We saw in pages 51...55 that after a change of variables we can always assume that \( v(x) \equiv 1 \), and, if \( p(x) \equiv 0 \) on \((a, b)\), then we can even assume \( p(x) \equiv 1 \).

Moreover, this change of variables is a unitary transform between \( L^2(a, b), w(x) dx \) and \((a, b), \rho(x) \equiv 1 \), so it preserves the eigenvalues of the operator, and takes eigenfunctions into eigenfunctions (of equal norm).

If \( w \equiv 0 \) we can safely assume \( v(x) \equiv 1 \) if we so wish.

Oh, not, if we have concrete problems and we do not want to bother with changes of variables, or if we are interested in some \( v(x) \) for which \( v(a) = 0 \) or \( v(b) = 0 \).

Recall:

Theorem

Let \( L = \frac{d}{dx} \left( \frac{d}{dx} \left( p \frac{d}{dx} \right) + q \right) \), \( p, v > 0 \) on \((a, b)\) and \( p, v, p, q \) on \((a, b)\).

\[ D(L) = \left\{ u \in L^2(a, b), w(x) dx \left| u', u'' \in H, \left. p(au'' + bu') \right|_{x=a} = 0 \right. \right\} \]

\[ \left( i \right) \left. p(au'' + bu') \right|_{x=b} = 0 \]

with \( q, p, a, b \), \( p, q \) real.

(Hint: if \( p(a) = 0 \) then no condition is needed at \( x=a \); similarly at \( x=b \).)

Then \( L \) is SA on \( D(L) \) and \( D(L) \) is dense in \( H \) therefore

1. All eigenvalues of \( L, D(L) \) are real
2. All eigenvalues are bounded below: \( \lambda_n \geq \lambda_1 \in \mathbb{R} \)
3. \( \lambda_n \neq \lambda_k \) if \( \lambda_n \neq \lambda_k \)
4. All \( \lambda_n \) have multiplicity 1: \( \lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \ldots \)
5. \( \lambda_n \to +\infty \) and \( u_1, u_2, \ldots \) form a basis in \( H \).
Moreover

If the BC are not regular, but mixed:
\[ \alpha u(a) + \alpha' u'(a) + \alpha_2 u(b) + \alpha_2' u'(b) = 0 \]
\[ \beta u(b) + \beta' u'(b) + \beta_2 u(a) + \beta_2' u'(a) = 0 \]

then similar results hold except that the eigenvalues may not be simple: \( \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \) but each eigenspace is finite-dimensional.

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Existence of the Green's function for
SA Sturm Liouville problems

Theorem. Let \((L, D(L))\) be a regular SA problem, as in p. T1.

1. If \( \lambda = 0 \) is not an eigenvalue then the problem admits a Green's function.

2. The Green's function is unique.

3. If \( u_1, u_2, \ldots, u_n \) are the eigenfunctions of \((L, D(L))\) then

\[ G(x,t) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{\lambda_n} u_n(x) u_n(t) \]

(the generalized Fourier series of \( G(x,t) \))

4. and the solution of \( Ly = f \) is

\[ y(x) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \langle u_n, f \rangle u_n(x) \].
5. Furthermore, \( w(t) \sum_{n} \overline{u_n(t)} u_n(x) = \delta(x-t) \)

Remark: since \( L \) has real coefficients, and so do \( BC \), and since the eigenvalues are real, from the theory of ODE there are real eigenfunctions. Of course, any scalar multiple of them is also an eigenfunction, so we may wish/need to work with complex-valued eigenfunctions for SA operators.
Proof

1. We construct the Green’s function:

\[ \frac{1}{w_0} \left[ -\frac{d}{dx} \left( \rho(x) \frac{dG}{dx} \right) + q(x) G \right] = \delta(x-t), \quad x \in (0, b) \]

\[-\frac{d}{dx} \left( \rho(x) \frac{dG}{dx} \right) + q(x) G = \frac{w(x) \delta(x-t)}{w(t) \delta(x-t)}\]

\[ G(x,t) = \begin{cases} y_a(x) & \text{for } x < t \\ y_b(x) & \text{for } x > t \end{cases}\]

where \( y_a, y_b \) are solutions of \( Ly = 0 \), \( \alpha y_a(a) + \alpha' y_a'(a) = 0 \) and \( \beta y_b(b) + \beta' y_b'(b) = 0 \).

These two must be matched:

\[ (\rho y_a)(t) = (\rho y_b)(t) \]

\[ D(t) y_b'(t) - \left( D(t) - \frac{\rho}{w(t)} \right) y_b(t) = \frac{w(t)}{\rho(t)} \]

If \( y_a, y_b \) are linearly dependent \( \Rightarrow y_b = \alpha y_a \) and \( y_a \Rightarrow y_a \) satisfies the BC at \( x = b \). Also \( \Rightarrow Ly = 0, y_a \in D(t) \Rightarrow y_a \) is an eigenfunction in \( D = 0 \), contradiction.

