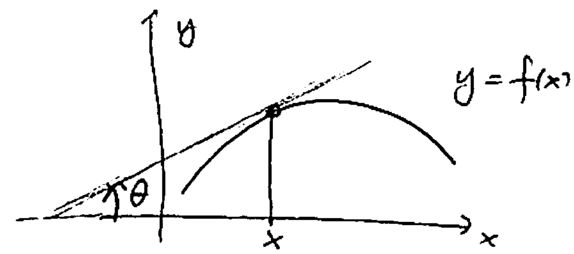
When do Fourier series converge (point-wise)?

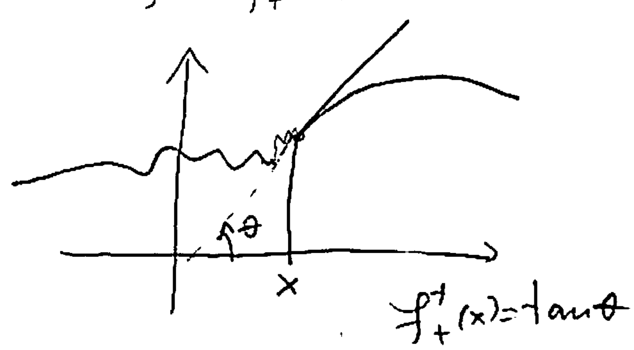
Recall that f is called differentiable at x if
$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \text{ exists; then } = f'(x)$$
and $f'(x)$ = slope of the tan line to the graph of f at the point $(x, f(x))$.

$$\tan\theta = f'(x)$$



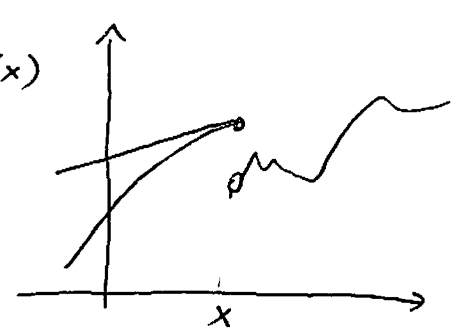
Def. f is said to have a right-hand derivative if $\lim\limits_{\substack{h\to 0\\ h>0}} \frac{f(x+h)-f(x)}{h}$ exists; $= f'_+(x)$

slope of the tan line to the part of the graph to the right of x.

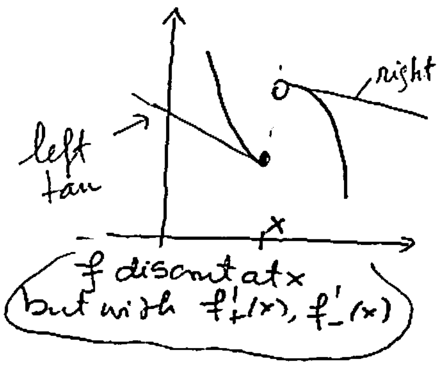


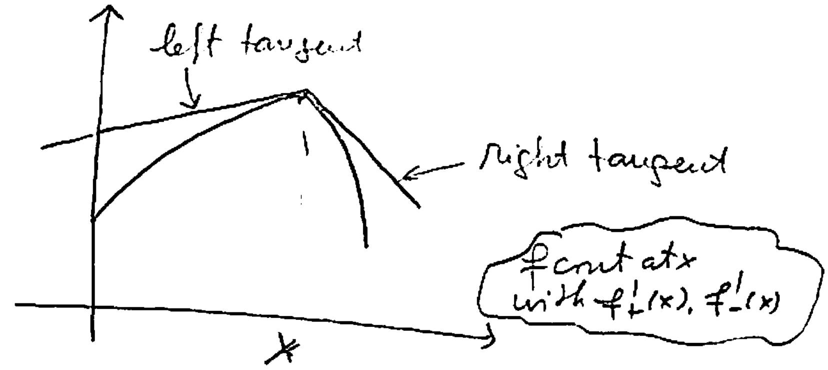
Similarly

Def f has a left-hand derivative if $\lim\limits_{\substack{h\to 0\\ h<0}} \frac{f(x+h)-f(x)}{h}$ exists; $= f'_-(x)$

