Green's functions of linear, nonhomogeneous differential equations.

Main idea : suppose one has a lin, nonhom. ODE or PDE
$$L = L(x, D) \quad , \quad Ly = f(x)$$

for example: $L = \frac{d}{dx} + a(x)$ $\qquad \frac{dy}{dx} + a(x)y = f(x)$

$L = \frac{1}{w(x)}\left[\frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x)\right] \qquad \frac{1}{w(x)}\left[(p(x)y')' + qy\right] = f$

$L = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \qquad y_t - y_{xx} = f(x,t)$

etc

then if y_1, y_2 solve $Ly_1 = f_1$, $Ly_2 = f_2$ then
$$L(c_1 y_1 + c_2 y_2) = c_1 f_1 + c_2 f_2$$
superposition of solutions corresponds to superposition of forcing terms,

and if $Ly_n = f_n$ then if $y = \sum_n c_n y_n$ then $Ly = f = \sum_n c_n f_n$

Note : the same is true if we have some homogeneous B.C. or I.C. since i.e.g. say $Ly_n = f_n$, $y_n(a) = \boxed{0}$

then if $y = \sum c_n y_n$ we have $y(a) = \sum c_n y_n(a) = \boxed{0}$ and $Ly = \sum c_n f_n$

Why not do a "continuous superposition" ?

After all. $f(x) = \int f(t)\delta(x-t)\,dt$

So solve $LG = \delta(x-t)$ for each t parameter,

of course $G = G(x,t)$ then superimpose : $y(x) = \int f(t) G(x,t)\,dt$

and expect that $Ly = \int f(t)\,\delta(x-t)\,dt$

so $Ly = f(x)$

If we have a <u>homogeneous problem</u> (BC or ic)

then find G with $LG = \delta(x-t)$ & the same BC or iC

and then y will satisfy the problem.

$G(x,t)$ solution of $LG = \delta(x-t)$ is called a <u>Green's function</u>

<u>Example 1</u> Solve $\begin{cases} Ly = f(x) \\ y(x_0) = 0 \end{cases}$, $L = \frac{d}{dx} + a(x)$ (E1)

Find its Green's functions: solve $Ly = \delta(x-t)$

so $\frac{dG}{dx} + a(x)G = \delta(x-t)$. Integrating factor $e^{\int_{x_0} a(x)\,dx}$

$$\frac{d}{dx}\left(e^{\int_{x_0}^{x} a} G(x)\right) = e^{\int_{x_0}^{x} a}\,\delta(x-t) \quad (*)$$

Of course, $g(x)\,\delta(x-t) = g(t)\,\delta(x-t)$ if g is smooth

[since for any test function φ $(g(x)\delta(x-t), \varphi) = \int g(x)\varphi(x)\delta(x-t)$
$= g(t)\varphi(t) = g(t)\,(\delta(x-t), \varphi) = (g(t)\delta(x-t), \varphi)$]

so $(*)$ becomes

$$\frac{d}{dx}\left[e^{\int_{x_0}^{x} a} G(x)\right] = e^{\int_{x_0}^{t} a}\,\delta(x-t) \quad (**)$$

Recall that $\boxed{\frac{d}{dx} H(x) = \delta(x)}$

[Since $\left(\frac{d}{dx}H, \varphi\right) := -\left(H, \frac{d\varphi}{dx}\right) = -\int_{-\infty}^{\infty} H(x)\varphi'(x)\,dx = -\int_{0}^{\infty} \varphi'(x)\,dx$
$= -\varphi(x)\Big|_0^{\infty} = \varphi(0) = (\delta, \varphi)$]

and similarly $\boxed{\frac{d}{dx} H(x-t) = \delta(x-t)}$

So integrating (**) we get

$$e^{\int_{x_0}^{x} a} G(x,t) = e^{\int_{x_0}^{t} a}\left[H(x-t) + C(t)\right] \leftarrow \text{C may depend on the parameter } t$$

$$\text{so, } G(x,t) = \underbrace{e^{-\int_t^x a(s)} H(x-t)}_{\text{particular sol}} + \underbrace{e^{-\int_t^x a(s)ds} C(t)}_{\text{gen. sol. of the homog. eq.}}$$

