The Green's functions of SA Sturm-Liouville problems

We saw in pages S1...S5 that after a change of variables we can always assume that $w(x) \equiv 1$, and, if $p(x) \neq 0$ on $[a,b]$, then we can even assume $p(x) \equiv 1$.

Moreover, this change of variables is a unitary transf. between $L^2([a,b], w(x)dx)$ and (some) $L^2([c,d], dz)$ so it preserves the eigenvalues of the operator, and takes eigenfunctions into eigenfunctions (of equal norm).

If $w \neq 0$ we can safely assume $w(x) \equiv 1$ if we so wish.

Or not, if we have concrete problems and we do not want to bother with changes of variables, or if we are interested in some $w(x)$ for which $w(a)=0$ or $w(b)=0$.

Recall:

$\boxed{\text{Theorem}}$ Let $L = \frac{1}{w}\left[-\frac{d}{dx}\left(p\frac{du}{dx}\right) + qu\right]$; $p, w > 0$ on (a,b), p, w, p', q cont on $[a,b]$.

$$D(L) = \left\{ u \in \underbrace{L^2((a,b), w(x)dx)}_{\equiv \mathcal{H}} \;\middle|\; u', u'' \in \mathcal{H},\ \begin{cases} p(\alpha u + \alpha' u')\big|_{x=a} = 0 \\ p(\beta u + \beta' u')\big|_{x=b} = 0 \end{cases} \right\}$$

with $q(x), \alpha, \alpha', \beta, \beta'$ real.

(note: if $p(a)=0$ then no condition is needed at $x=a$; similarly at $x=b$)

Then L is SA on $D(L)$ and $D(L)$ is dense in $\mathcal{H}$ therefore

1°. All eigenvalues of $L, D(L)$ are real
2°. All eigenvalues are bounded below: $\lambda_n \geqslant \lambda_1 \in \mathbb{R}$
3°. $u_n \perp u_k$ if $\lambda_n \neq \lambda_k$
4°. All λ_n have multiplicity 1: $\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \ldots$
5°. $\lambda_n \longrightarrow +\infty$ and $u_1, u_2 \ldots u_n \ldots$ form a basis for $\mathcal{H}$.

Moreover

If the BC are not regular, but mixed:

$$\alpha u(a) + \alpha' u'(a) + \alpha_2 u(b) + \alpha_2' u'(b) = 0$$
$$\beta u(b) + \beta' u'(b) + \beta_2 u(a) + \beta_2' u'(a) = 0$$

then similar results hold <u>except</u> that the eigenvalues may not be simple: $\lambda_1 \le \lambda_2 \le \lambda_3 \le \dots$
but each eigenspace is finite-dimensional.

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<u>Existence of the Green's function for SA Sturm Liouville problems</u>

<u>Theorem</u> Let $(L, D(L))$ be a regular SA problem, as in p. T_1.

1. If $\lambda = 0$ is not an eigenvalue then the problem admits a Green's function.
2. The Green's function is unique.
3. If $u_1, u_2 \dots u_n \dots$ are the eigenfunctions of $(L, D(L))$ then

$$G(x,t) = w(t) \sum_{n=1}^{\infty} \frac{1}{\lambda_n} u_n(x)\, u_n(t)$$

(the generalized Fourier series of $G(x,t)$)

4. and the solution of $Ly = f$ is
$$y(x) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \langle u_n, f \rangle_w \, u_n(x).$$

5. Furthermore, $w(t)\sum_n \overline{u_n(t)}\,u_n(x) = \delta(x-t)$

<u>Remark</u> since L has real coefficients, and so do BC, and since the eigenvalues are real, from the theory of ODE there are real eigenfunctions. Of course, any scalar multiple of them is also an eigenfunction, so we may wish/need to work with complex-valued eigenfunctions for SA operators.

Proof

1. We construct the Green's function:

Solve $\frac{1}{w(x)}\left[-\frac{d}{dx}\left(p(x)\frac{dG}{dx}\right)+q(x)G\right]=\delta(x-t) \quad, x\in(a,b)$

$$-\frac{d}{dx}\left(p(x)\frac{dG}{dx}\right)+q(x)G=\underbrace{w(x)\,\delta(x-t)}_{= w(t)\,\delta(x-t)}$$

$$G(x,t)=\begin{cases} C(t)\,y_a(x) & \text{for } x<t \\ D(t)\,y_b(x) & \text{for } x>t\end{cases}$$

where: y_a, y_b are nonzero sol. of $Ly=0$, $\alpha y_a(a)+\alpha' y_a'(a)=0$
and $\beta y_b(b)+\beta' y_b'(b)=0$.

