The Green's function of the adjoint problem

I. Real-valued functions

Suppose we want to solve  

\[ p(x) \frac{du}{dx} = f(x) \]

\[ u(0) = 0 \]

\[ \frac{du}{dx} \text{ in } x \in [0,1], \ p(x) > 0, \text{ positive} \]

We know that \( u(x) = \int_0^x \frac{f(\xi)}{p(\xi)} d\xi \) is the solution, but let us pretend we do not know and use the Green's function method.

Find \( G(x,t) \) so that \( p(x) \frac{dG}{dx} = \delta(x-t), \ G(0,t) = 0 \)

\[ \Rightarrow \frac{dG}{dx} = \frac{1}{p(x)} \delta(x-t) = \frac{1}{p(t)} \delta(x-t) \] and integrate

\[ \Rightarrow G(x,t) = \frac{1}{p(t)} H(x-t) + c(t) \]

\( G(0,t) = \begin{cases} 1 & (\text{for } t > 0) \\ 0 & \end{cases} \Rightarrow G(x,t) = \frac{1}{p(t)} H(x-t) \)

Solution \( u(x) = \int_0^x G(x,t)f(t)dt = \int_0^x \frac{f(t)}{p(t)} dt \)

Now the adjoint problem:

\[ L = p(x) \frac{du}{dx}, \quad D(L) = \left\{ u \in L^2(0,1) \mid u' \in L^2(0,1), \ u(0) = 0 \right\} \]

Find \( L^* : \) for real-valued functions, the distribution pairing is the \( L^2 \)-inner product.
so the $L^2$-adjoint is the transpose of an operator between dual spaces.

Find $L^*$: we want $\langle Lu, v \rangle = \langle u, L^*v \rangle$

for all $u \in \mathcal{D}(L)$ and all $v \in \mathcal{D}(L^*)$ to be determined.

\[
\langle Lu, v \rangle = \int_0^1 (Lu)(x)v(x) \, dx = \int_0^1 \left( p(x) \frac{d}{dx} u(x) \right) v(x) \, dx = (by \text{ parts})
\]

\[
= \left. p(x)u(x)v(x) \right|_0^1 - \left[ u(x) \left( \frac{d}{dx} p(x)v(x) \right) \right]_0^1 \, dx
\]

\[
= (p(x)u(x)v(x) - p(0)v(0)u(0))_0^1 + \int_0^1 u(x) \left[ -\frac{d}{dx} (p(x)v(x)) \right] \, dx
\]

must be equal to $L^*v$

\[
\downarrow
\]

$v(1) = 0$

\[
\downarrow
\]

$\mathcal{D}(L^*) = \{ v \in L^2[0,1], v' \in L^2[0,1], v(1) = 0 \}$

Let us find the Green's function for the dual problem:

\[
\left\{ \begin{array}{ll}
L^*v & = f(x) \\
v(1) = 0 &
\end{array} \right.
\]

\[
\Downarrow
\]

\[
\frac{d}{dx} \left[ p(x) \tilde{G}(x,t) \right] = \delta(x-t)
\]

\[
\tilde{G}(1,t) = 0 \text{ for all } t \in (0,1)
\]

\[
\Rightarrow \frac{d}{dx} \left[ p(x) \tilde{G}(x,t) \right] = -\delta(x-t) \text{ in density}
\]

\[
p(x) \tilde{G}(x,t) = -H(x-t) + C(t)
\]

\[
p(x) \tilde{G}(1,t) = -1 + C(t) ( \text{ for } x < 1 ) = 0 \quad \text{Note!}
\]

\[
\Rightarrow \tilde{G}(x,t) = \frac{1}{p(x)} \left[ 1 - H(x-t) \right] = \frac{1}{p(x)} H(t-x) \downarrow G(t,x)
\]
Note this general fact:

**Theorem**

\[ G(x, t) \text{ is the Green's function of a problem} \]

\[ \Rightarrow G(t, x) \text{ is the Green's function of the adjoint problem.} \]

II. Complex-valued functions

**Example** take \( p(x) = i \) in the example before

\[ Lu = i \frac{d}{dx} u \]

\[ u(0) = 0 \]

Here we have to define distributions as conjugate-linear.

Distributions of type-function act as

\[ (\phi, f) = \int \phi(x) f(x) \, dx = \langle \phi, f \rangle \]

(Note that \( \bar{\delta} = \delta \) since \( \delta = \lim \delta_n \) with \( \delta_n \) real-valued.)

