Monday
April 30
2012

For possible revision of Sect. 4.8.2

The given b. v. problem is

$$\Big\{\underbrace{-\frac{1}{\rho}\Big[\frac{d}{dx}p\frac{d}{dx}-q(x)\Big]}_{\mathcal{L}}-\lambda\Big\}u=\frac{f(x)}{\rho(x)} \iff Lu=-f(x)$$

where

$$L=\frac{d}{dx}p\frac{d}{dx}-q+\lambda\rho(x)$$

with $B_a(u)=0$

$B_b(u)=0$

The corresponding S-L problem is

$$\mathcal{L}(u_n)=\lambda_n u_n$$

$$B_a(u_n)=0$$

$$B_b(u_n)=0$$

The corresponding adjoint S-L problem is

$$\mathcal{L}^*(v_n)=\overline{\lambda}_n v_n$$

$$B_a^*(v_n)=0$$

$$B_b^*(v_n)=0$$

Its solutions satisfy the "biorthonormality" relation

$$\langle v_{n'}, u_n\rangle \equiv \int_a^b \overline{v}_{n'}(\xi)\, u_n(\xi)\,\rho(\xi)\,d\xi=\delta_{n'n}$$

Consequently the solution to the given b.v. problem is

$$u(x) = \sum u_n(x) \frac{\langle v_n, f/\rho \rangle}{\lambda - \lambda_n}$$

The corresponding Green's fn problem is

$$\left\{ -\frac{1}{\rho}\left[\frac{d}{dx} p \frac{d}{dx} - q\right] - \lambda \right\} G(x-\xi) = \frac{\delta(x-\xi)}{\rho(x)} \left(= \frac{\delta(x-\xi)}{\rho(\xi)} \right).$$

For this case $f = \delta(x-\xi)$, so that

$$\langle v_n, \frac{f}{\rho} \rangle = \int_a^b \bar{v}_n(\xi) \frac{\delta(x-\xi)}{\rho(x)} \rho(\xi)\, d\xi$$

$$= \bar{v}_n(x)$$

The Green's fn is therefore

$$\boxed{G(x;\xi) = \sum_n \frac{u_n(x)\, \bar{v}_n(x)}{\lambda - \lambda_n}}$$

For a self-adjoint boundary value problem this becomes simply

$$G(x;\xi) = \sum \frac{u_n(x)\, \bar{u}_n(x)}{\lambda - \lambda_n}.$$