The Green's function of the adjoint problem

I. Real-valued functions

Suppose we want to solve $p(x)\frac{du}{dx} = f(x)$

$u(0)=0$

for $x \in [0,1]$, $p(x)>0$, p continuous.

We know that $u(x) = \int_0^x \frac{f(s)}{p(s)}\,ds$ is the solution,

but let us pretend we do not know and use the Green's function method

Find $G(x,t)$ so that $p(x)\frac{dG}{dx} = \delta(x-t)$, $G(0,t)=0$

$$\Rightarrow \frac{dG}{dx} = \frac{1}{p(x)}\delta(x-t) = \frac{1}{p(t)}\delta(x-t) \quad \text{and integrate}$$

$$\Rightarrow G(x,t) = \frac{1}{p(t)}H(x-t) + C(t)$$

$$G(0,t) = C(t) \ (\text{for } t>0) = 0 \Rightarrow G(x,t) = \frac{1}{p(t)}H(x-t)$$

Solution $u(x) = \int_0^1 G(x,t)f(t)\,dt = \int_0^x \frac{f(t)}{p(t)}\,dt$

Now the adjoint problem:

$$L = p(x)\frac{d}{dx}, \quad D(L) = \{u \in L^2(0,1) \mid u' \in L^2(0,1),\ u(0)=0\}$$

Find L^*: for real-valued functions, the distribution pairing $\equiv$ the L^2-inner product, so the L^2-adjoint $\equiv$ the transpose of an operator between dual spaces

Find L^*: we want $\langle Lu, v\rangle = \langle u, L^*v\rangle$
for all $u \in D(L)$ and all $v \in D(L^*)$ ← to be determined.

$$\langle Lu, v\rangle = \int_0^1 (Lu)(x)\,v(x)\,dx = \int_0^1 p(x)\frac{du}{dx}\,v(x) = (\text{by parts})$$

$$= p(x)u(x)v(x)\Big|_0^1 - \int_0^1 u(x)\frac{d}{dx}[p(x)v(x)]\,dx$$

$$= \underbrace{p(1)u(1)v(1)}_{\text{must }=0} - p(0)v(0)\underset{\overset{\|}{0}}{u(0)} + \int_0^1 u(x)\underbrace{\left[(-)\frac{d}{dx}(p(x)v(x))\right]}_{\underset{\text{must be}}{=} L^*v}dx$$

$$\Downarrow$$
$$v(1)=0$$
$$\Downarrow$$
$$D(L^*) = \{v \in L^2[0,1] \,/\, v' \in L^2[0,1],\ v(1)=0\}$$

Let us find the Green's function $\tilde{G}$ for the dual problem

$$\begin{cases} L^*v \\ v(1)=0 \end{cases} \qquad \begin{cases} -\dfrac{d}{dx}\left[p(x)\tilde{G}(x,t)\right] = \delta(x-t) \\ \tilde{G}(1,t)=0 \text{ for all } t\in[0,1) \end{cases}$$

$$\Rightarrow \frac{d}{dx}\left[p(x)\tilde{G}(x,t)\right] = -\delta(x-t) \quad \text{integrate} \Rightarrow$$

$$p(x)\tilde{G}(x,t) = -H(x-t) + C(t)$$

$$p(1)\tilde{G}(1,t) = -1 + C(t)\ (\text{for } t<1) = 0 \qquad \underline{\text{Note!}}$$

$$\Rightarrow \tilde{G}(x,t) = \frac{1}{p(x)}[1-H(x-t)] = \frac{1}{p(x)}H(t-x) \overset{\downarrow}{=\!=} G(t,x)$$

Note this general fact:

Theorem $G(x,t)$ is the Green's function of a problem
$\Longrightarrow$ $G(t,x)$ is the Green's function of the adjoint pb.

II. Complex-valued functions

Example take $p(x) = i$ in the example before

$$Lu = i\frac{d}{dx}u$$
$$u(0) = 0$$

Here we have to define distributions as conjugate-linear.
Distributions of type-functions f act as

$$(\varphi, f) = \int \varphi(x) f(x)\,dx = \langle \bar{\varphi}, f\rangle$$

(Note that $\bar{\delta} = \delta$ since $\delta = \lim_n f_n$ with f_n real-valued.)

