Stationary phase method

For a rapidly oscillating integrand:

$$\dot{I}(\rho) = \int_a^b e^{i\rho\varphi(t)} g(t)\,dt \qquad \text{with } \varphi(t), g(t) \text{ real}, \quad \rho \to +\infty$$

(if g is not real, just split $g = g_1 + i g_2$)

As $\rho \to +\infty$, $e^{i\rho\varphi(t)}$ oscillates more and more rapidly (unless $f(t) = \text{const}$) and we expect $\dot{I}(\rho) \to 0$ (as $\rho \to +\infty$)

Indeed, recall the following version of the Riemann-Lebesgue lemma

<u>Theorem</u> Let $g \in L^1(a,b)$. Then $\int_a^b e^{i\rho t} g(t)\,dt \to 0$ as $\rho \to +\infty$.

<u>How fast</u>? Integrate by parts:

if g is differentiable then

$$\int_a^b e^{i\rho t} g(t)\,dt = \frac{1}{i\rho} e^{i\rho t} g(t)\Big|_a^b - \frac{1}{i\rho}\int_a^b e^{i\rho t} g'(t)\,dt$$

$$= \frac{1}{i\rho}\left[e^{i\rho b} g(b) - e^{i\rho a} g(a)\right] - \frac{1}{(i\rho)^2} e^{i\rho t} g'(t)\Big|_a^b + \frac{1}{(i\rho)^2}\int_a^b e^{i\rho t} g''(t)\,dt$$

$$= \frac{1}{i\rho}\left[e^{i\rho b} g(b) - e^{i\rho a} g(a)\right] + O\left(\frac{1}{\rho^2}\right) \quad \underline{\underline{\text{like}}}\ \frac{1}{\rho}$$. See p. SP 1B

Consider now the integral $\dot{I}(\rho)$, slightly more complicated than the one in the theorem.

<u>Attempt the same for $\dot{I}(\rho)$</u>:

$$\int_a^b e^{i\rho\varphi(t)} g(t)\,dt = \int_a^b \frac{d}{dt}\left(e^{i\rho\varphi(t)}\right) \frac{g(t)}{i\varphi'(t)}\,dt = e^{i\rho\varphi(t)} \frac{g(t)}{i\varphi'(t)}\Big|_a^b + O(\rho^2)$$

<u>if</u> $\varphi'(t) \neq 0$ for $t \in [a,b]$.

The more derivatives of φ we take into account, the more terms we obtain in the asymptotic expansion!

Proposition

Assume $F \in C^n[a,b]$. Then

$$\int_a^b e^{i\rho t} F(t)\,dt = e^{i\rho t} \sum_{k=0}^{n-1} (-1)^k \frac{F^{(k)}(t)}{(i\rho)^{k+1}} \Bigg|_{t=a}^{b} + O(\rho^{-n})$$

Proof is just integration by parts and Riemann-Lebesgue Lemma, using the Taylor polynomial of F.

~~What about $\int_a^b e^{i\rho\varphi(t)} g(t)\,dt$? (where φ, F are real)~~

~~If there are no stationary phase points (i.e. $\varphi'(t) \neq 0$ for all $t \in [a,b]$) then φ is monotone, hence one-to-one, and we can reduce to the integral above by substituting $\varphi(t) = \tau$.~~

~~Break the integral:~~

$$\cancel{\int_a^b = \int_a^{t_1} + \int_{t_1}^{t_2} + \int_{t_2}^{b}}$$

~~and each of the 3 integrals must be treated separately.~~

~~Take for example $\int_a^{t_1} e^{i\rho\varphi(t)} g(t)\,dt$ where $\varphi'(t_1) = 0$, $\varphi'(t) \neq 0$ for all $t \in [a, t_1)$.~~

And indeed, if $\varphi'(t) \neq 0$ for $t \in [a,b]$ this means that φ is monotone and one-to-one and we can change variables in $I(\rho)$: set $s = \varphi(t)$ and

$$I(\rho) = \int_{\varphi(a)}^{\varphi(b)} e^{i\rho s} \frac{g(\varphi^{-1}(s))}{\varphi'(\varphi^{-1}(s))} ds$$

and we are in the situation of the Theorem, where, if g and φ have enough derivatives, $|I(\rho)| = O(\frac{1}{\rho})$

But if there are points t_s where $\varphi'(t_s) = 0$ the situation is very different. Intuitively, $\varphi(t) \approx$ const for $t \approx t_s$

A point t_s with $\varphi'(t_s) = 0$ is called a stationary point.

Say there is only one such point in $[a,b]$ (otherwise we break the interval $[a,b]$ into several pieces, each one containing just one stationary point).