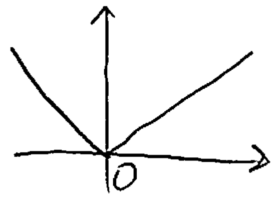


Example of a function which has both left and right derivatives, but it is not differentiable:





Ex. $f(x) = |x|$ has right and left derivatives at $x=0$



$f'_+(0) = 1$ and $f'_-(0) = -1$

Theorem 1 Suppose f is piecewise continuous on $(-\pi, \pi)$ and periodic on $\mathbb{R}$ with period 2π.

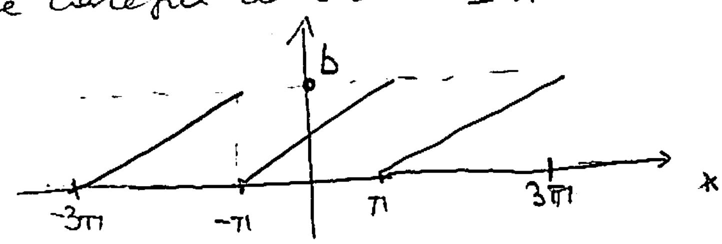
Then at each point x where $f'_+(x)$, $f'_-(x)$ exist

$$\lim_{N\to\infty} S_N(f)(x) = \frac{f(x+) + f(x-)}{2}$$

In particular, if f is continuous at x then $\lim_{N\to\infty} S_N f(x) = f(x)$

Remark be careful about $x = \pm\pi$

the function



is periodic, period 2π

but the points $x=-\pi$ and $x=\pi$ are points of discontinuity (for its periodic continuation from $[-\pi, \pi)$ to $\mathbb{R}$) and $\lim_{N\to\infty} S_N f(\pm\pi) = \frac{f(\pi-) + f(-\pi+)}{2} = \frac{b}{2}$.

<u>Remark</u> f piecewise continuous $\Rightarrow f \in L^2(-\pi,\pi)$
so f has a Fourier series.

<u>Note</u>

- There are many theorems of this type: if f is a nice function in this and that sense, then its Fourier series converges to that.
  But this theorem is quite useful in applications.
- If f, or x, does not satisfy the hypothesis of the theorem, it does <u>not</u> mean that the Fourier series does not converge at x! (The theorem gives sufficient conditions for convergence, they are not necessary.)

For the proof, we first need two lemmas

The first lemma is a special case of

FYI: $\boxed{\text{Riemann-Lebesgue Lemma:}}$

if f is integrable then $\lim_{n\to\pm\infty} \int_a^b f(x)\sin(nx)\,dx = 0$

$$\lim_{n\to\pm\infty} \int_a^b f(x)\cos(nx) = 0$$

or, together, $\lim_{n\to\pm\infty} \int_a^b f(x)e^{inx}\,dx = 0$

$\boxed{\text{Read}}$: if $f \in L^1(a,b)$ then $\hat{f}_n \to 0$ as $n \to +\infty$ or $n \to -\infty$

We will discuss later the duality: the more regular f is, the more rapidly $\hat{f}_n$ go to zero as $n \to \pm\infty$

<u>Digression</u>

it was assumed that $\int_{-\pi}^{\pi} |f(x)| < \infty$,

that is, $f \in L^1(-\pi,\pi)$ (and not in L^2!) <u>This is a more stringent cond</u>

For example $\frac{1}{x^{2/3}} \in L^1(-\pi,\pi)$ but $\notin L^2(-\pi,\pi)$

In general, note for <u>bounded intervals</u>

$$f \in L^2(a,b) \Rightarrow f \in L^1(a,b)$$

(because by Cauchy-Schwartz

$$\|f\|_{L^1} = \int_a^b |f(x)|\,dx = \langle |f|, 1 \rangle \le \underbrace{\|f\|_{L^2}}_{<+\infty} \underbrace{\|1\|_{L^2}}_{= b-a}$$)

so: $L^1(a,b) \underset{\neq}{\subset} L^2(a,b)$

Lemma 1

If $G(x)$ is a function piecewise continuous on $(0,\pi)$

then $\lim_{N\to\infty} \int_0^\pi G(x) \sin(N+\frac{1}{2})x \, dx = 0$ (for $N \in \mathbb{Z}_+$)

