Recall self-adjoint eigenvalue problems for second order diff eqs.

\[-P(x) \frac{d^2 u}{dx^2} - Q(x) \frac{du}{dx} - R(x) u = 2u\]

\[L(x, \frac{d}{dx}) u\]

Substitute \( P(x) = \frac{p(x)}{w(x)} \), \( Q(x) = \frac{p'(x)}{w(x)} \), \( R(x) = -\frac{q(x)}{w(x)} \)

\[\frac{p(x)}{w(x)} = \exp \int \frac{Q}{P} dt, \quad w = \frac{1}{P} p, \quad q = -\frac{P}{P} p\]

so that to bring the eq to a formal SA form

\[\frac{1}{w(x)} \left[ -\frac{d}{dx} \left( \frac{p(x)}{w(x)} \frac{du}{dx} \right) + q(x) u \right] = 2u\]

\[L(x, \frac{d}{dx}) u\]

formal SA in \( L^2 (\cdot, w(x) \, dx) \)

We can simplify even further (so that \( w(x) = 1, p(x) = 1 \)) using the Liouville transformation as follows.

Substitute \( z = \int \frac{\sqrt{w(t)}}{p(t)} \, dt \), denote \( u(x) = y(z) \) in \( \bigcirc \)

\[\Rightarrow -\frac{d^2 y}{dz^2} - \frac{1}{2wp} \frac{d}{dt} (wp) \frac{dy}{dz} + \frac{q}{w} y = 2y\]

and with \( y(z) = (wp)^{-\frac{1}{4}} N(z) \)

\[\Rightarrow -\frac{d^2 (wp^{-\frac{1}{4}} N(z))}{dz^2} + g(z) N(z) = 4N\]

Normal form

where \( g(z) = \frac{1}{4} \left( \frac{wp}{w} \right)^{\frac{1}{2}} + \frac{1}{6} \left( \frac{wp}{w} \right)^2 \frac{q}{w} \)
\[
\frac{d}{dx} = \frac{d}{dt} \frac{d^2}{dx^2} = \sqrt{\frac{w(x)}{p(x)}} \frac{d}{dt}
\]

So \( L u = \frac{1}{w} \left[ - \frac{d}{dx} \left( \frac{\sqrt{w}}{\sqrt{p}} \frac{d g}{d x^2} \right) + \frac{\hat{q} y}{w} \right] \)

\[
= \frac{1}{w} \left[ - (\sqrt{p w})' \frac{d g}{d x^2} - \sqrt{p w} \cdot \sqrt{\frac{w}{p}} \frac{d^2 g}{d x^2} + \frac{\hat{q} y}{w} \right]
\]

\[
= - \frac{d^2 g}{d x^2} - \frac{1}{w} \frac{1}{2\sqrt{p w}} \frac{d}{dx} (p w) \frac{d g}{d x^2} + \frac{\hat{q} y}{w}
\]

\[
= - \frac{d^2 g}{d x^2} - \frac{1}{w} \frac{1}{2\sqrt{p w}} \sqrt{\frac{w}{p}} \frac{d}{d x} (p w) \frac{d g}{d x^2} + \frac{\hat{q} y}{w}
\]

\[
= - \frac{d^2 g}{d x^2} - \frac{1}{2 w p} \frac{d}{d x} (p w) \frac{d g}{d x^2} + \frac{\hat{q} y}{w}.
\]

Then, with \( \psi = \varphi \sqrt{w} \Rightarrow \)

\[
\psi'' \psi + 2 \psi' \psi' - \psi'' \psi - \frac{1}{2 w p} (p w) \left( \psi'' \psi + \frac{\psi' \psi}{w} \right) + \frac{\hat{q} \psi \sqrt{w}}{w} = \lambda \psi \sqrt{w}
\]

\[
\psi'' + \psi' \left( \frac{2 \psi'}{\psi} - \frac{1}{2 w p} (p w) \right) + \psi \left( - \frac{\psi''}{\psi} - \frac{1}{2 w p} (p w) \frac{\psi'}{w} + \frac{\hat{q}}{w} \right) = \lambda \psi
\]