Green's function: \( \forall \phi(x) \; i \frac{d}{dx} G(x, t) = \delta(x-t) \)

No \[ G(x, t) = -i \frac{H(x-t)}{t} + C(t) \]

\[ G(0, t) = 0 \Rightarrow C(t) = 0 \; \text{so} \]

\[ G(x, t) = -i \frac{H(x-t)}{t} \]
Now find the adjoint problem:

\[ L = i \frac{d}{dx}, \quad D(L) = \{ u \in L^2[0,1] | u' \in L^2[0,1], u(0) = 0 \} \]

\[ \left< u, v \right> = \int_0^1 \overline{u(x)} v(x) \, dx = \int_0^1 i \left( \overline{u'(x)} v'(x) \right) \, dx \] (by parts)

\[ = -i u(x) v'(x) \bigg|_0^1 + \int_0^1 \overline{u(x)} v''(x) \, dx \]

\[ = -i u(1) v'(1) + i \overline{u(0)} v'(0) + \left< u, \frac{i d}{dx} v \right> \]

\[ \text{no } D(L^*) = \{ v \in L^2 : \overline{v'} \in L^2, v(0) = 0 \} \]

\[ \text{Green's function} \quad \int \frac{i d}{dx} = \delta(x-t) \]

\[ \tilde{G}(1,t) = 0 \quad \text{for all } t \in (0,1) \]

\[ \tilde{\Phi}(t) = -i H(x,t) + C(t) \]

\[ \tilde{\Phi}(1,t) = -i + C(t) = 0 \implies C(t) = i \]

\[ \tilde{G}(x,t) = i \left( 1 - H(x-t) \right) = i H(t-x) = \Theta(t-x) \]

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**Theorem** The Green's function of the adjoint problem

\[ \text{is } G(t,x) \text{ if } G(x,t) \text{ is the Green's function of the problem.} \]
Proof of the Theorem

Let \( L(x, \frac{d}{dx}) = \sum_{k=0}^{n} a_k(x) \frac{d^k}{dx^k} \) be a linear differential operator.

Find its formal adjoint (i.e., we assume all the boundary values cancel out nicely): if \( \psi, \phi \in \mathcal{C} \)

\[
\langle L\psi, \phi \rangle = \int \frac{1}{2} \sum_{k=0}^{n} a_k(x) \frac{d^k}{dx^k} \phi(x) \psi(x) \, dx = \sum_{k=0}^{n} (-1)^k \int \frac{d^k}{dx^k} \left( a_k(x) \phi(x) \right) \psi(x) \, dx
\]

by parts

\[
\Rightarrow \quad \langle \psi, L^* \phi \rangle
\]

Thus, \( L^* \phi = \sum_{k=0}^{n} (-1)^k \frac{d^k}{dx^k} (a_k(x) \phi(x)) \) is the formal adjoint.

The domain of \( L \): if \( a_n \neq 0 \) then we need \( n \) conditions:

\( n \) independent linear functionals \( B_i[u] = 0, i=1, 2, \ldots, n \)

(recall that we can always assume homogeny for non-homogeny problems)

\( D(L) = \left\{ u \in L^2 \mid u, u^{(n)} \in L^2, B_i[u] = 0, i=1, \ldots, n \right\} \)

\( D(L^*) \) will include exactly those conditions (and no more) that ensure the boundary terms which come from integration by parts vanish.

They can be written down, but they are long formulas.
For example, \( L = \frac{1}{\nu} \left[ -\frac{d}{dx}(\rho(x) \frac{d}{dx}) + \zeta(x) \right] \)

with \( \nu > 0 \), but \( \rho \) complex valued, \( \rho(x) \neq 0 \) in \( (a, b) \)

\( D(L) = \left\{ y \in L^2[a, b] \mid y', y'' \in L^2[a, b], \alpha y(a) + \alpha' y'(a) = 0, \beta y(b) + \beta' y'(b) = 0 \right\} \)

Find \( L \), \( D(L^*) \)

\[
<Lu, v> = \int_a^b (-(\rho \bar{u}'') + \bar{\zeta} u) v \, dx = \int_a^b (\rho \bar{u}'') v + \int_a^b \bar{\zeta} u v
\]

\[
= -\rho \bar{u}' |^b_a + \int_a^b \rho \bar{u}' v + \int_a^b \bar{\zeta} u v
\]

\[
= -\rho \bar{u}' |^b_a + \rho \bar{u} v |^b_a + \int_a^b -\bar{u} (\rho v')' + \int_a^b -\bar{u} (\bar{\zeta} v)
\]

\[
= \bar{\rho} (\bar{u} v' - \bar{u}' v) |^b_a + \int_a^b \bar{u} \left[ -(\rho v')' + \bar{\zeta} v \right] \quad \text{must} = 0 \quad \text{in all} \quad u \in D(L), \, v \in L^2
\]

\[
p(b)(\bar{u} v' - \bar{u}' v) |^b_a - p(a)(\bar{u} v' - \bar{u}' v) |^a_0 = 0 \quad \text{in all} \quad u \in D(L), \, v \in D(L^*)
\]

We must now simplify this condition.

