Green's function is the kernel for the resolvent.

Complex integration formulas

Recall that in the SA case we found that if $(L, D(L))$ is SA then the sol. of $(L-z)y = f$

Note: poles of $\mathcal{G}$ = eigenvalues of

is $y = (L-z)^{-1} f = \int_a^b G(x,t,z) f(t)\,dt = \sum_{n=1}^{\infty} \frac{1}{\lambda_n - z} \langle u_n, f \rangle u_n(x)$

The Green's function is the kernel for the resolvent

Since $u_1, u_2 \ldots u_n \ldots$ form an orthonormal basis for our $\mathcal{H}$

note that $P_{\lambda_n} f = \langle u_n, f \rangle u_n$ is the orthonormal projection of f onto u_n hence

$$\mathcal{G}_{(z)} =: (L-z)^{-1} = \sum_{n=1}^{\infty} \frac{1}{\lambda_n - z} P_n \qquad \text{for all } z \neq \lambda_1, \lambda_2 \ldots \lambda_n \ldots$$

→ the spectral resolution of the resolvent

which is the natural extension to infinite dim of the formula we had for matrices (see Ch. Self-adjointness, Sec. 3.7)

Use Cauchy's integral formula (Thm. 34 in), $f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta$ if z is inside γ, if z is outside γ) then also in inf dim we have;

[as in fin.-dim, formula (25), (25) and next one]

Recall: $\phi(L) = \frac{1}{2\pi i} \oint_\gamma \phi(z)(z-L)^{-1}\,dz = -\frac{1}{2\pi i} \oint_\gamma \phi(z)(L-z)^{-1}\,dz$

$$= -\frac{1}{2\pi i} \sum_{n=1}^{\infty} \oint_\gamma \frac{\phi(z)}{\lambda_n - z}\,dz$$

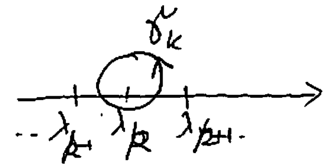
if $\phi(z)$ is an analytic function.

$\mathcal{G}(z)$ = operator-valued analytic function of z

and we can calculate $\oint_\gamma \mathcal{G}(z)\,dz$

Suppose $\gamma \equiv \gamma_k$ encircles only one simple eigenvalue λ_k

then $\oint_{\gamma_k} \mathcal{G}(z)\,dz = \sum_{n=1}^{\infty} \underbrace{\oint_{\gamma_k} \frac{1}{\lambda_n - z}\,dz}_{=0 \text{ for } n\neq k} P_n$



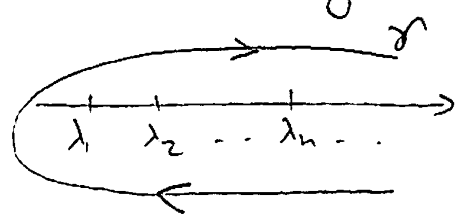
since for $n \neq k$, $f(z) = \frac{1}{\lambda_n - z}$ is analytic inside γ_k

$$= -\oint_{\gamma_k} \frac{dz}{z-\lambda_k} P_k = -2\pi i\, P_k$$

so $-\frac{1}{2\pi i}\oint_{\gamma_k} \mathcal{G}(z)\,dz = P_k$ = the $\perp$ proj onto u_k.

so that $\boxed{-\frac{1}{2\pi i}\oint_{\gamma_k} \mathcal{G}(z)\,dz\, f = P_k f = \langle u_k, f\rangle u_k}$

Now take γ a simple "closed" curve encircling all $\lambda_1, \lambda_2 \ldots$



then $-\frac{1}{2\pi i}\oint_\gamma \mathcal{G}(z)\,dz = \sum_{k=1}^{\infty} P_k = I$

so that $-\frac{1}{2\pi i}\oint_\gamma \mathcal{G}(z)\,dz\; f = f$

Having the Green's function the eigenfunctions are found as

$$\boxed{\langle u_k, 1\rangle u_k = -\frac{1}{2\pi i}\oint_{\gamma_k} \mathcal{G}(z)\,dz\; 1 = -\frac{1}{2\pi i}\oint_{\gamma_k} dz \int_a^b dt\; G(x,t,z)}$$

