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Another Application of Green's Functions: Integral Equations

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4.8 Another Application of Green's Function: Integral Equations. —1—

The process of solving the inhomogeneous boundary value problem

$$\frac{d}{dx} p \frac{du}{dx} + [q(x) + \lambda \rho(x)] u = - f(x) \qquad a < x < b$$
$$B_1(u) = d$$
$$B_2(u) = e$$

is somewhat awkward from a numerical and even a conceptual point of view. Solving the differential equation is a local process: one determines the function and its properties at $x+dx$ from those at x. One repeats this step-like process until one has found $u(x)$ for $a \le x \le b$. Upon completion one checks whether the boundary conditions B_1 and B_2 have been satisfied. If

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not, one alters the function u at the point where one started solving the differential equation and the starts all over again. Thus one might have to solve the differential equation many times before one finally obtain the desired solution.

It is evident that this undesirable drudgery is due to the fact that the boundary conditions, which determine the qualitatively important features of the solution u, are stated separately and are not an intrinsic part of the differential equation.

This deficiency can be removed by recasting the boundary value problem in the form of an integral equation. The one-dimensional Sturm-Liouville system with, say, inhomogeneous Dirichlet boundary conditions,

$$\frac{d}{dx} p \frac{du}{dx} + [q(x) + \lambda \rho(x)]\, u = 0$$

$$u(a) = e$$

$$u(b) = f\,,$$

illustrates the general principle. To convert this system into a single integral equation, one considers the corresponding Green's function problem

$$\left[\frac{d}{dx} p \frac{d}{dx} + q(x)\right] G(x;\xi) = -\delta(x-\xi)$$

$$G(a;\xi) = 0$$

$$G(b;\xi) = 0$$

One transposes the term $\lambda \rho(x) u(x)$ to the right hand side of the S-L equation and considers it as an inhomogeneous equation. Multiply this equation by $G(x;\xi)$, multiply the Green's function equation by $u(x)$, subtract, integrate and obtain

$$p(x)\left[G(x;\xi)\frac{du(x)}{dx} - \frac{dG(x;\xi)}{dx}u(x)\right]_{x=a}^{x=b} = -\lambda \int_a^b G(x;\xi)\rho(x)u(x)\,dx + u(\xi)$$

Using reciprocity relation $G(x;\xi) = G(\xi;x)$, and switching variables one finds

$$u(x) = \lambda \int_a^b G(x;\xi)\rho(\xi)u(\xi)\,d\xi + p(a)u(a)\left.\frac{dG(\xi;x)}{d\xi}\right|_{\xi=a} - p(b)u(b)\left.\frac{dG(\xi;x)}{d\xi}\right|_{\xi=b} \qquad (*)$$

This is an integral equation for $u(x)$. Note that the boundary conditions for $u(x)$ are an intrinsic part of the equation: the boundary conditions do not have to be stated separately. Also note that if $u(x)$ satisfies the homogeneous Dirichlet conditions $u(a)=0$, $u(b)=0$, then the integral equation becomes

$$u(x) = \lambda \int_a^b G(x;\xi)\,\rho(\xi)\,u(\xi)\,d\xi \qquad (**)$$

which is an eigenvalue equation for the function u,

4.8.1 Eigenfunctions and Their Integral Equations

To illustrate this integral equation, conside the integral equation satisfied by several of the familiar orthogonal functions.

(i) Trigonometric fuctions:

$$\frac{d^2u}{dx^2} + \lambda u = 0 \qquad u(0) = u(\ell) = 0$$

$$u(x) = \lambda \int_0^\ell G(x;\xi)\,u(\xi)\,d\xi$$

$$G(x;\xi) = \frac{1}{\ell}\begin{cases} x(\ell-\xi) & \text{when } x<\xi \\ \xi(\ell-x) & \text{when } \xi<x \end{cases}$$

Eigenfunctions: $u_n(x) = \sin\frac{n\pi x}{\ell}$; $\lambda = \left(\frac{n\pi}{\ell}\right)^2$; n = intege

(ii) Bessel functions:

$$\frac{1}{x}\frac{d}{dx}x\frac{du}{dx}+\left(\lambda-\frac{n^2}{x^2}\right)u=0 \qquad u \text{ finite at } x=0,\infty$$

$$u(x)=\lambda\int_0^\infty G(x;\xi)\,\xi\,d\xi$$

$$G(x;\xi)=\frac{1}{2n}\begin{cases}\left(\frac{x}{\xi}\right)^n & \text{when } x<\xi\\ \left(\frac{\xi}{x}\right)^n & \text{when } \xi<x\end{cases}$$

Eigenfunction: $u_n(x)=J_n(\sqrt{\lambda}\,x)$; $\quad 0<\lambda<\infty$

(iii) Legendre polynomials

$$\frac{d}{dx}(1-x^2)\frac{du}{dx}+\lambda u=0 \qquad u=\text{finite at } x=\pm1$$

$$u(x)=\lambda\int_{-1}^{1}G(x;\xi)\,u(\xi)\,d\xi-\frac{1}{2}\int_{-1}^{1}u(\xi)\,d\xi$$

$$G(x;\xi)=\frac{1}{2}\begin{cases}\ln\left(\frac{1+x}{1-\xi}\right) & \text{when } x<\xi\\ \ln\left(\frac{1+\xi}{1-x}\right) & \text{when } \xi<x\end{cases}$$

$$u_n(x)=P_\ell(x);\quad \lambda=\ell(\ell+1)\quad;\ \ell=\text{integer.}$$

4.8.2 Types of Integral Equations.

It is evident that different types of boundary value problems give rise to different types of integral equations.

Fredholm Equations

The inhomogeneous boundary value problem gave rise to Eq (*), whose form is

$$u(x)=\lambda\int_a^b K(x;\xi)\,u(\xi)\,d\xi+\varphi(x)\,.\qquad (***)$$

In this case $K(x;\xi)=G(x;\xi)\rho(\xi)$, $\varphi(x)$ is a known function, and $u(x)$ is the unknown function.

The integration limits a and b are fixed. An integral equation for $u(x)$ of the form (***) is called an <u>inhomogeneous Fredholm equation</u> of the <u>second kind</u>.

The expression $K(x;\xi)$ is called the integration kernel of the integral equation.

A <u>homogeneous</u> <u>Fredholm equation</u> <u>of the second kind</u> is obtained by dropping the function $\varphi(x)$,

$$u(x) = \lambda \int_0^b K(x;\xi)\, u(\xi)\, d\xi .$$

Equation (**) and the subsequent eigen value equations are examples of such equations.

A Fredholm equation of the first kind has the form

$$\varphi(x) = \int_0^b K(x;\xi)\, u(\xi)\, d\xi$$

whenever $\varphi(x)$ is a known function and

$u(\xi)$ is the unknown function.

<u>Volterra Equations</u>

Fredholm equations are based on definite integrals. If the integration limits are variable, then the corresponding integral equations are <u>Volterra</u> <u>equations</u>. An <u>inhomogeneous</u> <u>Volterra equation</u> <u>of the second kind</u>, corresponding to Eq. (***), has the form

$$u(x) = \int_0^x K(x;\xi)\, u(\xi)\, d\xi + \varphi(\xi) .$$

If $\varphi = 0$ then one has a homogeneous equation. A <u>Volterra equation of the</u> <u>first kind</u> has the form

$$\varphi(x) = \int_a^x K(x;\xi)\, u(\xi)\, d\xi ,$$

where φ is known and u is the unknown function.