Impose the initial condition: $G(x_0,t)=0$

$$\Rightarrow H(x_0-t) + C(t) = 0 \text{ so } C(t) = -H(x_0-t)$$

and $$G(x,t) = e^{-\int_t^x a(s)ds}\left[H(x-t) - H(x_0-t)\right]$$

Suppose $x>x_0$. Then $H(x-t) - H(x_0-t) = \begin{cases} 0 & \text{for } t>x \\ 1 & \text{for } x_0 \le t < x \\ 0 & \text{for } t < x_0 \end{cases}$

$$= \chi_{[x_0,x]}(t)$$

so $$G(x,t) = e^{-\int_t^x a(s)ds}\chi_{[x_0,x]}(t)$$ and

$$y(x) = \int_{-\infty}^{\infty} f(t)G(x,t)\,dt = \int_{x_0}^{x} dt\, f(t)\, e^{-\int_t^x a}$$

[Check: using the integrating factor $\frac{d}{dx}\left[e^{\int_{x_0}^x a} y\right] = e^{\int_{x_0}^x a} f(x)\,dx$

so $y(x) = e^{-\int_{x_0}^x a}\left[C + \int_{x_0}^{x} e^{\int_{x_0}^t a} f(t)\,dt\right]$ where $C=0$ since $y(x_0)=0$

$$= e^{-\int_{x_0}^x a}\int_{x_0}^{x} e^{\int_{x_0}^t a} f(t)\,dt = \int_{x_0}^{x} e^{\int_{x_0}^t a + \int_x^{x_0} a} f(t)\,dt = \int_{x_0}^{x} e^{\int_x^t a} f(t)\,dt \ \checkmark$$]

<u>Note</u>: the Green's function of a first order operator : $\frac{dG}{dx} + a(x)G = \delta(x-t)$

<u>satisfies</u>: • $G(x,t)$ is a solution of the homogeneous equation $\frac{dG}{dx} + a(x)\, G = 0$ for $x<t$.

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Fundamental solutions:

Remark If L is autonomous (i.e. the coefficients do not depend on x) then $G(x,t) = F(x-t)$ is one particular Green function and $y = f * F$ is a particular sol to $Ly = f$.

Then it suffices to find some solutions of $\boxed{LF = \delta(x)}$ and then $y_{part} = \int_{-\infty}^{\infty} f(t)\, F(x-t)\, dt$ is a particular sol to $Ly = f$.

Example 2 Take in Example 1 $a(x) = a = \text{const}$.

Then solve: $\frac{dF}{dx} + aF = \delta(x)$ ⊛

$$\Rightarrow \frac{d}{dx}\left[e^{ax} F\right] = \underbrace{e^{ax}\delta}_{= \delta} \quad \Rightarrow e^{ax} F(x) = H(x) + C$$

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Example 4 Non-homogeneous conditions

solve $\frac{dy}{dx}+ay=f(x)$, $y(x_0)=y_0$

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Example 4.4

Find the fundamental sol of

$$L = \frac{d}{dx}\left[p(x)\frac{d}{dx}\right] \quad \text{where } p, p' \text{ continuous}, p(x) \neq 0$$

$$(LG, \varphi) = (G, L^t\varphi) = (G, L\varphi)$$

$$= \int_{-\infty}^{t} G\,(p\varphi')' + \int_{t}^{\infty} G\,(p\varphi')'$$

$$= Gp\varphi'\Big|_{-\infty}^{t} - \int_{-\infty}^{t} G'p\varphi' + Gp\varphi'\Big|_{t}^{\infty} - \int_{t}^{\infty} G'p\varphi'$$

$$= G(t-)(p\varphi')(t) - G(t+)(p\varphi')(t) - G'p\varphi\Big|_{-\infty}^{t} - G'p\varphi\Big|_{t}^{\infty} + \underbrace{\int_{-\infty}^{t} (G'p)'\varphi}_{=0} + \underbrace{\int_{t}^{\infty} (G'p)'\varphi}_{=0}$$