Let $\mathcal{G}(z) f = (L-z)^{-1} f = \int_a^b G(x,t,z) f(t)\, dt$

Then

$$-\frac{1}{2\pi i} \oint_{\gamma_k} \mathcal{G}(z) f \, dz = \langle U_k, f \rangle U_k \quad \text{for all } f \in L^2 \quad (*)$$

// ← we mean

$$-\frac{1}{2\pi i} \oint_{\gamma_k} dz \int_a^b G(x,t,z) f(t)\, dt$$

We can also plug in $\delta(x-t)$ for $f(x)$ in (*)

[Because Let $f_n \to \delta_{(x-s)}$ a fundamental sequence for $\delta(x-s)$

We can take limits (in the sense of distributions) in

$$-\frac{1}{2\pi i} \oint_{\gamma_k} dz \int_a^b G(x,t,z) \underbrace{f_n(t)}_{\downarrow\ \delta(t-s)} dt = \Big(\int_a^b \overline{U_k} \underbrace{f_n(x)}_{\downarrow\ \delta(x-s)} \Big) U_k \quad]$$

So taking $\delta(x-s)$ for $f(x)$ in (*) $\Rightarrow$

$$-\frac{1}{2\pi i} \oint_{\gamma_k} dz \int_a^b G(x,t,z)\, \delta(t-s)\, dt = \Big(\int_a^b \overline{U_k(x)}\, \delta(x-s)\, dx \Big) U_k(x)$$

or

$$\boxed{-\frac{1}{2\pi i} \oint_{\gamma_k} dz \; G(x,s,z) = \overline{U_k(s)}\, U_k(x)}$$

and adding up

and for γ containing all $\lambda_1, \lambda_2 \ldots \lambda_n$ -

$$\Rightarrow \quad -\frac{1}{2\pi i} \oint_{\gamma} dz\, G(x,s,z) = \sum_k \overline{U_k(s)}\, U_k(x) = \frac{\delta(x-s)}{w(s)}$$

↑ by T2B

(Recall: $(\varphi, \psi) = \langle \overline{\varphi}, \psi \rangle$

Also, when using a weight w, (real) write $\langle \overline{\varphi}, \psi \rangle_w = (w\varphi, \psi) = (\varphi, w\psi)$)

[Use L^T for transpose, L^* for adjoint]

Non-SA problem

Let $L = -\dfrac{d^2}{dx^2} + q(x)$ for $x \in [a, b]$, $q(x) \in \mathbb{C}$

[We saw that any 2nd order L is unitarily equivalent to the form above if p, w do not vanish at the endpoints; a linear affine change in x can move any $x \in [a,b]$ to any other $x \in [c,d]$.]

If $q(x) \in \mathbb{R}$ and BC $\in \mathbb{R}$ then L is SA, but not otherwise.

Existence of the Green's function for the non SA case

part 1 and 2 of the Theorem for SA does apply, yielding that $G(x,t,z)$ for $L-z$ has the form, if $z \neq$ eigenvalue,

$$G(x,t,z) = \begin{cases} \dfrac{1}{W(z)}\, y_b(t,z)\, y_a(x,z) & \text{for } x<t \\ \dfrac{1}{W(z)}\, y_a(t,z)\, y_b(x,z) & \text{for } x>t \end{cases}$$

where $y_a(x,z)$ satisfies $(L-z)y_a = 0$, BC at $x=a$
$y_b(x,z)$ $\qquad (L-z)y_b = 0$, BC at $x=b$

and $W(z) = W[y_a, y_b]$ does not depend on x.