Proof

$$\int_0^\pi G(x) \sin(Nx + \tfrac{x}{2})\, dx = \underbrace{\int_0^\pi G(x) \cos\tfrac{x}{2} \sin Nx \, dx}_{= a_N} + \underbrace{\int_0^\pi G(x) \sin\tfrac{x}{2} \cos Nx \, dx}_{= b_N}$$

where a_N, b_N are coeff in the Fourier series on $G\cos$, respectively $G\sin$

Since the Fourier coeff $\in \ell^2(\mathbb{Z}) \Longrightarrow a_N \to 0, b_N \to 0$.

$\square$

Recall $\delta_N(x) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{inx} = \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}$

Note that $\int_0^\pi \delta_N(x)\,dx = \frac{1}{2\pi} \sum_{n=-N}^{N} \int_0^\pi e^{inx}\,dx = \frac{1}{2\pi} \cdot \pi = \frac{1}{2}$

Lemma 2

Suppose $g(x)$ is piecewise cont and $g'_+(0)$ exists

Then $\lim_{N\to\infty} \int_0^\pi g(x)\, \delta_N(x)\, dx = \frac{1}{2} g(0+)$

Proof

Write $g(x) = [g(x) - g(0+)] + g(0+)$

then $\int_0^\pi \delta_N(x)\, g(0+)\,dx = \frac{1}{2} g(0+)$, while

C_3

$$\int_0^N [g(x)-g(0+)]\,\delta_N(x) = \int_0^N \underbrace{\frac{g(x)-g(0+)}{\sin\frac{x}{2}}}_{=G(x)} \sin\left(N+\tfrac{1}{2}\right)x\,dx$$

$= G(x)$, piecewise continuous

since $\displaystyle\lim_{x\to 0+} \frac{g(x)-g(0+)}{\sin\frac{x}{2}}$

$$= \lim_{x\to 0+} \frac{g(x)-g(0+)}{x}\,\frac{x}{\sin\frac{x}{2}} = g_+'(0)\cdot 2$$

so by Lemma 1 has zero limit

$\square$

Proof of the Theorem

Recall

Def A function f is called piecewise continuous on (a,b) if there are a finite number of intervals, divided at

$a \equiv a_0 < a_1 < a_2 < a_3 < \dots < a_p \equiv b$

so that f is continuous on each interval $(a, a_1), (a_1, a_2), \dots, (a_{p-1}, \underset{b}{a_p})$

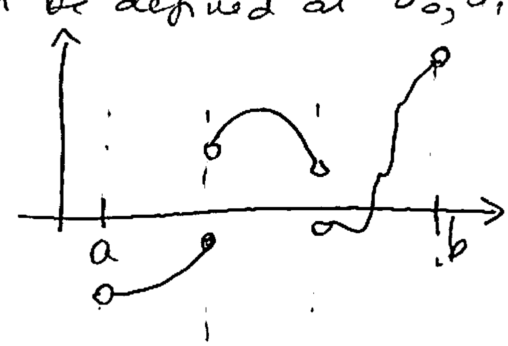
and f has lateral limits at each $a_0, a_1, \dots a_p$:

there exist (and are finite) $\displaystyle\lim_{\substack{x\to a_k\\ x>a_k}} f(a_k) := f(a_k+)$

and $\displaystyle\lim_{\substack{x\to a_k\\ x<a_k}} f(a_k) := f(a_k-)$

(f need not be defined at $a_0, a_1, \dots a_p$)

Ex.