Green's function: solve $i\frac{d}{dx}G(x,t) = \delta(x-t)$

so $G(x,t) = -iH(x-t) + C(t)$

$G(0,t) = 0 \rightarrow C(t) = 0$ so

$G(x,t) = \quad -iH(x-t)$

Now find the adjoint problem:

$$L = i\frac{d}{dx}, \quad D(L) = \{u \in L^2[0,1] \mid u' \in L^2[0,1], u(0)=0\}$$

$$\langle Lu, v\rangle = \int_0^1 \overline{Lu(x)}\, v(x)\,dx = \int_0^1 -i\,\overline{u'(x)}\, v(x)\,dx = \text{(by parts)}$$

$$= -i\,\overline{u(x)}v(x)\Big|_0^1 + \int_0^1 \overline{u(x)}\; i\frac{dv}{dx}\,dx$$

$$= -i\,\overline{u(1)}\underbrace{v(1)}_{\text{must be } 0} + i\,\underbrace{\overline{u(0)}}_{=0}v(0) + \langle u, \underbrace{i\tfrac{d}{dx}v}_{L^*v}\rangle$$

so $D(L^*) = \{v \in L^2 \mid v' \in L^2,\ v(1)=0\}$

Green's function $\begin{cases} i\dfrac{d\tilde{G}}{dx} = \delta(x-t) \\ \tilde{G}(1,t) = 0, \text{ for all } t \in [0,1] \end{cases}$

$\tilde{G}(x,t) = -iH(x-t) + C(t)$

$\tilde{G}(1,t) = -i + C(t) = 0 \Rightarrow C(t) = i$

so $\tilde{G}(x,t) = i(1 - H(x-t)) = i\,H(t-x) = \overline{G(t,x)}$

<u>Theorem</u> The Green's function of the adjoint problem is $\overline{G(t,x)}$ if $G(x,t)$ is the Green's function of the problem.

Proof of the Theorem

Let $L(x, \frac{d}{dx}) = \sum_{k=0}^{n} a_k(x) \frac{d^k}{dx^k}$ be a linear diff. op.

Find its formal adjoint (i.e. we assume all the boundary values cancel out nicely; if $\varphi, \psi \in C_0^\infty$

$$\langle L\varphi, \psi\rangle = \int \sum_{k=0}^{n} a_k(x) \frac{d^k}{dx^k}\varphi \; \overline{\psi(x)}\, dx \underset{\text{by parts}}{=} \sum_{k=0}^{n} (-1)^k \int \overline{\varphi(x)}\, \frac{d^k}{dx^k}\left(\overline{a_k(x)}\, \psi(x)\right) dx$$

$= \langle \varphi, L^*\psi\rangle$

if $L^*\psi = \sum_{k=0}^{n} (-1)^k \frac{d^k}{dx^k}\left(\overline{a_k(x)}\,\psi\right)$ is the formal adjoint

The domain of L: if $a_n \neq 0$ then we need n conditions:
n independent linear functionals $B_j[u] = 0$, $j = 1, 2, \ldots n$
(recall that we can always assume homog BC for nonhomog DE)

$D(L) = \{u \in L^2 \mid u', \ldots u^{(n)} \in L^2,\ B_j[u] = 0,\ j = 1 \ldots n\}$

$D(L^*)$ will include exactly those conditions (and no more) that insure the boundary terms which come from integration by parts vanish.

They can be written down, but there are long formulas, ...

For example, $L = \frac{1}{w}\left[-\frac{d}{dx}(p(x)\frac{d}{dx}) + q(x)\right]$ SF, SB

with $w>0$, but p, q complex valued, $p(x) \neq 0$ on $[a,b]$

$$D(L) = \{ y \in L^2[a,b] \mid y', y'' \in L^2[a,b] \ , \ \alpha y(a) + \alpha' y'(a) = 0, \ \beta y(b) + \beta' y'(b) = 0 \}$$

Find $L, D(L^*)$

$$\langle Lu, v\rangle = \int_a^b (-(\bar p \bar u')' + \bar q \bar u)\, v\, dx = \int_a^b -(\bar p \bar u')' v + \int_a^b \bar q \bar u v$$

$$= -\bar p \bar u' v\Big|_a^b + \int_a^b \bar p \bar u' v' + \int_a^b \bar q \bar u v$$

$$= -\bar p \bar u' v\Big|_a^b + \bar p \bar u v'\Big|_a^b + \int_a^b -\bar u (\bar p v')' + \int_a^b \bar u (\bar q v)$$

$$= \underbrace{\bar p (\bar u v' - \bar u' v)\Big|_a^b}_{\substack{\text{must } = 0 \\ \text{for all } u \in D(L),\, v \in D(V)}} + \int_a^b \bar u \underbrace{\left[-(\bar p v')' + \bar q v\right]}_{= L^* v}$$

$$p(b)(\bar u v' - \bar u' v)\big|_{x=b} - p(a)(\bar u v' - \bar u' v)\big|_{x=a} = 0 \text{ for all } u \in D(L),\ v \in D(L^*)$$

We must now simplify this condition.