$$= -\left[G(t_+) - G(t-)\right](p\varphi')(t) + \left[G'(t+) - G'(t-)\right](p\varphi)(t)$$

which must equal $\varphi(t)$ for all φ. $\Rightarrow G(t_+) = G(t-)$

and $\left[G'(t+) - G'(t-)\right]p(t) = 1$

so G continuous at $t=p$

G' has a jump discont. $G'(t+) - G'(t-) = \frac{1}{p'(t)}$

Example 4.5 Find the fundamental solutions of a second order linear operator

$$L = P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y$$

with $P(x) \neq 0$, P, Q, R continuous.

$G(x,t)$ satisfies $LG = \delta(x-t)$, or

$$P(x)\frac{d^2G}{dx^2} + Q(x)\frac{dG}{dx} + R(x)G = \delta(x-t)$$

Hence for $x<t$: $P(x)G'' + Q(x)G + R(x)G = 0$

and also for $x>t$. $\boxed{\text{Theorem}}$

Let y_1, y_2 be two indep. sol. of $Ly=0$

Then for $x<t$: $G(x) = C_1 y_1(x) + C_2 y_2(x)$ — where $C_{1,2}$, $D_{1,2}$ may depend on t!

for $x>t$: $G(x) = D_1 y_1(x) + D_2 y_2(x)$

at $x=t$: $\begin{cases} G(t+) = G(t-) \\ G'(t+) - G'(t-) = \dfrac{1}{P(t)} \end{cases}$

Indeed: for any test function φ we must have

$$(LG, \varphi) = (\delta(x-t), \varphi)$$

Note: $(LG, \varphi) = \left(P(x)\frac{d^2G}{dx^2} + Q(x)\frac{dG}{dx} + R(x)G, \varphi\right)$

$$= \left(\frac{d^2G}{dx^2}, P(x)\varphi\right) + \left(\frac{dG}{dx}, Q(x)\varphi\right) + (R(x)G, \varphi)$$

$$= \left(G, \frac{d^2}{dx^2}(P(x)\varphi)\right) - \left(G, \frac{d}{dx}(Q(x)\varphi)\right) + (G, R(x)\varphi)$$

$$\equiv (G, L^t\varphi)$$ where

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If $L = P(x)\dfrac{d^2}{dx^2} + Q(x)\dfrac{d}{dx} + R(x)$

then $L^t\varphi = \dfrac{d^2}{dx^2}\left[P(x)\varphi\right] - \dfrac{d}{dx}\left[Q(x)\varphi\right] + R(x)\varphi$

is the <u>transpose</u> of L ; $\boxed{\text{Note}}$ if $L = \frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x)$ then $L^t = L$.

<u>So</u>
$$\underbrace{(G, L^t\varphi)}_{\parallel} = \underbrace{(\delta(x-t), \varphi)}_{=\varphi(t)}$$

$$\int_{-\infty}^{\infty} G(x)(L^t\varphi)(x)\,dx$$

$$\int_{-\infty}^{t} G(x)(L^t\varphi)(x)\,dx + \int_{t}^{+\infty} G(x)(L^t\varphi)(x)\,dx$$

$$= \int_{-\infty}^{t}\left(C_1y_1(x) + C_2y_2(x)\right)(L^t\varphi)(x) + \int_{t}^{+\infty}\left(D_1y_1(x) + D_2y_2(t)\right)(L^t\varphi)(x)\,dx$$