want \( \Rightarrow \) \( \frac{\psi'}{\psi} = - \frac{1}{4} \frac{1}{w p} (p w)' \Rightarrow \psi = (p w)^{-\frac{1}{4}} \)

indeed

and since from \( \Rightarrow \) \( \frac{\psi''}{\psi} - \frac{\psi'^2}{\psi^2} = - \frac{1}{4} \left( \frac{(p w)'}{w p} \right)^2 \) the case of \( \psi \) is

\[
\frac{1}{4} \left( \frac{(p w)'}{w p} \right)^2 - \frac{1}{16} \left( \frac{(p w)'}{w p} \right)^4 + \frac{1}{8} \left( \frac{(p w)'}{w p} \right)^2 + \frac{\hat{q}}{w} \equiv G(\xi)
\]
So: if \( p(x) > 0 \) on \([a, b]\)
\[\frac{w(x)}{p(x)} > 0\]
\( p, p', q, w \) continuous

then the change of variables \((x, w) \rightarrow (z, \nu)\)
takes equation \(\star\) to \(\star\)
an interval \([a, b]\) into an interval \([0, c]\) one-to-one
and onto (since \(z/x\) is increasing on \([a, b]\))
and we have to change the boundary conditions using
the same transformation:

it suffices to study the eigenvalue problem
\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{d^2 \nu}{dz^2} + G(z) \nu = \lambda \nu, \quad z \in (0, c) \\
\nu(0) = \nu(c) = 0
\end{array} \right.
\end{align*}
\]

In fact, this change of variables is an isomorphism
of Hilbert spaces between \(L^2([a, b], w(x) \, dx)\) and \(L^2(0, c, \nu \, dx)\)

(Indeed, it is unitary since
\[
< u_1, u_2 >_w = \int_{a}^{b} \frac{ \overline{u_1(x)} \, u_2(x) \, w(x)}{p(x)} \, dx 
\]

\[
= \int_{0}^{c} \left( \frac{w(x)}{p(x)} \right)^{1/2} \nu_1(z) \overline{\nu_2(z)} \nu_2(z) \, \frac{\nu_2}{\sqrt{w}} \, w \, dz 
= \int_{0}^{c} \sqrt{w} \, \overline{\nu_2} \, \nu_2 \, dz = < \nu_1, \nu_2 >
\]
We showed that if \( p(x) \to 0 \) in

\[
L = \frac{1}{w} \left[ -\frac{d}{dx} \left( \rho(x) \frac{d}{dx} \right) + \omega(x) \right]
\]

after a change of variables we can bring \( L \) to the form

\[
-\frac{d^2}{dz^2} + G(z)
\]

by a unitary transformation, hence preserving the eigenvalues.

If \( p(a) = 0 \) so \( p(b) = 0 \) then the Liouville transformation may blow up at \( x = a \) or at \( x = b \). In this case, we obtain

\[
\tilde{z} = \int_a^x \sqrt{w(t)} \, dt, \quad \tilde{u}(x) = w^{-\frac{1}{4}} \tilde{u}(\tilde{z})
\]

(This gets rid of the weight \( w \)) and \( L \) becomes

\[
\tilde{L}(\tilde{z}, \frac{d}{d\tilde{z}}) \tilde{v} = -(\rho \tilde{v})' + \tilde{G}(\tilde{z}) \tilde{v}
\]

(see next page for calculation.)

Therefore, the linear transformation

\[
L^2([a, b], w(x)dx) \rightarrow L^2([a, b], d\tilde{z})
\]

\[
u(x) \rightarrow \tilde{v}(\tilde{z})
\]

is a unitary transformation, and \( \tilde{L} \) is \( L \) after this transformation.

