$\boxed{\text{Simpler forms for second order diff operators}}$

Recall **Self-adjoint Eigenvalue problems for second order diff eqs.**

$$\underbrace{-P(x)\frac{d^2u}{dx^2} - Q(x)\frac{du}{dx} - R(x)u}_{L(x,\frac{d}{dx})u} = \lambda u$$

Substitute $P(x) = \dfrac{p(x)}{w(x)}$, $Q(x) = \dfrac{p'(x)}{w(x)}$, $R(x) = -\dfrac{q(x)}{w(x)}$

$\left(\text{so that } p(x) = \exp\int_{x_0}^{x}\frac{Q}{P},\ w = \frac{1}{P}\,p,\ q = -\frac{R}{P}\,p\right)$

to bring the eq to a formal SA form

$$\underbrace{\frac{1}{w(x)}\left[-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u\right]}_{L(x,\frac{d}{dx})u} = \lambda u \qquad (*)$$

formal SA in $L^2(\cdot\,,\ w(x)\,dx)$

We can simplify even further (so that $w(x)=1$, $p(x)=1$)

using the Liouville transformation as follows.

Substitute $\boxed{z = \int_a^x \sqrt{\frac{w(t)}{p(t)}}\,dt}$. denote $u(x) = y(z)$ in $(*)$.

$$\Rightarrow\quad -\frac{d^2y}{dz^2} - \frac{1}{2wp}\frac{d}{dz}(wp)\frac{dy}{dz} + \frac{q}{w}y = \lambda y$$

and with $\boxed{y(z) = (wp)^{-1/4}\,v(z)} \Rightarrow \boxed{-\frac{d^2v}{dz^2} + G(z)v = \lambda v}$ $(**)$

Normal form

where $G(z) = \frac{1}{4}\left(\frac{(wp)'}{wp}\right)' + \frac{1}{16}\left(\frac{(wp)'}{wp}\right)^2 + \frac{q}{w}$

since $\frac{d}{dx} = \frac{d}{dz}\frac{dz}{dx} = \sqrt{\frac{w(x)}{p(x)}}\,\frac{d}{dz}$

so $Lu = \frac{1}{w}\left[-\frac{d}{dx}\left(p\,\frac{\sqrt{w}}{\sqrt{p}}\,\frac{dy}{dz} \right) + q\,y \right]$

$$= \frac{1}{w}\left[-(\sqrt{pw})'\,\frac{dy}{dz} - \sqrt{pw}\cdot\sqrt{\frac{w}{p}}\,\frac{d^2y}{dz^2} + q y \right]$$

$$= -\frac{d^2y}{dz^2} - \frac{1}{w}\,\frac{1}{2\sqrt{pw}}\,\frac{d}{dx}(pw)\,\frac{dy}{dz} + \frac{q}{w}y$$

$$= -\frac{d^2y}{dz^2} - \frac{1}{w}\,\frac{1}{2\sqrt{pw}}\,\sqrt{\frac{w}{p}}\,\frac{d}{dz}(pw)\,\frac{dy}{dz} + \frac{q}{w}y$$

$$= -\frac{d^2y}{dz^2} - \frac{1}{2wp}\,\frac{d}{dz}(wp)\,\frac{dy}{dz} + \frac{q}{w}y.$$

then with $y = \varphi v \implies$

$$-v''\varphi - 2\varphi'v' - \varphi''v - \frac{1}{2wp}(wp)'(\varphi v' + \varphi' v) + \frac{q}{w}\varphi v = \lambda \varphi v$$

$$-v'' + v'\underbrace{\left(-\frac{2\varphi'}{\varphi} - \frac{1}{2wp}(wp)' \right)}_{\text{want } = 0} + v\left(-\frac{\varphi''}{\varphi} - \frac{1}{2wp}(wp)'\frac{\varphi'}{\varphi} + \frac{q}{w} \right) v = \lambda v$$

so $\frac{\varphi'}{\varphi} = -\frac{1}{4}\,\frac{1}{wp}(wp)' \implies \varphi = (wp)^{-1/4}$ indeed

and since from $\implies \frac{\varphi''}{\varphi} - \frac{\varphi'^2}{\varphi^2} = -\frac{1}{4}\left(\frac{(wp)'}{wp}\right)'$ the coeff of v is

$$\frac{1}{4}\left(\frac{(wp)'}{wp}\right)' - \frac{1}{16}\left(\frac{(wp)'}{wp}\right)^2 + \frac{1}{8}\left(\frac{(wp)'}{wp}\right)^2 + \frac{q}{w} \equiv G(z)$$

So : if $p(x) > 0$ on $[a,b]$
$w(x) > 0$
p, p', q, w continuous

then the change of variables $(x,u) \longmapsto (z,v)$

takes equation (*) to (**)
an interval $[a,b]$ into an interval $[0,c]$ one to one and onto (since $z(x)$ is increasing on $[a,b]$)
and we have to change the boundary conditions using the same transformations;

it suffices to study the eigenvalue problem for

$$\begin{cases} -\dfrac{d^2v}{dz^2} + G(z)v = \lambda v \quad , z \in [0,c] \\ \oplus \text{ BC} \end{cases}$$

In fact, this change of variables <u>is an isomorphism</u> of Hilbert spaces between $L^2([a,b], w(x)\,dx)$ and $L^2([0,c], dx)$

(Indeed, it is unitary since

$$\langle u_1, u_2 \rangle_w = \int_a^b \overline{u_1(x)}\, u_2(x)\, w(x)\, dx \quad \left(\text{since } \frac{dz}{dx} = \frac{\sqrt{w}}{\sqrt{p}}\right)$$

$$= \int_0^c (wp)^{-1/2}\, \overline{v_1(z)}\, v_2(z)\, \frac{\sqrt{p}}{\sqrt{w}}\, w\, dz = \int_0^c \overline{v_1(z)}\, v_2\, dz = \langle v_1, v_2 \rangle \Big)$$

We showed that if $p(x) \neq 0$ in

$$L = \frac{1}{w}\left[-\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)\right]$$

after a change of variables we can bring L to the form $-\frac{d^2}{dz^2} + G(z)$

by a unitary transformation, hence preserving the eigenvalues.