These two must be matched: $C(t)y_a(t)=D(t)y_b(t)$
$$D(t)y_b'(t)-C(t)y_a'(t)=\frac{w(t)}{p(t)}$$

If y_a, y_b are linearly dependent $\Rightarrow y_b=\text{const}\, y_a \Rightarrow y_a$ satisfies the BC at $x=b$ also $\Rightarrow Ly_a=0$, $y_a\in D(L) \Rightarrow y_a$ is an eigenfunction for $\lambda=0$, contradiction

So y_a, y_b are lin indep. $\Rightarrow W[y_a,y_b]:=y_a y_b'-y_a' y_b\neq 0$

Now match them at $x=t$: solve

$$\begin{cases} C(t)y_a(t)-D(t)y_b(t)=0 \\ C(t)y_a'(t)-D(t)y_b'(t)=-\frac{w(t)}{p(t)}\end{cases}$$

$$\Rightarrow C(t)=\frac{\begin{vmatrix} 0 & -y_b \\ -\frac{w(t)}{p(t)} & -y_b'\end{vmatrix}}{W[y_a,y_b]}=\frac{w(t)y_b(t)}{W[y_a,y_b]p(t)}, \quad D(t)=\frac{\begin{vmatrix} y_a & 0 \\ y_a' & +\frac{w}{p}\end{vmatrix}}{W}=\frac{+w(t)y_a(t)}{W[y_a,y_b]\,p(t)}$$

On the other hand we can also calculate the Wronskian

y_a, y_b satisfy $\quad -py'' - p'y' + qy = 0 \quad (*)$

$$y'' = \frac{q}{p}y - \frac{p'}{p}y'$$

$$\begin{pmatrix} y \\ y' \end{pmatrix}' = \begin{bmatrix} 0 & 1 \\ \frac{q}{p} & -\frac{p'}{p} \end{bmatrix} \begin{pmatrix} y \\ y' \end{pmatrix} \qquad \text{Recall } W = \text{const}\, e^{\int \mathrm{Tr} M} = \frac{\text{const}}{p}$$

so $W(x)p(x) = \text{const} \equiv \frac{1}{\mathcal{K}}$ note : $\mathcal{K}$ does not depend on t

$\Rightarrow \quad C(t) = \mathcal{K}\, w(t)\, y_b(t) \quad , D(t) = \mathcal{K}\, w(t)\, y_a(t) \quad \Rightarrow$

$$G(x,t) = \begin{cases} \mathcal{K}\, w(t)\, y_b(t)\, y_a(x) & \text{for } x<t \\ \mathcal{K}\, w(t)\, y_a(t)\, y_b(x) & \text{for } x>t \end{cases}$$

2. Uniqueness : suppose G_1, G_2 are two Green's functions $\Rightarrow \tilde{y} = G_1 - G_2$ solves $(*)$ and satisfies the BC. How many solutions $y_a \neq 0$ satisfy $(*)$ and $\alpha y(a) + \alpha' y'(a) = 0$? Since not both α and α' are 0 $\Rightarrow$ there is a one parameter family of solutions : any Cy_a. $\Rightarrow \tilde{y} = Cy_a$ Similarly $\tilde{y} = Dy_b$

so $Cy_a = Dy_b \Rightarrow$ either $C = D = 0 \Rightarrow \tilde{y} = 0 \Rightarrow G_1 = G_2$ uniqueness,

or y_a satisfies the BC at $x=b$ as well $\Rightarrow$ y_a is an eigenfunction corresponding to $\lambda = 0$, contradiction.

3. The eigenfunctions $u_1, u_2 \ldots u_n \ldots$ form a basis for $\mathcal{H}$. Assume $\|u_n\|_w = 1$

$G(\cdot, t) \in \mathcal{H}$ (since it is continuous) so $G(x,t) = \sum_{n=1}^{\infty} c_n(t) u_n(x)$ with $\{c_n(t)\}_n \in \ell^2$

where $c_n = \langle \bar{u}_n, G(\cdot, t) \rangle_w$. Apply L in $\uparrow$

(Comment:
We want
$$LG(x,t) = L\left(\sum_{n=1}^{\infty} c_n(t) u_n\right) \underset{?}{=} \sum_{n=1}^{\infty} c_n(t) L u_n = \sum c_n(t) \lambda_n u_n$$

Note: L is not bounded, this means L is not continuous, so I cannot say that $L\left(\lim_{N\to\infty} \sum_{n=1}^{N} c_n u_n\right) = \lim_{N\to\infty} L\left(\sum_{1}^{N} c_n u_n\right)$. We need to be careful here.)