Note that $W(z) = 0 \iff z =$ eigenvalue of $(L, D(L))$ and for all other values G is analytic in z (since $L-z$ is, by general theorems in ODEs)

So: $\boxed{\text{eigenvalues of } (L, D(L)) = \text{poles of } G(x,t,z) = \text{zeroes of } W(z)}$

Since $L-z$ depends analytically on z, by general ODE theorems, so do y_a, y_b and therefore $W(z)$ and therefore $G(x,t,z)$ is analytic in z, except at its poles (= eigenvalues of L). $\Rightarrow$ the only limit point of eval can be ∞

* * *

It is convenient to assume the interval $x \in [0,\pi]$

$$L = -\frac{d^2}{dx^2} + q(x) \qquad \text{for } x \in [0,\pi] \qquad , q(x) \in \mathbb{C}$$

If $q(x) \in \mathbb{R}$ then L is SA, but if $q(x) \notin \mathbb{R}$, or the BC have nonreal coeff then L cannot be SA.

Let $G(x,t,z)$ be the Green's function for $(L-z, D(L))$ and $G_0(x,t,z)$ for the "unperturbed" operator $L_0 = -\frac{d^2}{dx^2}$ $((L_0-z), D(L))$ $\to$ Same BC as L!

Since $\frac{\partial G}{\partial x}$ and $\frac{\partial G_0}{\partial x}$ are cont. for $x<t$, and for $x>t$, and have the same jump at $x=t$ $\Rightarrow$ $\frac{\partial G}{\partial x} - \frac{\partial G_0}{\partial x}$ = continuous on $[0,\pi]$

(e.g. $f(x) = \begin{cases} a, & \text{for } x<t \\ a+N, & \text{for } x>t \end{cases}$, $g(x) = \begin{cases} b, & \text{for } x<t \\ b+N & \text{for } x>t \end{cases}$ $\Rightarrow f-g = a-b$)

In fact $G(\cdot,t,z)$ and $G_0(\cdot,t,z)$ are C^2 for $x<t$ and for $x>t$ and certainly C^1 on $[0,\pi]$. In fact, G, G_0 are C^2 also at $x=t$ for all z which are not poles of G or of G_0, because

$$L_0(G-G_0)-z(G-G_0) = -q(x)\,G(x,t,z)$$

so, using the Green's function for $L_0 - z$ then

$$(G-G_0)(x,t,z) = -\int_0^{\pi} G_0(x,s,z)\, q(s)\, G(s,t,z)\, ds$$

for all z which are not poles of G or of G_0

* * *

Let $L = -\frac{d^2}{dx^2} + q(x)$, $q(x) \in \mathbb{C}$, $x \in [a,b]$

$B_a[u] := \alpha u(a) + \alpha' u'(a) = 0$
$B_b[u] := \beta u(b) + \beta' u'(b) = 0$ } for short $B[u]=0$

so $D(L) = \{ y \in L^2(a,b) \mid y', y'' \in L^2(a,b), B[y]=0 \}$

<u>Find the adjoint of L</u> : quite obviously, $L^* = -\frac{d^2}{dx^2} + \overline{q(x)}$

and we need $D(L^*)$ so that $\langle u, Ly\rangle = \langle L^* u, y\rangle$ for all $y \in D(L)$, $u \in D(L^*)$

so $$\int_a^b \overline{u(x)}\left(-y''(x) + q(x) y(x)\right) dx = \int_a^b \left[-\overline{u}''(x) + \overline{q}(x) u(x)\right] y(x)\, dx$$

but $\;\; ||$

$$-\overline{u}y'\Big|_a^b + \int_a^b \overline{u}'y' + \int_a^b \overline{u}\, q y = \left(-\overline{u}y' + \overline{u}'y\right)\Big|_a^b + \int_a^b \left(-\overline{u}'' + q\overline{u}\right) y\, dx$$

so we need $\left(-\overline{u}y' + \overline{u}'y\right)\Big|_a^b = 0$ for all $y \in D(L)$ and $u \in D(L^*)$

