Friday

Lecture 6 : The 1st half. 6.1

The Fourier Integral Theorem

The Fourier Integral Theorem is a statements which relates the representation of a function in the given (time) domain to its represention in the Fourier (frequency) domain.

Theorem

Given: (a) a function f which piecewise continuous on every bounded interval on the real line.

(b) f is absolutely integrable on $(-\infty,\infty)$, i.e

$$f \in L^1(-\infty,\infty): \int_{-\infty}^{\infty} |f(x)|\, dx < \infty$$

Conclusion: 6.2

$$\lim_{r\to\infty} \int_{-r}^{r} dk \frac{e^{ikx}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \frac{e^{-ikt}}{\sqrt{2\pi}} f(t) = \frac{1}{2}\left[f(x^+)+f(x^-)\right]$$

$$\left|\hat{f}(k)\right| \le \int_{-\infty}^{\infty} dt \left|\frac{e^{-ikt}}{\sqrt{2\pi}} f(t)\right| < \infty$$

for every $x \in (-\infty,\infty)$ where the one-sided derivatives $f_R'(x)$ and $f_L'(x)$ exist

Comment:

(i) If f is continuous at x, then this conclusion is summarized by the following Fourier xform pair:

$$\boxed{\begin{aligned} \hat{f}(k) &= \int_{-\infty}^{\infty} dt \frac{e^{-ikt}}{\sqrt{2\pi}} f(t) \\ f(x) &= \int_{-\infty}^{\infty} dk \frac{e^{ikx}}{\sqrt{2\pi}} \hat{f}(k) \end{aligned}}$$

(ii) Integrating the exponentials over k yields

$$\frac{1}{2\pi}\int_{-r}^{r} dk\; e^{ik(x-t)} = \frac{1}{2\pi}\left.\frac{e^{ik(x-t)}}{i(x-t)}\right|_{-r}^{r}$$

$$= \frac{1}{\pi}\,\frac{\sin r(x-t)}{(x-t)}$$

Let $r = \frac{1}{\alpha}$. Then

$$\delta_\alpha(u) = \frac{1}{\pi}\,\frac{\sin\frac{u}{\alpha}}{u} = \delta_\alpha(-u),\quad u = x-t$$

"$\delta_\alpha(u)$ is an even fn"

Has the property

1. $$\int_{-\infty}^{\infty}\delta_\alpha(u)\,du = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\frac{u}{\alpha}}{u/\alpha}\,\frac{du}{\alpha}$$

$$= \frac{2}{\pi}\int_0^{\infty}\frac{\sin y}{y}\,dy$$

$$= \frac{2}{\pi}\,\frac{\pi}{2} = 1 \qquad \forall\alpha$$

2. $\lim_{\alpha\to 0}\int_{\epsilon}^{\infty}\delta_\alpha(u)f(u)\,du = \lim_{\alpha\to 0}\int_{\epsilon}^{\infty}\sin\frac{u}{\alpha}\,\frac{f(u)}{u}\,du = 0$ (Why?!)

3. Similarly: $\lim_{\alpha\to 0}\int_{-\infty}^{-\epsilon}\delta_\alpha(u)f(u)\,du = 0$

4. a $\lim_{\alpha\to 0}\int_{-\infty}^{\infty}\delta_\alpha(u)f(u)\,du = f(0)$

4. b $\lim\int^{\infty}\delta(t-x)f(t)\,dt = f(x)$

Go to page 6.5

Comment: An α-parametrized family of function $\{\delta_\alpha(u)\}$ with the above four properties is called a <u>delta-convergent series</u>.

We need to determine

$$\lim_{\alpha\to 0}\int_{-\infty}^{\infty}\delta_\alpha(x-t)\,f(t)\,dt = \lim_{\alpha\to 0}\int_{-\infty}^{\infty}\delta_\alpha(u)\,f(u+x)\,du$$

$u = t - x$

To this end one observes that

$$\lim_{\alpha\to 0}\int_{-\infty}^{\infty}\delta_\alpha(u)g(u)\,du = \lim_{\alpha\to 0}\Big\{\underbrace{\int_{-\infty}^{-\epsilon}}_{=\int_{-\infty}^{-\epsilon}\sin\frac{u}{\alpha}\frac{g(u)}{u}du\to 0 \text{ as } \alpha\to 0}+\int_{-\epsilon}^{\epsilon}+\int_{\epsilon}^{\infty}\Big\}\delta_\alpha(u)\,g(u)\,du$$

$\int_{-\infty}^{-\epsilon}\sin\frac{u}{\alpha}\frac{g(u)}{u}du \to 0$ as $\alpha\to 0$ (Why?!)

Similarly

$$\lim_{\alpha\to 0}\int_{\epsilon}^{\infty}\delta_\alpha(u)\,g(u)\,du = 0$$

Thus

$$\lim_{\alpha\to 0}\int_{-\infty}^{\infty}\delta_\alpha(u)\,g(u)\,du = \lim_{\alpha\to 0}\int_{-\epsilon}^{\epsilon}\delta_\alpha(u)\,f(u+x)\,du$$

$$= f(\xi)\lim_{\alpha\to 0}\int_{-\epsilon}^{\epsilon}\delta_\alpha(u)\,du \quad \text{where } x-\epsilon\le\xi\le x+\epsilon$$

$$= f(\xi)\lim_{\alpha\to 0}\int_{-\infty}^{\infty}\delta_\alpha(u)\,du$$

$$= f(\xi)\cdot 1$$

$= f(x)$ whenever f is continuous at x.

Thus with $\delta_\alpha(u)$ as a <u>delta-convergent sequence</u> we have

$$\lim_{\alpha\to 0}\frac{1}{2\pi}\int_{-\frac{1}{\alpha}}^{\frac{1}{\alpha}}e^{ik(x-t)}\,dk \equiv \lim_{\alpha\to 0}\delta_\alpha(x-t) = \delta(x-t)$$

whose limit is the Dirac delta fn where it is understood that one integrate <u>first</u>, and <u>then</u> take the limit:

$$\int_{-\infty}^{\infty}\delta(x-t)\,f(t)\,dt \equiv \lim_{\alpha\to 0}\int_{-\infty}^{\infty}\delta_\alpha(x-t)\,f(t)\,dt\ \left[=f(x)\right]$$

Next: Lecture 6: <u>The 2nd half</u>

(1) "Dirac-delta" normalization

Completeness: Parcevals identity: $\langle f, f\rangle = \langle \hat{f}, \hat{f}\rangle$

(2) Fourier xform of a generalized function ("distribution", e.g. a ticking clock)