We integrate by parts in each integrals;

$$\int_{t}^{\infty}\left[D_1y_1(x) + D_2y_2(x)\right]\left[\frac{d^2}{dx^2}(P(x)\varphi) - \frac{d}{dx}(Q(x)\varphi) + R(x)\varphi\right]$$

$$= \underbrace{\left(D_1y_1(x) + D_2y_2(x)\right)}_{=G}\left[\underbrace{\frac{d}{dx}(P(x)\varphi - Q(x)\varphi}_{=P'\varphi + P\varphi' - Q\varphi}\right]\Bigg|_{t}^{+\infty} - \int_{t}^{\infty}\frac{d}{dx}\underbrace{\left(D_1y_1 + D_2y_2\right)}_{G}\left(\frac{d}{dx}(P\varphi) - Q\varphi\right)$$

$$+ \int_{t}^{\infty}\left(D_1y_1 + D_2y_2\right)R\varphi$$

$=$

$$= -G(t+)\left[P'(t)\varphi(t) + P(t)\varphi'(t) - Q(t)\varphi(t)\right]$$

$$\underbrace{-\int_t^\infty \frac{dG}{dx}\frac{d}{dx}(P\varphi)}_{\text{by parts}} + \int_t^\infty \frac{dG}{dx} Q\varphi + \int_t^\infty G R\varphi$$

$$= -G(t+)\left[P'\varphi + P\varphi' - Q\varphi\right](t) - \frac{dG}{dx}P\varphi\Big|_t^\infty + \int_t^\infty \frac{d^2G}{dx^2}P\varphi + \int_t^\infty \frac{dG}{dx}Q\varphi + GR\varphi$$

$$= -G(t+)\left[P'\varphi + P\varphi' - Q\varphi\right](t) + G'(t+)P(t)\varphi(t) + \int_t^\infty \underbrace{(LG)}_{=0 \text{ for } x>t}\varphi$$

Similarly for the other integral, only instead of $-G(t+)$ we get $+G(t-)$.

All in all we obtain

$$(LG,\varphi) = -\left[G(t+) - G(t-)\right](P'\varphi + P\varphi' - Q\varphi)(t) + \left[G'(t+) - G'(t-)\right]P(t)\varphi(t) \underset{\substack{\uparrow \\ \text{must equal} \\ \text{for all } \varphi}}{=\!=} \varphi(t)$$

$$\Rightarrow \quad G(t+) - G(t-) = 0 \quad \text{so } G \text{ cont. at } x=t$$

$$G'(t+) - G'(t-) = \frac{1}{P(t)} \qquad \square$$

What about boundary value problems?

Example 5 A Sturm–Liouville problem.

solve $\frac{d^2y}{dx^2} = f(x)$ with homogeneous B.C.

say $y(0)=0$
$y(1)=0$.

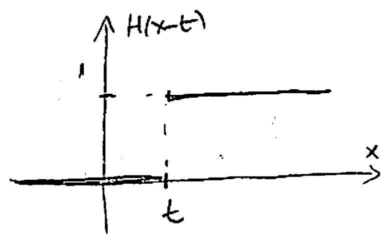
First find Green's function with the given B.C.
Then, any superposition will satisfy the same B.C.
because they are homogeneous: if $y_1, y_2 \ldots y_n \ldots$
satisfy $y_n(0)=0; y_n(1)=0$ then so would $\sum_n c_n y_n$;
same with continuous superposition.

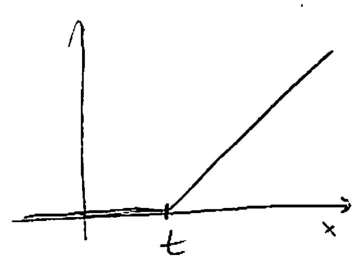
Solution I

Find $G(x,t)$ so that $\frac{d^2G}{dx^2} = \delta(x-t)$, $\left.\begin{matrix} G(0,t)=0 \\ G(1,t)=0 \end{matrix}\right\}$ for all t