Suppose $\alpha' \neq 0, \beta' \neq 0 \Rightarrow y'(a) = -\frac{\alpha}{\alpha'} y(a)$

$$y'(b) = -\frac{\beta}{\beta'} y'(b)$$

so $+\bar{u}(b) \frac{\beta}{\beta'} y(b) + \bar{u}'(b) y(b) = \bar{u}(a) \frac{\alpha}{\alpha'} y(a) + \bar{u}'(a) y(a)$

for all $y(a), y(b) \Rightarrow \begin{cases} \alpha \bar{u}(a) + \alpha' \bar{u}'(a) = 0 \\ \beta \bar{u}(b) + \beta' \bar{u}'(b) = 0 \end{cases} \Rightarrow \begin{cases} \bar{\alpha} u(a) + \bar{\alpha'} u'(a) = 0 \\ \bar{\beta} u(b) + \bar{\beta'} u'(b) = 0 \end{cases}$

So $D(L^*) = \left\{ u \in L^2(a,b) \,\middle|\, u', u'' \in L^2,\ \begin{matrix} \bar{\alpha} u(a) + \bar{\alpha'} u'(a) = 0 \\ \bar{\beta} u(b) + \bar{\beta'} u'(b) = 0 \end{matrix} \right\}$ forshort $B^*[u] = 0$

(if α' or β' are 0 we get the same conditions, easy to see by taking limits $\alpha' \to 0$)

$\#$

Clearly, the adjoint of $(L - z, D(L))$ is $(L^* - \bar{z}, D(L^*))$

$L = -\frac{d^2}{dx^2} + q(x) \qquad L^* = -\frac{d^2}{dx^2} + \overline{q(x)}$

$\boxed{\text{Note}}$: λ, u eval/func of $L \Leftrightarrow \bar{\lambda}, \bar{u}$ eval/func of L^*

If L is not SA then the eigenfunctions corresp to distinct eigenvalues are not orthogonal. However, they are orthogonal to the eigenfunctions of the adjoint:

(!) while this is always true in fin-dim, it is not necessarily true in inf-dim (Yes for L = 2nd order diff op. with regular BV. If L is not in normal form the efunct is v (not necessarily $\bar{u}$)

Theorem Let $Lu = \lambda u$ $(u \neq 0)$ and $L^* v = \mu v$ $(v \neq 0)$

If $\mu \neq \bar{\lambda}$ then $\langle u, v \rangle = 0$

Proof $\bar{\lambda} \langle u, v \rangle = \langle \lambda u, v \rangle = \langle Lu, v \rangle = \langle u, L^* v \rangle = \langle u, \mu v \rangle$
$= \mu \langle u, v \rangle$

so $(\bar{\lambda} - \mu) \langle u, v \rangle = 0$ so if $\bar{\lambda} - \mu \neq 0$ then $\langle u, v \rangle = 0$.

$\square$

We will see soon that under fairly general conditions, on L does have enough eigenfunctions to form a basis for $\mathcal{H}$ (though not an orthogonal one) and in this case it has a Green's function (solution to $LG = \delta(x-t)$, $G(\cdot,t) \in D(L)$) then the same is true for L^*.

In this case $G(x,t) = \sum_{n=1}^{\infty} c_n(t)\, u_n(x)$ and we will calculate $c_n(t)$.

Assume all $\lambda_n \neq 0$. Recall: $L^* \bar{u}_n = \bar{\lambda}_n \bar{u}_n$

Then $\langle G(x,t), \bar{u}_k \rangle = \langle \sum_{n=1}^{\infty} c_n(t)\, u_n(x), \bar{u}_k \rangle = \bar{c}_n(t) \langle u_k, \bar{u}_k \rangle$

and $\rightarrow$ //

$$\overline{\langle \overline{G(x,t)}, u_k \rangle} = \overline{\frac{1}{\lambda_k} \langle \overline{G(x,t)}, \lambda_k u_k \rangle} = \overline{\frac{1}{\lambda_k} \langle \overline{G(x,t)}, L u_k \rangle}$$

$$= \overline{\frac{1}{\lambda_k} \langle L^* \overline{G(x,t)}, u_k \rangle} = \frac{1}{\lambda_k}$$