Indeed:

\[
\langle \tilde{v}_1, \tilde{v}_2 \rangle = \int_a^b \overline{\tilde{v}_1(\tilde{z})} \tilde{v}_2(\tilde{z}) \, d\tilde{z} = \int_a^b w^{\frac{1}{2}} \bar{u}_1(x) u_2(x) \frac{d}{dx} dx
\]

\[
= \int_a^b w^{\frac{1}{2}} \bar{u}_1(x) u_2(x) \sqrt{w} \, dx = \int_a^b \bar{u}_1 u_2 w \, dx = \langle u_1, u_2 \rangle_w
\]

\( \implies \) There is no loss of generality assuming \( w(x) = 1 \).
Calculations: \( \tilde{z} = \int_a^x \sqrt{w(t)} \, dt = \frac{a}{dx} = \sqrt{w} \frac{d}{dx} \), \( u(x) = \tilde{y}(x) \)

\[ \infty \quad Lu = \frac{1}{w} \left[ -\frac{d}{dx} \left( p \sqrt{w} \frac{d\tilde{y}}{dx} \right) + \tilde{y} \right] \]

\[ = \frac{1}{w} \left[ -\frac{d}{dz} (p \sqrt{w}) \cdot \sqrt{w} \cdot \frac{d\tilde{y}}{dz} - p \sqrt{w} \frac{d^2\tilde{y}}{dz^2} \sqrt{w} + \tilde{y} \right] \]

\[ = -p \frac{d\tilde{y}}{dz} - \frac{1}{\sqrt{w}} \frac{d}{dz} (p \sqrt{w}) \frac{d\tilde{y}}{dz} + \frac{\tilde{y}}{w} \tilde{y} \]

Then let \( \tilde{y} = \tilde{\phi} \tilde{\eta} \)

\[-p (\tilde{\phi} \tilde{\eta}'' + 2 \tilde{\phi}' \tilde{\eta}'' + \tilde{\phi}'' \tilde{\eta}) - \frac{1}{\sqrt{w}} (p \sqrt{w})' (\tilde{\phi} \tilde{\eta}' + \tilde{\phi}' \tilde{\eta}) + \frac{\tilde{\phi}}{w} \tilde{\phi} \tilde{\eta} = \lambda \tilde{\phi} \tilde{\eta} \]

\[-p \tilde{\phi}'' + \tilde{\phi}' (-\frac{2p \tilde{\phi}'}{\tilde{\phi}} - \frac{1}{\sqrt{w}} (p \sqrt{w})') + (-p \tilde{\phi}' - \frac{1}{\sqrt{w}} (p \sqrt{w}) \frac{\tilde{\phi}}{\tilde{\phi}} + \frac{\tilde{\phi}}{w}) \tilde{\phi} \]

\[ \text{Ward} = -p' \]

so find \( \phi \) so that \( \frac{2p \tilde{\phi}'}{\tilde{\phi}} + \frac{1}{\sqrt{w}} (p \sqrt{w})' = p' \)

\[ \frac{\tilde{\phi}'}{\tilde{\phi}} + \frac{1}{2p \sqrt{w}} (p' \sqrt{w} + p (\sqrt{w})') = \frac{p'}{2p} \]

\[ \frac{\tilde{\phi}'}{\tilde{\phi}} = -\frac{(\sqrt{w})'}{2 \sqrt{w}} \quad \text{so} \quad \frac{\tilde{\phi}'}{\tilde{\phi}} = -\frac{w'}{4 \sqrt{w}} \quad \text{take} \quad \phi = w^{-\frac{1}{4}} \]

Then \( \frac{\tilde{\phi}''}{\tilde{\phi}} - \frac{w''}{w} = \frac{1}{4} (w')^2 \quad \text{so} \quad \frac{\tilde{\phi}''}{\tilde{\phi}} = \frac{1}{16} (w')^2 - \frac{1}{4} (w')^2' \) and the coeff of \( \tilde{\eta}' \)

\[-p \tilde{\phi}'' \left[ \frac{1}{4} \frac{w''}{w^2} - \frac{w''}{w} + \frac{w''}{w^2} \right] + p \frac{w'}{4w} + \frac{\tilde{\phi}}{w} \]

\[-p \frac{1}{4} \left( \frac{3}{4} \frac{w''}{w^2} - \frac{w''}{w} \right) + \frac{p w'}{4w} + \frac{\tilde{\phi}}{w} \equiv \tilde{G}/2) \]