \pi) \to \mathbb{R} \) and

\[ \lim_{N \to \infty} S_N(f)(\pm \pi) = \frac{f(\pi) + f(-\pi)}{2} = \frac{b}{2} \].
Remark: If piecewise continuous \( f \in L^1(-\pi, \pi) \)
so \( f \) has a Fourier series.

Note:
- There are many theorems of this type: if \( f \) is a
  nice function in this and that sense, then its Fourier
  series converges to that.
  
  But this theorem is quite useful in applications.

- If \( f \) does not satisfy the hypothesis of the theorem,
  it does not mean that the Fourier series does not
  converge at \( x \)! (The theorem gives insufficient conditions
  for convergence, they are not necessary.)

For the proof, we first need two lemmas.

The first lemma is a special case of

\[ \text{Riemann-Lebesgue Lemma:} \]

If \( f \) is integrable then
\[
\lim_{n \to \infty} \int_a^b f(x) \sin(nx) \, dx = 0
\]
\[
\lim_{n \to \infty} \int_a^b f(x) \cos(nx) \, dx = 0
\]

or, together, \( \lim_{n \to \infty} \int_a^b f(x) e^{inx} \, dx = 0 \)

(Read: if \( f \in L^1(a, b) \) then \( f_n \to f \) as \( n \to \infty \))

We will discuss later the duality: the more regular \( f \) is,
the more rapidly \( f_n \to f \) at \( a, b \) as \( n \to \infty \).
Digression

It was assumed that \( \int_{-\pi}^{\pi} |f(x)| < \infty \),

that is, \( f \in L^1(-\pi, \pi) \) (and not in \( L^2 \)). This is a more stringent condition.

For example, \( \frac{1}{x^{2/3}} \in L^1(-\pi, \pi) \) but \( \notin L^2(-\pi, \pi) \).

In general, note for bounded intervals

\[ f \in L^2(a, b) \implies f \in L^1(a, b) \]

(because by Cauchy-Schwarz)

\[ \|f\|_1 = \int_a^b |f(x)| \, dx = \langle |f|, 1 \rangle \leq \|f\|_2 \cdot \|1\|_2 \leq \frac{\|f\|_2}{2} \cdot \frac{1}{2} = \frac{1}{4} (b-a) \]

\[ \implies \quad L^1(a, b) \subseteq L^2(a, b) \]
**Lemma 1**

If $G(x)$ is a function piecewise continuous on $(0, \pi)$ then

$$
\lim_{N \to \infty} \int_{0}^{\pi} G(x) \sin \left( (N+\frac{1}{2})x \right) \, dx = 0 \quad (\text{for } N \in \mathbb{Z}_+) 
$$

**Proof**

$$
\int_{0}^{\pi} G(x) \sin \left( (N+\frac{1}{2})x \right) \, dx = \int_{0}^{\pi} G(x) \cos \frac{x}{2} \sin N x \, dx + \int_{0}^{\pi} G(x) \sin \frac{x}{2} \cos N x \, dx 
$$

$$
= a_N 
$$

$$
= b_N 
$$

where $a_N$, $b_N$ are coeff in the Fourier series on $G \cos$, respectively $G \sin$.

Since the Fourier coeff $e^{-iNz} \Rightarrow a_N \to 0$, $b_N \to 0$.

**Lemma 2**

Recall $\delta_N(x) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{inx} = \frac{1}{2\pi} \frac{\sin \left( (N+\frac{1}{2})x \right)}{\sin \frac{x}{2}}$

Note that $\int_{0}^{\pi} \delta_N(x) \, dx = \frac{1}{2\pi} \sum_{n=-N}^{N} \int_{0}^{\pi} e^{inx} \, dx = \frac{1}{2\pi} \cdot \pi = \frac{1}{2}$

Suppose $g(x)$ is piecewise cont and $g'(0)$ exist.