Green's function for $q(x) \equiv 0$

"Unperturbed" operator $L_0 = -\frac{d^2}{dx^2}$ on $[0, \pi]$

$$\left.\begin{aligned} \alpha y_0(0) + \alpha' y_0'(0) = 0 \\ \beta y_\pi(\pi) + \beta' y_\pi'(\pi) = 0 \end{aligned}\right\} BC$$

Find its Green's function:

$-\frac{d^2}{dx^2} G = z G + \delta(x-t)$ ⊕ BC. Denote $z = \lambda^2$

$-\frac{d^2}{dx^2} G = \lambda^2 G$ for $x<t$ and for $x>t$

Gen. sol $y = c_1 e^{i\lambda x} + c_2 e^{-i\lambda x}$ (for $\lambda \neq 0$)

$\alpha y_0(0) + \alpha' y_0'(0) = 0 \overset{(e.g.)}{\Longrightarrow} c_1 = \left(\alpha' + \frac{i\alpha}{\lambda}\right), \ c_2 = \alpha' - \frac{i\alpha}{\lambda}$

$y_0(x) = c_1 e^{i\lambda x} + c_2 e^{-i\lambda x}$

then $y_\pi(x) = d_1 e^{i\lambda(x-\pi)} + d_2 e^{-i\lambda(x-\pi)}$, $d_1 = \beta' + \frac{i\beta}{\lambda}$, $d_2 = \beta' - \frac{i\beta}{\lambda}$

$W[y_0, y_\pi] = \text{const} = W[y_0, y_\pi]\big|_{x=0} = \begin{vmatrix} y_0(0) & y_\pi(0) \\ y_0'(0) & y_\pi'(0) \end{vmatrix}$

$$= \begin{vmatrix} 2\alpha' & d_1 e^{-i\lambda\pi} + d_2 e^{i\lambda\pi} \\ -2\alpha & i\lambda d_1 e^{-i\lambda\pi} - i\lambda d_2 e^{i\lambda\pi} \end{vmatrix} = 2 \begin{vmatrix} \alpha' & \frac{d_1}{\theta} + d_2\theta \\ -\alpha & \frac{i\lambda d_1}{\theta} - i\lambda d_2 \theta \end{vmatrix}$$

$$= 2\left[\alpha'\left(\frac{i\lambda d_1}{\theta} - i\lambda d_2\theta\right) + \alpha\left(\frac{d_1}{\theta} + d_2\theta\right)\right]$$

$$= 2\left[\frac{i\lambda\alpha' + \alpha}{\theta} d_1 + (-i\lambda\alpha' + \alpha)\theta d_2\right]$$

(Check: if $\alpha' = \beta' = 0$, $\alpha = \beta = 1$
then $c_1 = i/\lambda$, $c_2 = -i/\lambda$, $d_1 = \frac{i}{\lambda}$, $d_2 = -i/\lambda$
$W = 2\left(\frac{1}{\theta}\frac{i}{\lambda} - \frac{i}{\lambda}\theta\right) = \frac{2i}{\lambda\theta}(\theta^2 - 1)$
so $\lambda_n = n^2$ as we know)

So $G_0(x,t,z) = \dfrac{1}{W(z)}\, M(x,t,\lambda)$

where $M(x,t,\lambda) = \begin{cases} y_\pi(t)\, y_0(x) & \text{for } t<x \\ y_0(t)\, y_\pi(x) & \text{for } t>x \end{cases}$

$$= y_\pi(t) y_0(x) + \Big[\underbrace{y_0(t) y_\pi(x) - y_\pi(t) y_0(x)}_{\|}\Big] H(x-t)$$

$$\left(c_1 e^{i\lambda t} + c_2 e^{-i\lambda t}\right)\left(d_1 e^{i\lambda(x-\pi)} + d_2 e^{-i\lambda(x-\pi)}\right)$$
$$-\left(c_1 e^{i\lambda x} + c_2 e^{-i\lambda x}\right)\left(d_1 e^{i\lambda(t-\pi)} + d_2 e^{-i\lambda(t-\pi)}\right)$$