Integrate once $\Rightarrow \frac{dG}{dx} = H(x-t) + C(t)$ Integrate again

$$G(x,t) = C_{(t)}x + D_{(t)} + \int_0^x H(s-t)\,ds \qquad \text{where } H(s-t) = \begin{cases} 0 & \text{if } s<t \\ 1 & \text{if } s>t \end{cases}$$





$$\int_{-\infty}^{x} H(s-t)\,ds$$

For $t>0$:

$$\int_0^x H(s-t)\,ds = \begin{cases} \text{for } t\in[0,x] & = \int_t^x ds = x-t \\ \text{for } t>x & = 0 \end{cases} = (x-t)\cdot H(x-t)$$

$$G(x,t) = C(t)x + D(t) + (x-t)\,H(x-t)$$

$G(0,t) = 0$ for all $t\in[0,1] \Rightarrow D(t) = -t\,\underbrace{H(-t)}_{=0} = 0$

$G(1,t) = 0$ for all $t\in[0,1] \Rightarrow C + D + (1-t)\underbrace{H(1-t)}_{=1} = 0 \Rightarrow C = t-1$

$$G(x,t) = (t-1)x + (x-t)\,H(x-t)$$

and $y(x) = \int_0^1 G(x,t)\,f(t)\,dt$

$\boxed{\text{Solution II}}$ $L = \frac{d^2}{dx^2}$ has constant coeff. So we can use a fundamental sol:

solve $\frac{d^2F}{dx^2} = \delta(x) \Rightarrow \frac{dF(x)}{dx} = H(x) + C$

$\Rightarrow F(x) = \int_0^x H(t)\,dt + Cx + D = \begin{cases} Cx + D & \text{if } x < 0 \\ x + Cx + D & \text{if } x > 0 \end{cases}$

$\boxed{\int_0^x H(t)\,dt = xH(x)}$

choose $C = D = 0 \Rightarrow F(x) = x\,H(x)$

Then $y_p = f * F$ is a particular sol of $Ly = f$

$y_p(x) = \int_a^1 f(t)F(x-t)\,dt$. BECAUSE:

$\boxed{\text{Proposition}}$ Let L be a linear diff. op with constant coefficients, e.g. $Ly = \alpha y'' + \beta y' + \gamma y$.
Let F be a fundamental solution: $LF = \delta(x)$
Then a particular sol. of $Ly = f(x)$ for $x \in [a,b]$ is
$y_p(x) = f * F(x) = \int_a^b f(t)F(x-t)\,dt$ for $x \in [a,b]$

(in other words, it is like $f = 0$ outside $[a,b]$)

Proof Show that $Ly_p = f(x)$ for all $x \in [a,b]$.
Let φ be any test function with $\varphi^{(k)}(a) = \varphi^{(k)}(b) = 0$, $k = 0,1,2\ldots$ $\in C_0^\infty(a,b)$

Principle The test functions used must vanish on the boundary of the domain of interest so that after integration all boundary terms are zero

Show that $Ly_p = f$ in the sense of distributions, that is show that

$$(Ly_p, \varphi) = (f, \varphi) \quad \text{for all } \varphi \in C_c^\infty(a,b).$$

GF
6C

$$\boxed{(Ly_p,\varphi)} = (\alpha y_p'' + \beta y_p' + \gamma y_p, \varphi) = \alpha(y_p'', \varphi) + \beta(y_p', \varphi) + \gamma(y_p, \varphi)$$

$$= \alpha(y_p, \varphi'') - \beta(y_p, \varphi') + \gamma(y_p, \varphi) = \boxed{(y_p, L^t\varphi)} = (*)$$

where $L^t = \alpha \frac{d^2}{dx^2} - \beta \frac{d}{dx} + \gamma$ is $\boxed{\text{the transpose}}$ (or "adjoint") of L

$$(*) = \int_{-\infty}^{\infty} y_p(x)\, L^t\varphi(x) \underset{\substack{\uparrow \\ \varphi = 0 \text{ outside } [a,b]}}{=} \int_a^b y_p(x)\, L^t\varphi(x)$$

$$= \int_a^b dx \int_a^b dt\, f(t) F(x-t) L^t\varphi(x) = \int_a^b dt\, f(t) \int_a^b dx\; F(x-t) L^t\varphi(x)$$

$$= \int_a^b dt\; f(t) \underbrace{\int_a^b dx \underbrace{(LF)(x-t)}_{=\delta(x-t)} \varphi(x)\, dx}_{=\varphi(t)} = (f, \varphi) \qquad \square$$