Assume first that \( \alpha \neq 0, \beta \neq 0 \Rightarrow u(a) = -\frac{\alpha'}{\alpha} u'(a), \, u(b) = -\frac{\beta'}{\beta} u'(b) \)

Then we must have

\[
p(b) \left[ -\frac{\beta'}{\beta} v'(b) - v(b) \right] u'(b) - p(a) \left[ -\frac{\alpha'}{\alpha} u'(a) - v(a) \right] u'(a) = 0
\]

In any \( u(a), u'(b) \Rightarrow -\frac{\beta'}{\beta} v'(b) - v(b) = 0 \)

and \( -\frac{\alpha'}{\alpha} u'(a) - v(a) = 0 \)

\( \Rightarrow D(L^*) = \left\{ v \in L^2[a, b] \mid v', v'' \in L^2, \bar{\alpha} v(a) + \bar{\alpha'} v'(a) = 0, \, \bar{\beta} v(b) + \bar{\beta'} v'(b) = 0 \right\} \)
Recall: \( u \) satisfies the BC of \( L \) if it is reformulated as \( u \in D(L) \).

Also \( D(L^*) \) contains all those \( \nu \) for which

\[
<Lu, \nu> = <\nu, L\nu> \quad \text{for all} \quad \nu \in D(L)
\]

hence the adjoint \( BC \) are exactly those (minimal) in which the boundary terms vanish.

Let \( G(x, t) \) so that \( L(x, \partial_x) G(x, t) = \delta(x-t) \)

\[
BC (\tilde{G}) = 0 \quad \text{(translating: \( G(x,t) \in D(L) \))}
\]

Let \( \tilde{G}(x, t) \) so that \( L^*(x, \partial_x) \tilde{G}(x, t) = \delta(x-t) \)

\[
BC^* (\tilde{G}) = 0 \quad \text{(translating: \( \tilde{G}(\cdot, t) \in D(L^*) \))}
\]

\[
< L(x, \partial_x) G(x, t), \tilde{G}(x, \partial_t) > = < G(x, t), L^*(x, \partial_x) \tilde{G}(x, \partial_t) > \quad \text{so}
\]

\[
< \delta(x-t), \tilde{G}(x, \partial_t) > = < G(x, t), \delta(x-t) >
\]

\[
\langle \delta(x-t), \tilde{G}(x, \partial_t) \rangle = \langle \delta(x-t), G(x, t) \rangle
\]

\[
\| \tilde{G}(x, t) \| = \frac{1}{\| G(x, t) \|}
\]

So: \( \tilde{G}(x, \partial_t) = G(x, t) \)
Consequence: if $G(x,t)$ is the Green's function of a self-adjoint problem then $G(x,t) \in \mathcal{R}$ and $G(x,t) = G(t,x)$.

This is Exercise 4.4 from GFp. 5.05

\[ L = \frac{d}{dx} \left( p(x) \frac{dG}{dx} \right), \quad \text{u}(0) = 0, \quad \text{u}(1) = 0 \quad (\text{where} \ p, p' \ \text{cont}, \ p > 0) \]

Exercise: note $L G = \delta(x-t)$

$G(0,t) = 0, \ G(1,t) = 0$ for all $t \in [0,1]$

\[ -\frac{d}{dx} \left( p(x) \frac{dG}{dx} \right) = \delta(x-t) \quad \odot \]

**Solution**

\[ \frac{\partial G}{\partial x} \bigg|_{x=t} = 0 \quad \text{for} \quad x < t \quad \text{and} \quad \text{for} \quad x > t \]

Integrate

\[ p(x) \frac{dG}{dx} = c(x) \Rightarrow \frac{dG}{dx} = \frac{c(x)}{p(x)} \Rightarrow G(x,t) = \int_{x}^{t} \frac{c(t)}{p} + \text{const} \]

\[ G(x,t) = \begin{cases} c(t) \frac{1}{p} + D(t) \frac{x}{t} & \text{if} \ x < t \\ \int_{x}^{t} \frac{c(t)}{p} + \bar{G}(t) \frac{x}{t} & \text{if} \ x > t \end{cases} \]

$G(0,t) = D(t) = 0$, and $G(1,t) = \bar{G}(t) = 0$ for all $t \in [0,1]$

\[ G(x,t) = \begin{cases} c(t) \frac{1}{p} & \text{if} \ x < t \\ \int_{x}^{t} \frac{c(t)}{p} & \text{if} \ x > t \end{cases} \]

From $\odot$ we see that $G(0,t)$ is continuous at $x=t$

and $-p(x) \frac{dG}{dx}$ has a jump discontinuity of magnitude $1$.