$$= c_1 d_1 \cancel{e^{i\lambda(t+x-\pi)}} + c_2 d_1 e^{i\lambda(x-t-\pi)} + c_1 d_2 e^{-i\lambda(x-t-\pi)} + \cancel{c_2 d_2 e^{-i\lambda(x+t-\pi)}}$$
$$- c_1 d_1 \cancel{e^{i\lambda(x+t-\pi)}} \pm c_2 d_1 e^{-i\lambda(x-t+\pi)} - c_1 d_2 e^{i\lambda(x-t+\pi)} - \cancel{c_2 d_2 e^{-i\lambda(x-t+\pi)}}$$

$$= \Big(\underbrace{\frac{c_2 d_1}{\theta} - c_1 d_2 \theta}_{f_1}\Big) e^{i\lambda(x-t)} + \Big(\underbrace{c_1 d_2 \theta - \frac{c_2 d_2}{\theta}}_{f_2}\Big) e^{-i\lambda(x-t)}$$

$$= f_1 e^{i\lambda(x-t)} + f_2 e^{-i\lambda(x-t)}$$

So

$M(x,t,\lambda) = y_\pi(t) y_0(x) + F(x-t)$ where $F(\xi) = \left(f_1 e^{i\lambda\xi} + f_2 e^{-i\lambda\xi}\right) H(\xi)$
is the fd sol $\left((L_0 - z)F = \delta \right)$

Eigenvalues of L_0

Solve $W(z)=0$:

$$\frac{\pi}{2} W = (i\lambda\alpha'+\alpha)\, d_1 + (-i\lambda\alpha'+\alpha)\theta^2 d_2$$

$$(i\lambda\alpha'+\alpha)\left(\beta'+\frac{i\beta}{\lambda}\right) + (-i\lambda\alpha'+\alpha)\theta^2\left(\beta'-\frac{i\beta}{\lambda}\right)$$

$$= i\lambda\alpha'\beta' + \alpha\beta' - \alpha'\beta + \frac{i\alpha\beta}{\lambda} + \theta^2\left[-i\lambda\alpha'\beta' + \alpha\beta' - \alpha'\beta - \frac{i\alpha\beta}{\lambda}\right]$$

$$\frac{1}{2} W = \frac{1}{\lambda}\left\{\frac{1}{\theta}\underbrace{\left[i\alpha'\beta'\lambda^2 + (\alpha\beta'-\alpha'\beta)\lambda + i\alpha\beta\right]}_{P(-\lambda)} - \theta\underbrace{\left[i\alpha'\beta'\lambda^2 - (\alpha\beta'-\alpha'\beta)\lambda + i\alpha\beta\right]}_{P(\lambda)}\right\}$$

$$W(z) = \frac{2}{\lambda}\left[e^{-i\pi\lambda}P(-\lambda) - e^{i\pi\lambda}P(\lambda)\right] \quad \text{where } z=\lambda^2$$

For $\lambda\to 0$ $\quad W(z) \sim \frac{2}{\lambda}\left[(1-i\pi\lambda)(i\alpha\beta + (\alpha\beta'-\alpha'\beta)\lambda) - (1+i\pi\lambda)(i\alpha\beta - (\alpha\beta'-\alpha'\beta)\lambda)\right]$

$$= 2\left[-2i\pi\cdot i\alpha\beta + (\cancel{\alpha'\beta'-\alpha'\beta}) - \cancel{\ \ }\right] = 4\pi\alpha\beta + O(\lambda)$$

Therefore $W(\lambda)$ is entire in λ

$W(\lambda)$ has infinitely many zeroes:

Indeed Calculate asymptotically these zeroes for large λ

Denote $P(\lambda) = A\lambda^2 + B\lambda + C$

look for λ with $|\lambda| \gg 1$

$$W(\lambda)=0 \iff \underbrace{\frac{C\left(\frac{1}{\theta^2}-1\right)}{\lambda}}_{\substack{\text{much smaller}\\ \text{than the last term}\\ \text{unless } A=0}} + D\left(\frac{1}{\theta^2}+1\right) + A\lambda\left(\frac{1}{\theta^2}-1\right) = 0$$

much smaller than the previous

so $\frac{1}{\theta^2}D + A\lambda\left(\frac{1}{\theta^2}-1\right)$ = smaller terms.