Then $\lim_{N \to \infty} \int_{0}^{\pi} g(x) \delta_N(x) \, dx = \frac{1}{2} g'(0)$

**Proof**

Write $g(x) = [g(x) - g(0^+)] + g(0^+)$

Then $\int_{0}^{\pi} \delta_N(x) g(0^+) \, dx = \frac{1}{2} g(0^+)$, while
\[ \int_0^N [g(x) - g(x^+)] \, dN(x) = \int_0^N \frac{g(x) - g(x^+)}{\min\left(\frac{x}{N + \frac{1}{2}}\right)} \, dx = g(x), \text{ piecewise continuous} \]

Since \( \lim_{x \to 0^+} \frac{g(x) - g(x^+)}{\min\frac{x}{2}} \)

\[ = \lim_{x \to 0^+} \frac{g(x) - g(x^+)}{x} \cdot \frac{x}{\min\frac{x}{2}} = g'(0) \cdot \frac{1}{2} \]

so by Lemma 1 has zero limit

\[ \Box \]

**Proof of the Theorem**

Recall

**Def.** A function \( f \) is called **piecewise continuous** on \((a, b)\)

if there are a finite number of intervals, divided all

\[ a = a_0 < a_1 < a_2 < a_3 < \ldots < a_p = b \]

so that \( f \) is continuous on each interval \((a_i, a_{i+1})\), \(i = 0, 1, 2, \ldots, p\)

and \( f \) has lateral limits at each \( a_0, a_1, \ldots, a_p \):

- Thus exist (and are finite) \( \lim_{x \to a_k} f(x) = f(a_k^+) \)
- \( x > a_k \)

\( \lim_{x \to a_k} f(x) = f(a_k^-) \)

\( x \to a_k \)

\( x < a_k \)

(\( f \) need not be defined at \( a_0, a_1, \ldots, a_p \))

**Ex.**

\[ \text{piecewise continuous function} \]
The point is that on each subinterval 
\[ f : (a_{k-1}, a_k) \to \mathbb{R} \] is continuous 
and it could be extended to a continuous function on \([a_{k-1}, a_k]\), hence \(f\) has a max and a 
min on \([a_{k-1}, a_k]\) and it is Riemann-integrable:

\[
\int_a^b f(x) \, dx = \sum_{k=0}^{p-1} \int_{a_k}^{a_{k+1}} f(x) \, dx \quad \text{well defined!}
\]

So \(f \in L^2\) also, and \(f\) has a Fourier series

\[
f \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right)
\]

Consider the partial sums

\[
(S_Nf)(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left( a_n \cos nx + b_n \sin nx \right)
\]

We have

\[
(S_Nf)(x) = \delta_N * f = \int_{-\pi}^{\pi} f(x-y) \, \delta_N(y) \, dy =
\]

\[
= \int_{-\pi}^{\pi} f(x-y) \, \delta_N(y) \, dy = \int_{-\pi}^{\pi} f(x-y) \, \delta_N(y) \, dy
\]

and using Lemma 2

We have

\[
\lim_{N \to \infty} (S_Nf)(x) = \frac{1}{2} \, f(x+) + \frac{1}{2} \, f(x-)
\]

\(\square\)
Example

The Fourier series of

\[ f(x) = x + \pi \text{ for } x \in (-\pi, \pi) \]

then continued \(2\pi\)-periodic

\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{\pi} \cdot \frac{2\pi \cdot 2\pi}{2} = 2\pi \]

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{2\sin(n\pi)}{n} = 0 \]

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{\pi \sin(n\pi) - \cos(n\pi) \cdot n\pi}{n^2 \pi} = \frac{(-1)^n}{2n} \]

Fourier series

\[ f(x) = \sum_{n=1}^{\infty} (-1)^n \frac{2}{n} \sin(nx) \]

\[ \pi + \sum_{n=1}^{\infty} (-1)^n \frac{2}{n} \sin(nx) = \sqrt{\left( \frac{f(-\pi) + f(\pi)}{2} \right) - \frac{2\pi}{\pi} = \pi} \]
Corollary. If \( f \) is \( 2\pi \)-periodic on \( \mathbb{R} \), continuous, and has left and right derivatives at all points \( x \) then

\[
\hat{f}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{in all } x
\]