piecewise continuous function

The point is that on each subinterval

$f:(a_{k-1}, a_k) \to \mathbb{R}$ is continuous

and it could be extended to a continuous function on $[a_{k-1}, a_k]$, hence f has a max and a min on $[a_{k-1}, a_k]$ and it is integrable:

$$\int_a^b f(x)\,dx := \sum_{k=0}^{p} \int_{a_{k-1}}^{a_k} f(x)\,dx \quad \text{well defined!}$$

So $f \in L^2$ also, and f has a Fourier series

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Consider the partial sums

$$(S_N f)(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx))$$

We showed

$$(S_N f)(x) = f * \delta_N = \delta_N * f = \int_{-\pi}^{\pi} f(s-x)\,\delta_N(s)\,ds =$$

$$= \underbrace{\int_{-\pi}^{0} f(x-s)\,\delta_N(s)\,ds}_{\int_0^\pi f(x+t)\,\delta_N(t)\,dt} + \int_0^{\pi} f(x-s)\,\delta_N(s)\,ds$$

and using Lemma 2

We have

$$\lim_{N\to\infty} (S_N f)(x) = \frac{1}{2} f(x+) + \frac{1}{2} f(x-)$$

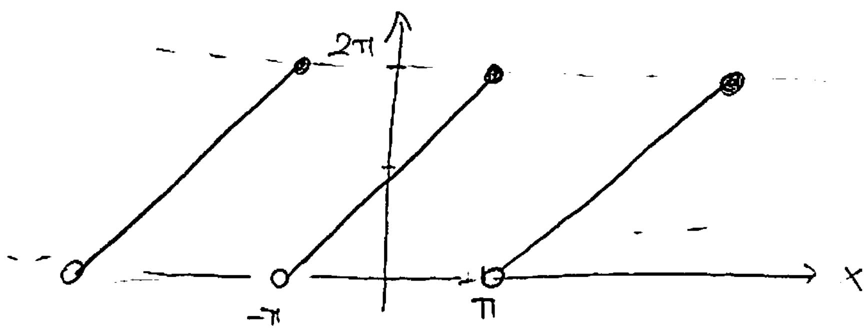
$\square$

Example

The Fourier series of

$$f(x) = x + \pi \text{ for } x \in (-\pi, \pi]$$

then continued 2π-periodic



is

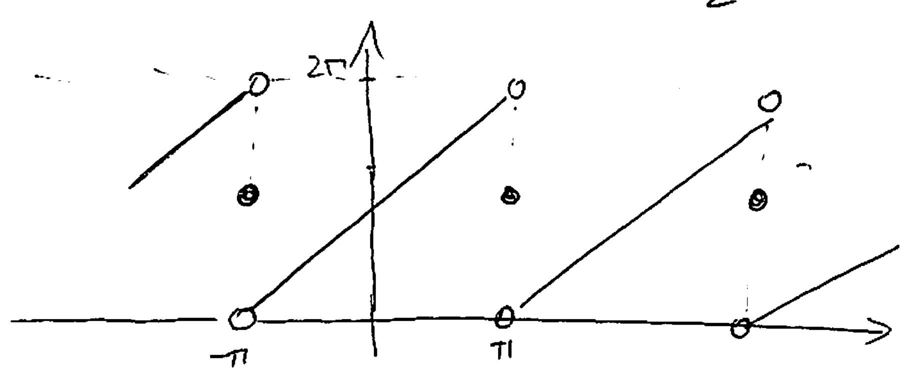
$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi} \cdot \frac{2\pi \cdot 2\pi}{2} = 2\pi$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{2 \sin(n\pi)}{n} = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{\sin(n\pi) - \cos(n\pi) \cdot n\pi}{n^2 \pi} \cdot 2 = \frac{(-1)^{n+1} 2}{n}$$

Fourier series

$$\pi + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin(nx) = \begin{cases} f(x) \text{ for } x \in (-\pi, \pi) \\ \frac{f(-\pi+) + f(\pi-)}{2} = \frac{0 + 2\pi}{\pi} = \pi \end{cases}$$



$\boxed{\text{Corollary}}$ If f is 2π periodic on $\underline{\underline{R}}$, continuous, and has left and right derivatives at all points x then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{for all } x$$