Assume first that $\alpha \neq 0, \beta \neq 0 \Rightarrow u(a) = -\frac{\alpha'}{\alpha} u'(a)$, $u(b) = -\frac{\beta'}{\beta} u'(b)$

(and $u'(a), u'(b)$ are arbitrary)

Then we must have

$$p(b)\left[-\frac{\bar\beta'}{\bar\beta} v'(b) - v(b)\right] u'(b) - p(a)\left[-\frac{\bar\alpha'}{\bar\alpha} v'(a) - v(a)\right] u'(a) = 0$$

for any $u'(a), u'(b) \Longrightarrow -\frac{\bar\beta'}{\bar\beta} v'(b) - v(b) = 0$

and $-\frac{\bar\alpha'}{\bar\alpha} v'(a) - v(a) = 0$

$$\Rightarrow D(L^*) = \{ v \in L^2(a,b) \mid v', v'' \in L^2, \ \bar\alpha v(a) + \bar\alpha' v'(a) = 0, \ \bar\beta v(b) + \bar\beta' v'(b) = 0 \}$$

Recall: u satisfies the BC of L:
is reformulated as $u \in D(L)$

Also $D(L^*)$ contains all those v for which

$$\langle Lu, v\rangle = \langle u, L v\rangle \quad \text{for all } u \in D(L)$$

hence (BC^* the adjoint BC) are exactly those (minimal) for which the boundary terms vanish.

Let $G(x,t)$ so that $L(x,\partial_x)\, G(x,t) = \delta(x-t)$

$$BC[G] = 0 \quad \Big(\text{translation}; \ \underset{\text{for all } t}{G(\cdot,t) \in D(L)}\Big)$$

Let $\tilde{G}(x,t)$ so that $L^*(x,\partial_x)\, \tilde{G}(x,t) = \delta(x-t)$

$$BC^*[\tilde{G}] = 0 \quad \Big(\text{translation}; \ \underset{\text{for all } t}{\tilde{G}(\cdot,t) \in D(L^*)}\Big)$$

$$\langle L(x,\partial_x)\, G(x,t), \tilde{G}(x,s)\rangle = \langle G(x,t), L^*(x,\partial_x)\, \tilde{G}(x,s)\rangle \quad \text{so}$$

$$\langle \delta(x-t), \tilde{G}(x,s)\rangle = \overline{\langle G(x,t), \delta(x-s)\rangle}$$

$$\underbrace{\langle \delta(x-t), \tilde{G}(x,s)\rangle}_{\tilde{G}(t,s)} = \overline{\langle \delta(x-s), G(x,t)\rangle} = \overline{G(s,t)}$$

So: $\overline{\tilde{G}(t,s)} = G(s,t)$

Consequence: if $G(x,t)$ is the Green's function of a self-adjoint problem then $G(x,t) \in \mathbb{R}$ and $G(x,t) = G(t,x)$

This is Ex 4.4 from GF p. 5.05

Ex. $L = -\frac{d}{dx}\left(p(x)\frac{d}{dx}\right)$, $u(0)=0$, $u(1)=0$ (where p, p' cont, $p>0$)

Find G: solve $LG = \delta(x-t)$

$G(0,t)=0$, $G(1,t)=0$ for all $t \in [0,1]$

$$-\frac{d}{dx}\left(p(x)\frac{dG}{dx}\right) = \delta(x-t) \qquad (*)$$

Solution I

so $-\frac{d}{dx}\left(p(x)\frac{dG}{dx}\right) = 0$ for $x<t$ and for $x>t$

Integrate $\underbrace{p(x)\frac{dG}{dx} = c(t)}_{(**)} \Rightarrow \frac{dG}{dx} = \frac{c(t)}{p(x)} \Rightarrow G(x,t) = \int_{\circ}^{x} \frac{c(t)}{p} + \text{const}$

any point

$$G(x,t) = \begin{cases} c(t)\int_0^x \frac{1}{p} + D(t) & \text{for } x<t \\ \tilde{c}(t)\int_1^x 1/p + \tilde{D}(t) & \text{for } x>t \end{cases}$$

$G(0,t) = D(t) = 0$ & $G(1,t) = \tilde{D}(t) = 0$ so

$$G(x,t) = \begin{cases} c(t)\int_0^x \frac{1}{p} & \text{for } x<t \\ \tilde{c}(t)\int_1^x 1/p & \text{for } x>t \end{cases}$$

From $(*)$ we see that $G(\cdot,t)$ is continuous at $x=t$ and $-p(x)\frac{dG}{dx}$ has a jump discontinuity of magnitude 1