A little asymptotic analysis:

If $A \neq 0$, $D \neq 0$ $\quad \lambda\left(\frac{1}{\theta^2}-1\right) = O\left(\frac{\lambda}{\theta^2}\right)$

$$A\lambda\left(e^{-2\pi i\lambda}-1\right) + D\left(e^{-2\pi i\lambda}\right) = \text{smaller}$$

so look for $\lambda = n+\Delta$ with $n \in \mathbb{Z}$, $\Delta \ll 1$

$$A(n+\Delta)\left(e^{-2\pi i n}(1-2\pi i\Delta)-1\right) + D(1-2\pi i\Delta) \approx D + \frac{C}{n+\Delta}\left(\frac{1}{\theta^2}-1\right)$$

$$A(n+\cancel{\Delta})(-2\pi i\Delta) + D(\cancel{1}-2\pi i\Delta) \sim D + \frac{C}{n}\left(1-\frac{\Delta}{n}\right)(-2\pi i\Delta)$$

Clearly $\Delta \sim \frac{\delta}{n}$

$$A\cancel{n}\left(-2\pi i\frac{\delta}{\cancel{n}}\right) + 2D - 2\pi i\frac{\delta}{n}D \sim \frac{C}{n}\left(-2\pi i\frac{\delta}{n}\right)$$

$$\Rightarrow \delta = \frac{2D}{2\pi i A}$$

So there are infinitely many zeros,

$$\lambda = \lambda_n = n - \frac{\delta}{n} + O\left(\frac{1}{n^2}\right), \quad n \in \mathbb{Z}$$

It can be proved there are no others.

//

The corresponding eigenfunctions $u_n(x)$ are, of course, the solutions $y_0(x;\lambda_n)$. Are they complete? How can we write an expansion in u_n using the Green's function?

From p. NS 2, by analogy with the SA case (even though in our non SA cases u_n are not mutually orthogonal)

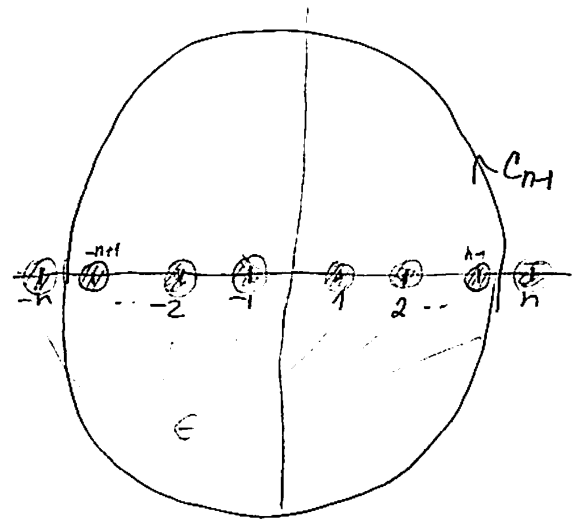
Consider C_n = circles $|\lambda| = n+\frac{1}{2}$. They avoid points in $\mathbb{C}$ with $|\lambda - n| < 1/4$ (see Fig next p.) and for large n, C_n does not intersect any λ_n (since $\lambda_n \sim n$ for n large). Let $\tilde{\gamma}_n = \{z \,/\, |z| = (n+\frac{1}{2})^2\}$ closed paths containing the eigenvalues $z_k = \lambda_k^2$ for $|k| \leq n$.