If $p(a)=0$ or $p(b)=0$ then the Liouville transformation may blow up at $x=a$ or at $x=b$. In this case, use only

$$\tilde{z} = \int_a^x \sqrt{w(t)}\, dt \quad , \quad u(x) = w^{-1/4}\, \tilde{v}(\tilde{z}) \qquad (***)$$

(this gets rid of the weight w) and L becomes

$$\tilde{L}\left(\tilde{z}, \frac{d}{d\tilde{z}}\right)\tilde{v} = -(p\,\tilde{v}')' + \tilde{G}(\tilde{z})\,\tilde{v}$$

(see next page for calculation)

<u>Theorem</u> The linear transformation $L^2([a,b], w(x)dx) \longrightarrow L^2([0,d], d\tilde{z})$

$$u(x) \longmapsto \tilde{v}(\tilde{z})$$

is a unitary transformation and $\tilde{L}$ is L after this transformation

<u>Indeed</u>: $\langle \tilde{v}_1, \tilde{v}_2 \rangle = \int_0^d \overline{\tilde{v}_1(\tilde{z})}\, \tilde{v}_2(\tilde{z})\, d\tilde{z} = \int_a^b w^{1/2}\, \overline{u_1(x)}\, u_2(x) \frac{d\tilde{z}}{dx}\, dx$

$$= \int_a^b w^{1/2}\, \overline{u_1(x)}\, u_2(x) \sqrt{w}\, dx = \int_a^b \overline{u_1}\, u_2\, w\, dx = \langle u_1, u_2 \rangle_w$$

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$\Rightarrow$ $\boxed{\text{There is no loss of generality assuming } w(x) \equiv 1.}$

Calculations: $\tilde{z} = \int_a^x \sqrt{w(t)}\, dt \Rightarrow \frac{d}{dx} = \sqrt{w}\, \frac{d}{d\tilde{z}}$, $u(x)=\tilde{y}(\tilde{z})$

so $Lu = \frac{1}{w}\left[-\frac{d}{dx}\left(p\sqrt{w}\,\frac{d\tilde{y}}{d\tilde{z}}\right) + q\tilde{y}\right]$

$= \frac{1}{w}\left[-\frac{d}{d\tilde{z}}(p\sqrt{w})\cdot\sqrt{w}\cdot\frac{d\tilde{y}}{d\tilde{z}} - p\sqrt{w}\,\frac{d^2\tilde{y}}{d\tilde{z}^2}\sqrt{w} + q\tilde{y}\right]$

$= -p\frac{d^2\tilde{y}}{d\tilde{z}^2} - \frac{1}{\sqrt{w}}\frac{d}{d\tilde{z}}(p\sqrt{w})\frac{d\tilde{y}}{d\tilde{z}} + \frac{q}{w}\tilde{y}$.

then let $\tilde{y} = \tilde{\varphi}\tilde{v}$

$-p(\varphi\tilde{v}'' + 2\tilde{\varphi}'\tilde{v}' + \tilde{\varphi}''\tilde{v}) - \frac{1}{\sqrt{w}}(\tilde{p}\sqrt{w})'(\tilde{\varphi}\tilde{v}' + \tilde{\varphi}'\tilde{v}) + \frac{q}{w}\tilde{\varphi}\tilde{v} = \lambda\tilde{\varphi}\tilde{v}$

$-p\tilde{v}'' + \tilde{v}'\underbrace{\left(-2p\frac{\varphi'}{\varphi} - \frac{1}{\sqrt{w}}(p\sqrt{w})'\right)}_{\text{Want } = -p'} + \left(-p\frac{\varphi''}{\varphi} - \frac{1}{\sqrt{w}}(p\sqrt{w})\frac{\varphi'}{\varphi} + \frac{q}{w}\right)v = \lambda v$

so find φ so that $2p\frac{\varphi'}{\varphi} + \frac{1}{\sqrt{w}}(p\sqrt{w})' = p'$

$\frac{\varphi'}{\varphi} + \frac{1}{2p\sqrt{w}}\left(\cancel{p'\sqrt{w}} + p(\sqrt{w})'\right) = \cancel{\frac{p'}{2p}}$

$\frac{\varphi'}{\varphi} = -\frac{(\sqrt{w})'}{2\sqrt{w}}$ so $\frac{\varphi'}{\varphi} = -\frac{w'}{4w}$ take $\varphi = w^{-1/4}$

then $\frac{\varphi''}{\varphi} - \frac{\varphi'^2}{\varphi^2} = -\frac{1}{4}\left(\frac{w'}{w}\right)'$ so $\frac{\varphi''}{\varphi} = \frac{1}{16}\left(\frac{w'}{w}\right)^2 - \frac{1}{4}\left(\frac{w'}{w}\right)'$ and the coeff

of $\tilde{v}$ is $-\frac{p}{4}\left[\frac{1}{4}\frac{w'^2}{w^2} - \frac{w''}{w} + \frac{w'^2}{w^2}\right] + p\frac{w'}{4w} + \frac{q}{w}$

$= -p\cdot\frac{1}{4}\left(\frac{3}{4}\frac{w'^2}{w^2} - \frac{w''}{w}\right) + \frac{p}{4}\frac{w'}{w} + \frac{q}{w} \equiv \tilde{G}(z)$