So \( y_a, y_b \) are linearly independent \( \Rightarrow W[y_a, y_b] = y_a y_b - y_a y_b \neq 0 \)

Now match them at \( x = t \): solve

\[ \begin{cases} (C(t) y_a(t) - D(t) y_b(t)) = 0 \\
C(t) y_a(t) - D(t) y_b(t) = \frac{w(t)}{\rho(t)} \end{cases} \]

\[ C(t) = \frac{r(t) y_a(t) - y_b(t)}{W[y_a, y_b]} = \frac{w(t) y_b(t)}{W[y_a, y_b]} \quad D(t) = \frac{\alpha y_a(a) + \alpha' y_a'(a)}{W[y_a, y_b]} \]

\[ W[y_a, y_b] = \frac{w(t) y_b(t)}{W[y_a, y_b]} \]
On the other hand we can also calculate the Wronskian

\[ y_n y_m' - p y'_n y_m = 0 \quad \star \]

\[ y'' = \frac{2}{p} y - \frac{p'}{p} y' \]

\[ \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{2}{p} & -\frac{p'}{p} \end{pmatrix} \begin{pmatrix} y' \\ y'' \end{pmatrix} \]

Recall \( W = \text{const} e^{-\int \frac{p'}{p} dt} = \text{const} \frac{1}{x} \)

\[ W(x) p(x) = \text{const} \frac{1}{x} \quad \text{note: } x \text{ does not depend on } t \]

\[ C(t) = x \int w(t) y_b(t) dt, \quad D(t) = x \int w(t) y_n(t) dt \]

\[ G(x,t) = \begin{cases} x \int w(t) y_b(t) y_n(x) & \text{if } x < t \\ x \int w(t) y_n(t) y_b(x) & \text{if } x > t \end{cases} \]

2. Uniqueness: Suppose \( G_1, G_2 \) are two Green's functions \( \Rightarrow \tilde{y} = G_1 - G_2 \)

Solves \( \star \) and satisfies the BC. How many such functions \( y \) can satisfy \( \star \) and \( \alpha y(x) + \beta y'(x) = 0 \)? Since not both \( \alpha \) and \( \beta \) are zero \( \Rightarrow \) there is a one parameter family of solutions: any \( Cy \). \( \Rightarrow \tilde{y} = Cy \). Similarly \( \tilde{y} = Dy \)

\( \Rightarrow \) either \( C = 0 \) or \( D = 0 \) \( \Rightarrow \tilde{y} = 0 \) \( \Rightarrow G_1 = G_2 \) uniqueness,

or \( \tilde{y} \) satisfies the BC at \( x = x_b \) as well \( \Rightarrow y \) is an eigenfunction corresponding to \( \lambda = 0 \), contradiction.

3. The eigenfunctions \( u_1, u_2, u_n \ldots \) form a basis for \( H \). Assume \( \| u_n \|_W = 1 \)

\[ G(x,t) \in \mathcal{H} \text{ (since it is continuous)} \Rightarrow G(x,t) = \sum_{n=1}^{\infty} c_n(t) u_n(x) \quad \text{with } \| c_n(t) \|_H \leq \| u_n \|_W \]

where \( c_n(t) = \langle u_n, G(x,t) \rangle \). Apply \( L \) in \( T \)
\[ G(x,t) = L \left( \sum_{n=1}^{\infty} c_n(t) u_n \right) = \sum_{n=1}^{\infty} c_n(t) \lambda_n u_n = \sum_{n=1}^{\infty} c_n(t) \lambda_n u_n \]

**Note:** \( L \) is not bounded, this means \( L \) is not continuous, so I cannot say that \( L \left( \lim_{N \to \infty} \sum_{n=1}^{N} c_n u_n \right) = \lim_{N \to \infty} L \left( \sum_{n=1}^{N} c_n u_n \right) \). We need to be careful here.

So we do it this way: \( Lu_n = \lambda_n u_n \)

\[ \Rightarrow c_n(t) = \lambda_n \langle u_n, G(\cdot, t) \rangle_w = \lambda_n \int_a^b u_n(x) G(x,t) w(x) \, dx \]

\[ = \int_a^b L u_n(x) G(x,t) w(x) \, dx = \int_a^b \overline{u_n(x)} \overline{L G(x,t) w(x)} \, dx \]

\[ = \int_a^b u_n(x) \delta(x-t) w(x) \, dx = \overline{u_n(t)} \overline{w(t)} \quad \Rightarrow \quad c_n(t) = \frac{\overline{u_n(t)} \overline{w(t)}}{\lambda_n} \]

\[ \Rightarrow G(x,t) = \sum_{n=1}^{\infty} \frac{u_n(x) \overline{u_n(t)} \overline{w(t)}}{\lambda_n} \]