So we do it this way: $Lu_n = \lambda_n u_n$ — L has real coefficients, $u_n, \overline{u_n}$ are efunc to λ_n

$$\Rightarrow \lambda_n c_n(t) = \lambda_n \langle u_n, G(\cdot,t)\rangle_w = \lambda_n \int_a^b \overline{u_n(x)}\, G(x,t)\, w(x)\, dx$$

$$= \int_a^b \overline{L u_n(x)}\, G(x,t)\, w(x)\, dx \underset{L=L^*}{=} \int_a^b \overline{u_n(x)}\, LG(x,t)\, w(x)\, dx$$

$$= \int_a^b \overline{u_n(x)}\, \delta(x-t)\, w(x)\, dx = \overline{u_n(t)}\, w(t) \implies c_n(t) = \frac{\overline{u_n(t)}\, w(t)}{\lambda_n}$$

$$\Rightarrow G(x,t) = \sum_{n=1}^{\infty} \frac{u_n(x)\overline{u_n(t)}\, w(t)}{\lambda_n}$$

4. Let $y \in D(L)$ solution of $Ly = f$, that is, let

$$y(x) = \int_a^b G(x,t) f(t)\, dt \quad (*)$$

Then $$y(x) = \int_a^b \sum_{n=1}^{\infty} \frac{u_n(x)\overline{u_n(t)}\, w(t)}{\lambda_n} f(t) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} u_n(x) \langle u_n, f\rangle_w$$ □

Remark: $L: D(L) \to R(L)$ (where $R(L) = \{ f \in \mathcal{H} \mid f = Ly$ for some $y \in D(L)\}$) is one-to-one and onto, and $(*)$ is the formula of its inverse!

5. On the other hand, (besides 4.) we can expand any $y \in L^2_w$ as

$$y = \sum_{n=1}^{\infty} d_n u_n \quad \text{where } d_n = \langle u_n, y \rangle_w = \int_a^b \bar{u}_n y w$$

$$\text{so } y = \sum_{n=1}^{\infty} \int_a^b \bar{u}_n(t)\, y(t) w(t)\, dt\; u_n(x) \quad = \text{(in distribution sense)}$$

$$= \int_a^b \left[\sum_{n=1}^{\infty} \bar{u}_n(t) u_n(x) \right] y(t) w(t)\, dt$$

therefore $w(t) \sum_{n=1}^{\infty} \overline{u_n(t)}\, u_n(x) = \delta(x-t)$

$\boxed{\text{Theorem}}$ If 0 is an eigenvalue of L, then $Ly=f$, $y\in D(L)$ has solutions if and only if $\langle v_0, f\rangle_w = 0$ (where $Lv_0=0, v_0\in D(L)$) and in this case the solution is not unique, rather there is a one-parameter family of solutions, namely

$$y = Cv_0(x) + \sum_{n\neq 0} \frac{\langle v_n, f\rangle_w}{\lambda_n} v_n(x)$$

Why We only need to retrace the previous proof and see what happens if 0 is an eigenvalue.

First we see that we cannot find a Green's function.

A direct verification shows that y above does solve $Ly=f$. Are there other solutions? Suppose that y_2 is also a solution $\Rightarrow y-y_2$ solve $L(y-y_2)=0$ $\Rightarrow$ $y-y_2=$ sol. of the homog. eq $\Rightarrow$ $y_2 = y +$ sol. of the homog eq.

and this y_h with $Ly_h=0$ also satisfies $BC[y_h]=0$. If it is lin dep of v_0, then y_2 has the same formula as y, only with a different c. Otherwise, y_h and v_0 are 2 indep sol of $Ly=0$ satisfying $[BC]$ $\Rightarrow$ any sol. of $Ly=0$ satisfies $BC[y]=0$. But this is not possible unless $\alpha=\alpha'=\beta=\beta'=0$, Cd!

$\square$

Moreover, we can use a similar line of proof to find the formula for the resolvent of f:

Let $z \in \mathbb{C}$, z not an eigenvalue of L. This means that

- 0 is not an eigenvalue of $L - z\dot{I}$; recall that its eigenvalues are $\lambda_n - z$! and the same evect u_n. Using the previous 2 theorems for the operator $L - z\dot{I}$ instead of L we find

$\boxed{\text{Theorem}}$ Let $(L, D(L))$ as before, and $z \neq \lambda_n$, $n = 1, 2 \ldots$

Then the Green function $G(x,t,z)$ of $L - zI$ has the expansion

$$G(x,t,z) = w(t) \sum_{n=1}^{\infty} \frac{1}{\lambda_n - z} u_n(x) u_n(t)$$

and the sol y of $Ly - zy = f$ is

$$y = (L - z\dot{I})^{-1} f = \sum_{n=1}^{\infty} \frac{1}{\lambda_n - z} \langle u_n, f \rangle_w \, u_n(x)$$

If $z = \lambda_k$ is an eigenvalue, then $Ly - \lambda_k y = f$, $y \in D(L)$ has solutions if and only if $\langle u_k, f \rangle_w = 0$ and in this case there is a one-parameter fam. of solutions, namely

$$y = C u_k + \sum_{\substack{n=1 \\ n \neq k}}^{\infty} \frac{\langle u_n, f \rangle}{\lambda_n - \lambda_k} u_n$$

The previous theorem can be found formulated as (an instance of) Fredholm's alternative:

Given $(L, D(L))$ <u>either</u> $(L-z)y=f$ has a unique sol. for every f

<u>or</u> there is a sol. $(\neq 0)$ of $(L-z)y=0$