So $G(t+,t) = \tilde{c}(t) \int_{0}^{t} \frac{c}{p} = G(t-,t) = c(t) \int_{0}^{t} \frac{c}{p}$

From $\odot$

\[ \left( -p(x) \frac{dG}{dx} \bigg|_{x=t} \right) - \left( -p(x) \frac{dG}{dx} \bigg|_{x=t-} \right) = -c(t) + c(t) = 1 \]
Solve:
\[
\begin{align*}
\int_{t}^{\infty} y \, P &= c \int_{t}^{\infty} y \, P \\
-\tilde{c} + c &= 1 \\
\Rightarrow c &= \frac{\int_{t}^{\infty} y \, P}{\int_{t}^{\infty} y \, P} \\
\tilde{c} &= \frac{\int_{1}^{t} y \, P}{\int_{1}^{\infty} y \, P}
\end{align*}
\]

Note: \( G(x, t) = G(t, x) \) (as it should, since the problem is in SA)

Solution II

Integrating (\( \star \))

\[
\begin{align*}
p(x) \frac{dG}{dx} &= -H(x-t) + A(t) \\
\Rightarrow G(x, t) &= A(t) \int_{0}^{x} \frac{1}{P} - \int_{0}^{\Delta t} \frac{1}{p_{B}(\Delta)} H(\Delta t) \, d\Delta + B(t) \\
&= \begin{cases} 
0 & \text{if } t \leq x < t \\
\int_{x}^{\infty} y \, P + B(t) 
\end{cases}
\]

\[G(0, t) = B(t) = 0 \]

\[G(1, t) = A(t) \int_{0}^{1} y \, P - \int_{0}^{1} y \, P = 0 \]

\[G(\infty, t) = \frac{\int_{t}^{\infty} y \, P}{\int_{t}^{\infty} y \, P} - B(1) x < t \]

The name as before, of course.
Translation - invariant equations

General linear equation: \( L(x, \frac{d}{dx}) = \sum_{k=0}^{n} a_k(x) \frac{d^k}{dx^k} \)

But if \( a_k(x) \equiv a_k \) are constants, then the linear op.
\( L(\frac{d}{dx}) = \sum_{k=0}^{n} a_k \frac{d^k}{dx^k} \) has constant coefficients

and if \( Lu = 0 \) then \( \phi(x) = u(x-c) \) is also a solution,
for any constant \( c \) (it is easy to check). So \( L(\frac{d}{dx}) \) is translation - invariant

If \( G(x,t) \) satisfies \( L(\frac{d}{dx}) G(x,t) = \delta(x-t) \)

then so does \( G(x-c, t-c) \) for any constant \( c \).

In particular, for \( c=t \) we get that \( G(x-t, 0) \) also satisfies \( \equiv F(x-t) \)

So if \( F \) is a fundamental solution: \( L(\frac{d}{dx}) F(x) = \delta \)

then \( G(x,t) = F(x-t) \) satisfies \( \bigcirc \)

So: Any Green's function of \( L(\frac{d}{dx}) + BC \) (if it exists) has
the form \( G(x,t) = F(x-t) + \text{sol. of homog. eq.} \)
and the sol. of \( L(\frac{d}{dx}) u = f + BC \) is \( u(x) = \int F(x-t) f(t) dt = (F \ast f)(x) \)

Example Consider the previous example with \( p(x) \equiv p = \text{const.} \)
\( L = -p \frac{d^2}{dx^2} , u(0) = 0, u(1) = 0. \)
Its Green's function is

\[ G(x,t) = \begin{cases} \frac{1}{p} (1-t)x & \text{if } x < t \\ \frac{1}{p} (1-x)t & \text{if } x > t \end{cases} \]

\[ = \frac{1}{p} (1-t)x + \frac{1}{p} \left[ (1-x)t - (1-t)x \right] H(x-t) \]

\[ = \frac{1}{p} (1-t)x - \frac{1}{p} (x-t) H(x-t) \]

Or, we could solve \(-p \frac{d^2 F}{dx^2} = \delta\)

which gives \(-p \frac{dF}{dx} = H(x)\) no \(F(x) = -\frac{1}{p} x H(x) + \text{any sol of the}\)

\(\text{homog eq.} (\equiv Cx+D)\)

\(\text{any}\)

Then \(G(x,t) = F(x-t) + \text{homog} = -\frac{1}{p} (x-t)H(x-t) + Cx + D\)

which is what we put. Then the solution of the homogeneous \(Cxd\)

is determined so that the BC are satisfied.