4. Let \( y \in D(L) \) solution of \( Ly = f \), that is, let

\[ y(x) = \int_a^b G(x,t) f(t) \, dt \]

Then

\[ y(x) = \int_a^b \sum_{n=1}^{\infty} \frac{u_n(x) u_n(t) \overline{w(t)} \overline{f(t)}}{\lambda_n} \, dt = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \int_a^b u_n(x) \overline{u_n(t)} f(t) \overline{w(t)} \, dt \]

\[ \Rightarrow G(x,t) = \sum_{n=1}^{\infty} \frac{u_n(x) \overline{u_n(t)} \overline{w(t)}}{\lambda_n} \]

Remark: \( L : D(L) \to R(L) \) (where \( R(L) = \{ f \in L^1 | f = Ly \text{ for some } y \in D(L) \} \))

is one-to-one and onto, and \( \Box \) is the formula of its inverse.
5. On the other hand, (besides 4.) we can expand any \( y \in L^2_w \) as

\[ y = \sum_{n=1}^{\infty} d_n \, u_n \]

where \( d_n = \langle u_n, y \rangle_w = \int_a^b u_n(x) \, y(x) \, dx \)

\[ y = \sum_{n=1}^{\infty} \int_a^b u_n(t) \, y(t) \, W(t) \, dt \, u_n(x) = (\text{in distribution sense}) \]

\[ = \int_a^b \left[ \sum_{n=1}^{\infty} u_n(t) \, u_n(x) \right] y(t) \, W(t) \, dt \]

Therefore

\[ W(t) \sum_{n=1}^{\infty} u_n(t) \, u_n(x) = \delta(x-t) \]
Theorem: If $\lambda$ is an eigenvalue of $L$, then $Ly = f$, $y \in \mathcal{D}(L)$ has solutions if and only if $\langle u_0, f \rangle = 0$ (where $L u_0 = 0, u_0 \in \mathcal{D}(L)$). And in this case the solution is not unique, rather there is a one-parameter family of solutions, namely

$$y = Cu_0(x) + \sum_{n \neq 0} \frac{\langle u_n, f \rangle}{\lambda_n} u_n(x)$$

We only need to retrace the previous proof and see what happens if $\lambda$ is an eigenvalue.

First we see that we cannot find a Green's function.

A direct verification shows that $y$ above does solve $Ly = f$. Are there other solutions? Suppose that $y_2$ is also a solution: $\Rightarrow y - y_2$ solves $L(y - y_2) = 0 \Rightarrow y - y_2 = \text{sol. of the homog. eq} \Rightarrow y_2 = y + \text{sol. of the homog. eq}$.

And this $y_2$ with $Ly_2 = 0$ also satisfies $\text{BC}[y_2] = 0$. If it is lin dep of $u_0$, then $y_2$ has the same form as $y$ only with a different $c$. Otherwise, $y_2$ and $u_0$ are lin indep sol of $Ly = 0$ making $\text{BC}[y_2]$ imply sol. of $Ly = 0$, satisfying $\text{BC}[y_2] = 0$. But this is not possible unless $\sigma = \sigma', \rho = \rho', C = D$.\]
Moreover, we can use a similar line of proof to find the formula for the resolvent of \( f \):

Let \( z \in \mathbb{C}, \, z \neq \text{an eigenvalue of } L. \) This means that

- \( 0 \) is not an eigenvalue of \( L - zI \); recall that its eigenvalues are \( \lambda_n - z \).

Using the previous theorem for the operator \( L - zI \) instead of \( L \) we find

**Theorem** Let \( (L, D(L)) \) as before, and \( z \neq \lambda_n, n = 1, 2, \ldots \).

Then the Green function \( G(x, t, z) \) of \( L - zI \) has the expansion

\[
G(x, t, z) = \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{1}{\lambda_n - z} U_n(x) U_n(t)
\]

and the solvability of \( Ly = f \) is

\[
y = (L - zI)^{-1} f = \sum_{n=1}^{\infty} \frac{1}{\lambda_n - z} <U_n, f>_W U_n(x)
\]

If \( z = \lambda_k \) is an eigenvalue, then \( Ly = \lambda_k y = f \) \( y \in D(L) \) has

solutions if and only if \( <U_k, f>_W = 0 \) and in this case there is a one-parameter family of solutions, namely

\[
y = C U_k + \sum_{n=1}^{\infty} \frac{<U_n, f>_W}{\lambda_n - \lambda_k} U_n
\]
The previous theorem can be found formulated as (an instance of) Fredholm's alternative:

Given \((L, d(L))\) either \((L-z)y = f\) has a unique sol. for every \(f\)

or there is a sol. of \((L-z)y = 0\)

\((\neq 0)\)