Uniqueness of the Green's function
The Fredholm alternative

There is no guarantee that a GF exists

Example $u'' = f$
$u'(0) = 0, \; u'(1) = 0$

Solve $G'' = \delta(x-t) \Rightarrow G' = H(x-t) + C(t) \Rightarrow$

$G(x,t) = (x-t)H(x-t) + C(t)x + D(t)$

$\partial_x G(0,t) = C(t) = 0$

$\partial_x G(1,t) = 1 \neq 0$ no solution!

To see what is going on, integrate directly $u'' = f$

$$u'(x) = \int_0^x f(s)\,ds + C$$

$u'(0) = 0 \Rightarrow C = 0$ and $u'(1) = \int_0^1 f(s)\,ds = 0$ only for special f!

Note: if $\int_0^1 f = 0$ then there are solutions: $u(x) = \int_0^x d\xi \int_0^\xi f(s)\,ds + D$ not unique!

But: **Theorem** Consider $L(x, \frac{d}{dx}) = \sum_{k=0}^{n} a_k(x) \frac{d^k}{dx^k}$

with n linearly indep. homogeneous B.C.

If the problem has a GF, then it is unique.

Why: Let G_1, G_2 be 2 GF: $LG_1 = \delta(x-t)$, $LG_2 = \delta(x-t)$, $BC[G_1] = 0$, $BC[G_2] = 0$

Then $G = G_1 - G_2$ solves $LG = 0$ and $BC[G] = 0$

From the theory of ODEs $\Rightarrow G \equiv 0$ so $G_1 = G_2$

<u>Theorem</u> The following are equivalent:

(i) The only sol. of the homog $\begin{array}{l} Lu=0 \\ BC[u]=0 \end{array}$ is $u=0$

(ii) $\begin{cases} Lu=f \\ BC[u]=0 \end{cases}$ has a sol. for every f

(iii) $\begin{cases} Lu \\ BC[u]=0 \end{cases}$ has a Green's function.

The proof is not given here.

Usually the first 2 are formulated as an instance of Fredholm's alternative: <u>either</u> $\begin{cases} Lu=f \\ BC[u]=0 \end{cases}$ has a unique sol. for all f

<u>or</u> $\begin{cases} Lu=0 \\ BC[u]=0 \end{cases}$ has nonzero solutions.

(these statements are mutually exclusive).