We then write

$$I(\rho) = \left(\int_a^{t_s-\delta} + \int_{t_s-\delta}^{t_s+\delta} + \int_{t_s+\delta}^{b} \right) \left(e^{i\rho\varphi(t)} g(t)\, dt \right)$$

are as before, are $O(\frac{1}{\rho})$

The middle integral: write the Taylor polynomial of φ at $t = t_s$

$$\varphi(t) \approx \varphi(t_s) + \underbrace{\varphi'(t_s)}_{=0}(t-t_s) + \frac{1}{2}\varphi''(t_s)(t-t_s)^2$$

if $\varphi''(t_s) = 0$ must go to $\varphi'''(t_s)$ etc.

& $g(t) = g(t_s) + O(t-t_s)$

so $$\int_{t_s-\delta}^{t_s+\delta} e^{i\rho\varphi(t)} g(t)\, dt \approx e^{i\rho\varphi(t_s)} g(t_s) \int_{t_s-\delta}^{t_s+\delta} e^{i\frac{\varphi''(t_s)}{2}(t-t_s)^2} dt$$

$$= e^{i\rho\varphi(t_s)} g(t_s) \int_{-\delta}^{\delta} e^{i\frac{\varphi''(t_s)}{2}s^2} ds$$

To find the asymptotic behavior of the integral above

$$J = \int_{-\delta}^{\delta} e^{iAs^2} ds \qquad \left(A = \tfrac{1}{2}\varphi''(t_s)\right)$$

- intuition: the behavior is collected from $s \approx 0$ (where the oscillations are not fast), so it is the same as for $\int_{-\infty}^{\infty} e^{iAs^2} ds$

- rigorous: $J = 2\int_0^{\delta} e^{iAs^2} ds$

and since $\int_{\delta}^{+\infty} e^{iAs^2} ds = O(\frac{1}{\rho})$ and we will see that $J = O(1)$

to dominant order, the asy of J is the same as that of

$$2\int_0^{+\infty} e^{iAs^2} ds \underset{s=\sqrt{\sigma}}{=} \int_0^{\infty} \sigma^{-1/2} e^{iA\sigma} d\sigma =$$

Remark Because $\sigma^{-1/2} \in L^1(0,+\infty)$ the formula for the Γ-function still applies and

$$= \frac{\Gamma(\frac{1}{2})}{(-i\rho A)^{1/2}}$$

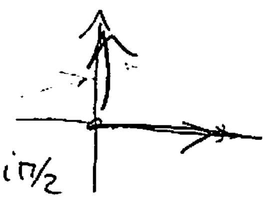
But what branch of $(-i)^{1/2}$?

To determine the branch, we need to prove this remark.

- if $A > 0$ then we can rotate the path of integration $[0,+\infty)$ clockwise, up to $i\mathbb{R}_+$ since in the first quadrant $\sigma = \sigma_1 + i\sigma_2$, with $\sigma_{1,2} > 0$ hence $iA\sigma = iA(\sigma_1 + i\sigma_2) = \underbrace{-A\sigma_2}_{<0} + iA\sigma_1$

$$= \int_0^{i\infty} \sigma^{-1/2} e^{iA\rho\sigma}\, d\sigma =$$

$\underline{A>0}$



here $\sigma = |\sigma| e^{i\pi/2}$

$$= \int_0^{i\infty} |\sigma|^{-1/2} e^{-i\pi/4} e^{iA\rho\sigma}\, d\sigma \quad ; \ \sigma = ix$$

$$= i \int_0^{\infty} x^{-1/2} e^{-i\pi/4} e^{-A\rho x}\, dx = e^{i\pi/4} \frac{\Gamma(\frac{1}{2})}{(A\rho)^{1/2}}$$

while if $A<0$ we rotate the path to $i\mathbb{R}_-$, where $\sigma = |\sigma| e^{-i\pi/2}$

and the integral is

$$= \int_0^{-i\infty} |\sigma|^{-1/2} e^{i\pi/4} e^{iA\rho\sigma}\, d\sigma \quad , \ \sigma = -ix$$

$$= -i \int_0^{\infty} x^{-1/2} e^{i\pi/4} e^{-(-A\rho)x}\, dx = e^{-i\pi/4} \frac{\Gamma(\frac{1}{2})}{(-A\rho)^{1/2}} = O(\rho^{-1/2})$$

$\gg \frac{1}{\rho}$ indeed

$\underline{\text{Note}}$ Another way to calculate $\int_0^{\sigma}$ is by writing

$$[0, \sigma] = i\mathbb{R}_+ + C_2$$

(for $A>0$)

then using the fact that

$$\int_{C_2} = O\left(\frac{1}{\sigma}\right)$$