Review: The Linear Algebra behind the Green's function concept.

In dealing with a linear system $A-\lambda B$ we are dealing with the following causal relation

$$\text{source } \vec{b} \rightarrow \boxed{A-\lambda B} \rightarrow \vec{u} \text{ response}$$

or input output

whose nature is found to be governed by the following equation

$$(A-\lambda B)\vec{u} = \vec{b}$$

The mathematical problem is this:

Given (a) $A-\lambda B: U \to V$; (b) any $b \in V$

Find $\vec{u} \in U$ such that $\boxed{(A-\lambda B)\vec{u} = \vec{b}}$ (✫)

Discussion: This linear algebra problem, it turns out, decomposes into answering two questions:

1. Can one find G such that
$$(A-\lambda B)G = I \quad \text{on } V.$$
2.) Can one find H on U
$$(A-\lambda B)^* H = I$$

1.) Yes, one can find a right inverse of $A - \lambda B$ <u>if</u> it has the property of being onto,
i.e. for any b, $\exists$ a $\vec{u}$ s.t.

$$(A - \lambda B)\vec{u} = b \quad \text{(existene)}$$

i.e. the columns of $A - \lambda B$ span V

2.) Yes, if one can find H^*;

$$\boxed{H^*(A - \lambda B) = I} \quad \text{on } U \quad (\star\star)$$

i.e can find a left inverse of $A - \lambda B$,
i.e $A - \lambda B$ has the property that

$$(A - \lambda B)u_1 = (A - \lambda B)u_2 \Rightarrow u_1 = u_2$$

$$(A - \lambda B)u = 0 \Rightarrow u = 0$$

i.e $A - \lambda B$ is 1-1, i.e,
i.e the columns of $A - \lambda B$ are independent,

3.) 1.) $\Rightarrow \underbrace{H^*(A - \lambda B)}G = H^* I$

2.) $\Rightarrow \quad I \quad G = H^*$

$$\left.\begin{matrix} \star\star \\ \star \end{matrix}\right\} \Rightarrow \boxed{\vec{u} = G\vec{b}}$$

SOL'N TO THE problem.