So $G(t+,t) = \tilde{c}(t)\int_1^t 1/p = G(t-,t) = c(t)\int_0^t 1/p$

From $(**)$ $\left(-p(x)\frac{dG}{dx}\Big|_{x=t+}\right) - \left(-p(x)\frac{dG}{dx}\Big|_{x=t-}\right) = -\tilde{c}(t) + c(t) = 1$

Solve: $\begin{cases} \tilde{C}\int_1^t 1/p = C\int^t 1/p \\ -\tilde{C}+C=1 \end{cases} \Rightarrow C = \dfrac{\int_t^1 1/p}{\int_0^1 1/p}, \quad \tilde{C} = \dfrac{\int_t^0 1/p}{\int_0^1 1/p}$

$$G(x,t) = \begin{cases} -\dfrac{\int_1^t 1/p}{\int_0^1 1/p}\int_0^x 1/p & \text{for } x<t \\[2ex] -\dfrac{\int_0^t 1/p}{\int_0^1 1/p}\int_1^x \dfrac{1}{p} & \text{for } x>t \end{cases}$$

Note: $G(x,t) = G(t,x)$ (as it should, since the problem in SA)

<u>Solution II</u> Integrate (*)

$$p(x)\frac{dG}{dx} = -H(x-t) + A(t) \Rightarrow$$

$$G(x,t) = A(t)\int_0^x \frac{1}{p} - \underbrace{\int_0^x \frac{1}{p(s)}\cdot \underbrace{H(s-t)}_{=0 \text{ for } s<t} ds}_{=0 \text{ for } x<t} + B(t)$$

$$= A(t)\int_0^x 1/p - H(x-t)\int_t^x 1/p + B(t)$$

$G(0,t) = B(t) = 0$

$G(1,t) = A(t)\int_0^1 1/p - \int_t^1 1/p = 0$

so $G(x,t) = \dfrac{\int_t^1 1/p}{\int_0^1 1/p}\int_0^x 1/p - H(x-t)\int_t^x 1/p = \begin{cases} \dfrac{\int_t^1 1/p}{\int_0^1 1/p}\int_0^x 1/p & \text{for } x<t \\[2ex] \dfrac{\int_t^1}{\int_0^1}\int_0^x - \int_t^x = \dfrac{\int_0^t \int_x^1}{\int_0^1} & \text{for } x>t \end{cases}$

the same as before, of course.

Translation-invariant equations

General linear equation: $L(x, \frac{d}{dx}) = \sum_{k=0}^{n} a_k(x) \frac{d^k}{dx^k}$

But if $a_k(x) \equiv a_k =$ constants, then the linear op.
$L(\frac{d}{dx}) = \sum_{k=0}^{n} a_k \frac{d^k}{dx^k}$ has constant coefficients

and if $Lu = 0$ then $v(x) = u(x-c)$ is also a solution, for any constant c (it is easy to check). So $L(\frac{d}{dx})$ is translation-invariant

If $G(x,t)$ satisfies $L(\frac{d}{dx})\, G(x,t) = \delta(x-t)$ ⊛

then so does $G(x-c, t-c)$ for any constant c

In particular, for $c=t$ we get that $\underbrace{G(x-t,0)}_{\equiv F(x-t)}$ also satisfies ⊛

So if F is a fundamental solution: $L(\frac{d}{dx})F(x) = \delta$
then $G(x,t) = F(x-t)$ satisfies ⊛

So: Any Green's function of $L(\frac{d}{dx})$ + BC (if it exists) has the form $G(x,t) = F(x-t) +$ sol. of homog. eq.
and the sol. of $L(\frac{d}{dx})u = f$ + BC is $u(x) = \int F(x-t) f(t)\,dt = (F * f)(x)$

Example Consider the previous example with $p(x) \equiv p$ = const
$L = -p\frac{d^2}{dx^2}$, $u(0) = 0, u(1) = 0$.

Its Green's function is

$$G(x,t) = \begin{cases} \frac{1}{p}(1-t)x & \text{for } x<t \\ \frac{1}{p}(1-x)t & \text{for } x>t \end{cases} = \frac{1}{p}(1-t)x + \frac{1}{p}\left[(1-x)t - (1-t)x\right]H(x-t)$$
$$= \frac{1}{p}(1-t)x - \frac{1}{p}(x-t)H(x-t)$$

Or, we could solve $-p\frac{d^2F}{dx^2} = \delta$

which gives $-p\frac{dF}{dx} = H(x)$. so $F(x) = -\frac{1}{p}xH(x)$ + any sol of the homog pb. $(\equiv Cx+D)$

Then any $G(x,t) = F(x-t) + \text{homog} = -\frac{1}{p}(x-t)H(x-t) \left[+ Cx + D\right]$

which is what we got. Then the solution of the homog eq $Cx+D$ is determined so that the BC are satisfied.