Uniqueness of the Green's Function

The Fredholm alternative

There is no guarantee that a GF exists

**Example**

\[ u'' = f, \quad u(0) = 0, \quad u'(1) = 0 \]

\[
\text{Solve } G'' = \delta(x-t) \Rightarrow G' = H(x-t) + C(t) \Rightarrow \\
G(x, t) = (x-t)H(x-t) + C(t)x + D(t) \\
G(0, t) = C(t) = 0 \\
G_x(1, t) = 1 \neq 0 \quad \text{no solution!}
\]

To see what is going on, integrate directly \( u'' = f \)

\[ u(x) = \int_0^x f(t) dt + C \]

\[ u'(0) = 0 \Rightarrow C = 0 \quad \text{and} \quad u'(1) = \int_0^1 f(t) dt = 0 \quad \text{only for special } f \]

**Note:** If \( \int_0^1 f = 0 \) then there are solutions: \( u(x) = \int_0^x \frac{3}{5} f(t) dt + D \)

**But:** **Theorem** Consider \( L(x, u, \frac{d}{dx}) = \sum_{k=0}^{n} a_k(x) \frac{d^k}{dx^k} u \)

with \( n \)-linearly independ homogeneous B.C.

If the problem has a GF, then it is unique.

**Why?** Let \( G_1, G_2 \) be 2 GF: \( LG_1 = \delta(x-t) \), \( LG_2 = \delta(x-t) \), \( BC[G_1] = 0 \), \( BC[G_2] = 0 \)

Then \( G = G_1 - G_2 \) satisfies \( LG = 0 \) and \( BC[G] = 0 \)

From the theory of ODEs \( \Rightarrow G \equiv 0 \quad \text{no } G_1 = G_2 \)
Theorem: The following are equivalent:

(i) The only sol. of the homog \( L \mathbf{u} = 0 \) \( \mathbf{Bc} \mathbf{u} = 0 \) is \( \mathbf{u} = 0 \)

(ii) \( L \mathbf{u} = \mathbf{f} \) has a sol. for every \( \mathbf{f} \)

(iii) \( L \) has a Green's function.

The proof is not given here.

Usually the first 2 are formulated as an instance of Fredholm's alternative: either \( \{ L \mathbf{u} = \mathbf{f} \} \) has a unique sol. for all \( \mathbf{f} \)

or \( \{ L \mathbf{u} = 0 \} \) has no non-zero solutions.

(These statements are mutually exclusive.)