So: back to our example

$$y_p(x) = \int_{-\pi}^{\pi} f(t) F(x-t)\, dt = \int_{-\pi}^{\pi} f(t)(x-t) H(x-t)\, dt$$

$$= \int_{-\pi}^{x} f(t)(x-t)\, dt$$

Gen. sol $y(x) = y_p(x) + y_{homg}(x)$

#

<u>Example6</u> A Sturm Liouville problem with nonhomogeneous BC:

$$\frac{d^2y}{dx^2} = f(x) \quad , \quad y(0) = A, \quad y(1) = B.$$

<u>Transform it into a problem with homogeneous B.C.</u>: find <u>any</u> function $g(x)$ so that $g(0) = A$ and $g(1) = B$. Then substitute $y = g + w$

and w will satisfy
$$\left.\begin{array}{l} w(0) = y(0) - g(0) = 0 \\ w(1) = y(1) - g(1) = 0 \end{array}\right\} \text{homogen. B.C}$$

and the equation

$$\frac{d^2w}{dx^2} - g''_{(x)} = f(x)$$

the simplest g: find g so that $g''(x) \equiv 0$ (sol. of the homog. eq)

We can do this here, look for $N(x)$ linear, $g(x) = mx + n$

and we must have
$$\begin{array}{l} g(0) = n = A \\ g(1) = m + n = B \end{array} \quad \text{so } g(x) = (B-A)x + A$$

So now we solve $\frac{d^2w}{dx^2} = f(x)$, $w(0) = 0$, $w(1) = 0$

as before, then $y = w + g$.

<u>Example 7</u> Not all non-homog. S-L problems have solutions; Sometimes the solution may not be unique.

Solve $\frac{d^2y}{dx^2} = f(x)$, $y'(0)=0$, $y'(1)=0$.

Find Green's function: as before, in Example 5,

$$G(x,t) = C(t)x + D(t) + (x-t)H(x-t) \quad \text{for } t\in[0,1]$$

Now we need to impose B.C. to find C and D but $(x-t)H(x-t)$ is not diff at $x=t$.

We found the gen. sol. $y(x) = y_{homg} + y_{part}$

$$y(x) = C_1 x + C_2 + \int_0^1 f(t)(x-t)H(x-t)\,dt$$

For example, say $f(t) = \text{const} = f$

then $y(x) = C_1 x + C_2 + f\int_0^1 (x-t)H(x-t)\,dt \overset{x-t=s}{=}$

$$= C_1x + C_2 + f\int_{x-1}^{x} sH(s)\,ds \underset{\text{for } x\in[0,1]}{=} C_1 x + C_2 + f\int_0^x s\,ds$$

$$= C_1 x + C_2 + f\frac{x^2}{2}$$

So $y'(x) = C_1 + fx$ and $y'(0) = C_1 = 0$

$y'(1) = C_1 + f = f \neq 0$! No solution!

To understand what is going on, integrate directly $\frac{d^2y}{dx^2} = f(x)$

$$\Rightarrow y'(x) = C_1 + \int_0^x f(s)\,ds \Rightarrow y(x) = C_1x + C_2 + \int_0^x dt \int_0^t f(s)\,ds$$

$y'(0) = C_1 = 0$

$y'(1) = \int_0^1 f(s)\,ds$ The problem has solution if and only if $\boxed{\int_0^1 f(s)\,ds = 0}$

Assume that $\int_0^1 f(s)\,ds = 0$

Then $y(x) = C_2 + \int_0^x dt \int_0^t f(s)\,ds$

is a one-parameter family of solutions!

(not unique!)

* * *

It is desirable to have a theorem which guarantees existence and uniqueness of solutions.

It can be proved (but not done here) that

<u>Theorem</u> The following are equivalent (Fredholm's alternative)

1. The only solution to the homogeneous BV is the zero function

2. The inhomog BVP has a unique sol. for any choice of f (forcing funct)

3. The BVP admits a Green's function.