The Fourier Integral

Any $f \in L^2(-A,A) \to \mathbb{R}$ or $\mathbb{C}$; piecewise cont, with lateral derivatives:

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}_n \, e^{in\frac{\pi x}{A}} \quad \text{where} \quad \hat{f}_n = \frac{1}{2A}\int_{-A}^{A} e^{-in\frac{\pi x}{A}} f(x)\,dx$$

It seems tempting to let $A \to \infty$.

Let f be piecewise smooth, with lateral derivatives

Also to integrate on $(-\infty,\infty)$ assume $\int_{-\infty}^{\infty}|f(x)|dx < \infty$ $(f \in L^1(\mathbb{R}))$

$$f(x) = \frac{1}{2A}\sum_{n=-\infty}^{\infty} e^{in\frac{\pi x}{A}} \int_{-A}^{A} e^{-in\frac{\pi t}{A}} f(t)\,dt = \frac{1}{2A}\sum_{n=-\infty}^{\infty}\int_{-A}^{A} e^{-in\frac{\pi}{A}(t-x)} f(t)\,dt$$

Since $\frac{1}{2A} \to 0$ this looks like a Riemann sum : denote $\frac{\pi}{A} = \varepsilon$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \varepsilon \int_{-\pi/\varepsilon}^{\pi/\varepsilon} e^{-in\varepsilon(t-x)} f(t)\,dt \qquad \xi_n = n\varepsilon,\ \Delta\xi = \xi_{n+1} - \xi_n = \varepsilon$$

and as $\varepsilon \to 0$ $\longrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} e^{-i\xi(t-x)} f(t)\,dt = f(x)$

Denoting $\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\xi t} f(t)\,dt$ = the Fourier Transform of f $= \mathcal{F}(f)$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\xi x}\hat{f}(\xi)\,d\xi$$ — the inverse Fourier Transf. $= \mathcal{F}^{-1}(\hat{f})$

By taking $A \to \infty$ in the discrete case $\Rightarrow \|\mathcal{F}(f)\|_{L^2}^2 = \|f\|_{L^2}^2$: $\int_{-\infty}^{\infty}|\hat{f}|^2 = \int_{-\infty}^{\infty}|f|^2$

In fact, $\mathcal{F}$ can be extended as a unitary operator $\mathcal{F}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$

but the integrals exist only in the cases above.

Note: a Fourier transform is a continuous version of a Fourier series

<u>Remark</u>: other normalizations you may find used:

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i x \xi} f(x)\,dx \ \& \ \mathcal{F}^{-1}(\hat{f}) = \int_{-\infty}^{\infty} e^{2\pi i x \xi} \hat{f}(\xi)\,d\xi$$

or

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-ix\omega} f(x)\,dx, \quad \mathcal{F}^{-1}(\hat{f})(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\omega} \hat{f}(\omega)\,d\omega$$

The advantage with the normalization here is that $\mathcal{F}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is <u>unitary</u> if

$$(\mathcal{F}f)(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ix\xi} f(x)\,dx$$

$$(\mathcal{F}^{-1}\hat{f}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ix\xi} f(x)\,dx$$

due to Parseval's identity:

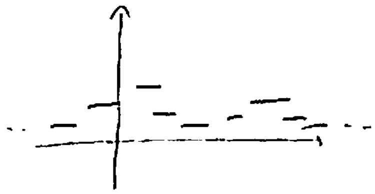
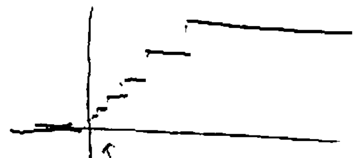
$$\|f\|_{L^2} = \|\hat{f}\|_{L^2} \quad \text{or} \quad \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi$$

Which implies also

$$\langle f, g\rangle_{L^2} = \langle \hat{f}, \hat{g}\rangle_{L^2} \quad \text{or} \quad \int_{-\infty}^{\infty} \overline{f(x)}\, g(x)\,dx = \int_{-\infty}^{\infty} \overline{\hat{f}(\xi)}\, g(\xi)\,d\xi$$

The Fourier Integral Theorem

Let f be piecewise continuous on every $[-M, M] \subset \mathbb{R}$

e.g.  but not 

not piecewise continuous on $[0, 100]$
(infinitely many pieces)

Assume that at each $x \in \mathbb{R}$ f has $f'(x+)$ and $f'(x-)$.

Assume $\lim_{N \to \infty} \int_{-N}^{N} |f(x)|\, dx := \int_{-\infty}^{\infty} |f(x)|\, dx < \infty$ (note: $f(x)$ decays as $x \to \pm\infty$, at least as $\frac{1}{x^2}$)

Then $\mathcal{F}^{-1}(\mathcal{F}(f))(x) = \dfrac{f(x+) + f(x-)}{2}$

The proof is similar to the discrete $\mathcal{F}$. Read the proof from Gerlach, sec 2.3.2.

One main difficulties in the theory of Fourier transforms and Fourier series is to establish when they converge. Yes, $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ unitary, but it may not be given as an integral! Here is a super-nice class of functions: $\underline{\mathcal{S}(\mathbb{R})}$ = the rapidly decaying functions on $\mathbb{R}$:= functions infinitely many times differentiable, which at $\pm\infty$ decay faster than any power, and so do all their derivatives. E.g. $f(x) = e^{-x^2}$.

Then $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is one to one and onto!
(Fourier Heaven)

(The proof does require some work! as well as the def of $\mathcal{S}$)