Idea of the proof (supplementary material, from C&L)

Let $E = \bigcup_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \{z \mid |z-n| < 1/4\}$

The circles $C_n = \{z \mid |z| = n+\frac{1}{2}\}$ do not intersect E

(and thus no λ_n for $|n|$ large)



— GO TO NEXT P., NS 13B —

Recall the problem $L_0 = -\frac{d^2}{dx^2}$ with BV $y'(0)=0$, $y'(\pi)=0$

(so $\alpha=\beta=0$, $\alpha'=\beta'=1$) has eval/vect $y_n = \cos(nx)$, $\lambda_n = n^2$, $n \geq 0$

This is a SA problem, hence $\cos(nx)$, $n=0,1,2\ldots$ form an orthogonal basis in $L^2[0,\pi]$, hence any $f\in L^2[0,\pi]$ can be uniquely written as a cos-series $f(x) = \sum_{n=0}^{\infty} c_n \cos(nx)$

where $c_n = \frac{\langle \cos nx, f\rangle}{\|\cos nx\|^2} = \frac{1}{\int_0^\pi \cos^2 nx\, dx} \cdot \int_0^\pi \cos nx\, f(x)\, dx$

— GO TO P. NS14 —

Recall that for SA problems the eigenfunctions u_n are complete and any $f \in L^2[0,\pi]$, $f = \sum_{n=1}^{\infty} \langle u_n, f \rangle u_n$

and each $\langle u_n, f\rangle u_n = -\frac{1}{2\pi i} \oint_{\gamma_n} \mathcal{G}(z)\,dz\, f = -\frac{1}{2\pi i} \oint_{\gamma_n} dz \int_0^{\pi} G(x,t,z) f(t)\,dt$

Then if C_N is a closed path encircling once $\lambda_1, \ldots \lambda_N$, but not containing the other λ's, we have for the partial sums

$$\sum_{n=1}^{N} \langle u_n, f\rangle u_n = -\frac{1}{2\pi i} \oint_{C_N} \mathcal{G}(z)\,dz\, f = -\frac{1}{2\pi i} \oint_{C_N} dz \int_0^{\pi} G(x,t,z) f(t)\,dt \;:= \sigma_N[f] \quad (*)$$

It turns out that for many nonSA problems which admit Green's functions this integral formula $\sigma_N[f]$ can be used to extend the partial sum of expansions in terms of eigenfunctions, (even if the coeff do not have the form $\langle u_n, f\rangle$)

Moreover, if $\alpha' \neq 0, \beta' \neq 0$ it turns out that $(*)$ is very good approximated by partial sums of cosine series.

— GO BACK TO P. NS 13 —

It can be proved that if α' and $\beta' \neq 0$ the efunc. expansion of a nonSA problem is asymptotically close to a cosine series (when $\alpha=\beta=0$). The proofs can be found in Coddington & Levinson, Ch.12, sec.2

<u>Theorem</u> If $\alpha'\beta' \neq 0$ then for f integrable on $[0,\pi]$ the difference between $\sigma_k[f]$ and the partial sum of the cosine series of f

$$S_N(x) = \sum_{j=0}^{N} c_k \cos kx \quad , c_0 = \frac{1}{\pi}\int_0^\pi f \ , \ c_k = \frac{2}{\pi}\int_0^\pi \cos(kx) f(x)\,dx$$

$\sigma_N[f] - S_N \to 0$ uniformly for $x \in [0,\pi]$ (i.e. in the sup norm)

Convergence also holds if $\alpha'\beta'=0$ but $\alpha\beta'-\alpha'\beta \neq 0$

<u>Theorem</u> If f is diff on $[0,\pi]$ then $\sigma_N'^{(x)} \to f'^{(x)}$ except possibly at $x=0, x=\pi$

$\boxed{\text{Back to } L = -\frac{d^2}{dx^2} + q(x)}$

We found its Green's function on p. NS3, G
We deduced an integral eq. for G in terms of G_0 on p. NS5

$$G(x,t,z) = G_0(x,t,z) - \int_0^{\pi} G_0(x,s,z)\, q(s)\, G(s,t,z)\, ds$$

Under good assumptions it will be shown from this that G has all the properties of a Green's function, and therefore must be the Green's function.