

Steepest descent

is for analytic functions ; find the asymptotic behavior of

$$I(p) = \int_C e^{p f(t)} g(t) dt \quad \text{where } C \text{ is a path in } \mathbb{C} \\ p \rightarrow +\infty \\ f, g \text{ analytic.}$$

Main idea deform the path of integration  $C$  so that along the new path  $C'$  we can use Watson's Lemma.

This means that along  $C'$  we have  $\operatorname{Im} f(t) = \text{const}$ .

Indeed, if  $f = u + iv$  and along  $C'$  :  $v(t) = c \Rightarrow$

$$I(p) = \int_{C'} e^{p(u(t) + ic)} g(t) dt = e^{ipc} \int_{C'} e^{p u(t)} g(t) dt \quad \text{and use WL,} \\ \text{and no oscillations on } C'$$

Paths along which  $\operatorname{Im} f = \text{const}$  are called steepest descent paths.  
Because along these  $\operatorname{Re} f$  varies most rapidly (hence the name).

Indeed Recall

Denote $z = x + iy$ , $f(z) = u(x, y) + i v(x, y)$
$f(z)$ is analytic if and only if $u, v$ satisfy the Cauchy-Riemann equations:
$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

On steepest descent paths  $v = \text{const}$ , curves  $\perp$  to  $\nabla v$ , hence (by  $\nabla u \cdot \nabla v = 0$ ),  $\parallel$  to  $\nabla u$ , hence curves along which  $u$  varies most rapidly.

Remark We could also deform  $C$  into  $C''$  along which  $\operatorname{Re} f(t) = \text{const}$  and use the method of stationary phase. However, Watson's lemma gives better approximations, and to any order (so it is preferable).

Example Find the asymptotic behavior of

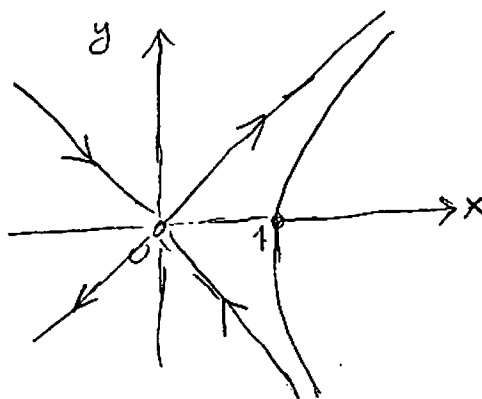
$$I(p) = \int_0^1 \frac{e^{ipz^2}}{z+1} dz \quad \text{as } p \rightarrow +\infty$$

Here  $f(z) = iz^2$ ,  $g(z) = \frac{1}{z+1}$

Find the lines of steepest descent:

$$z = x + iy, \quad f(x+iy) = i(x+iy)^2 = i(x^2 - y^2 + 2ixy) = -2xy + i(x^2 - y^2)$$

$$\operatorname{Im} f = \text{const} \Rightarrow x^2 - y^2 = k$$

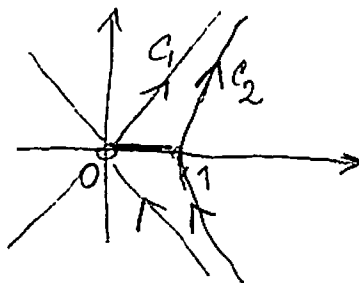


The curves through 0 are  $y = \pm x$  (there are 2 since 0 is a stationary point:  $f'(0) = 0$ ).

Along  $y = x$ ,  $\operatorname{Re} f = -2x^2$  it is larger when  $|x|$  is smaller. The arrows indicate directions of decrease of  $\operatorname{Re} f$ .

Along  $y = -x$ ,  $\operatorname{Re} f = 2x^2$ , decreases when  $|x|$  decreases

Through the point  $z=1$  the line of the steepest descent is the hyperbola  $x^2 - y^2 = 1$  (asymptotic to  $y = \pm x$ )



We deform  $[0, 1]$  by letting it "flow" as the arrows show and we end up with the deformation  $C_1 - C_2$

$$I(f) = \int_0^1 \frac{e^{ipz^2}}{z+1} dz = \underbrace{\int_{C_1} \frac{e^{ipz^2}}{z+1} dz}_{I_1} - \underbrace{\int_{C_2} \frac{e^{ipz^2}}{z+1} dz}_{I_2}$$

$$(1+i)^2 = 1 - 1 + 2i = 2i$$

$$\begin{aligned} C_1: z = x(1+i) \text{ so } I_1 &= (1+i) \int_0^\infty \frac{e^{ip(1+i)^2 x^2}}{x(1+i)+1} dx = (1+i) \int_0^\infty \frac{e^{-px^2}}{x(1+i)+1} dx \\ &\sim (1+i) \int_0^\infty e^{-px^2} \left[ 1 + \sum_{k=0}^\infty (1+i)^k x^k \right] dx \text{ and now use Watson's Lemma} \\ &\text{to get any number of terms} \\ &\text{in the asymptotics} \\ &\sim (1+i) \frac{\Gamma(\frac{1}{2})}{p^{1/2}} \left( 1 + O\left(\frac{1}{p}\right) \right) \end{aligned}$$

$C_2$ : along  $C_2$   $i\bar{z}^2 = -2xy + i$  (so  $i\bar{z}^2 - i \in \mathbb{R}_- \Leftrightarrow \bar{z}^2 - 1 \in i\mathbb{R}_+$ ) so substitute  $z = (1+ip)^{1/2}$

$$\begin{aligned} I_2 &= e^{ip} \int_0^\infty \frac{e^{-sp}}{(1+ip)^{1/2} + 1} \frac{i}{2(1+ip)^{1/2}} dp \text{ and we can use Watson's Lemma} \\ &= e^{ip} \int_0^\infty e^{-sp} \left( \frac{i}{4} + O(p) \right) dp \sim e^{ip} \frac{i}{4} \frac{1}{p} \end{aligned}$$

(and we can obtain many terms by expanding in Taylor series

$$\frac{i}{(1+ip)^{1/2} + 1} \frac{i}{2(1+ip)^{1/2}} = \sum_{k=0}^N c_k p^k \text{ then integrate.)}$$

# Example

Find the asymptotic behavior of  $H_\nu(p) = \frac{1}{\pi} e^{-i\pi\nu/2} \int_A^B e^{ip\omega z + i\nu z} dz$  ( $p \rightarrow +\infty$ )

$$\text{Let } I(p) = \int_A^B e^{ip\omega z + i\nu z} dz = \int_A^B e^{p f(z)} g(z) dz$$

$$\text{with } f(z) = i\omega z, g(z) = e^{i\nu z}$$

Since  $f, g$  are analytic, we can use steepest descent.

Stationary points:  $f'(z) = 0 \Rightarrow i\omega = 0 \Rightarrow \omega = 0 \Rightarrow z = k\pi, k \in \mathbb{Z}$

Find paths of steepest descent:

$$f(z) = \frac{1}{2}i(e^{iz} + e^{-iz}), f(x+iy) = \frac{i}{2}(e^{ix-y} + e^{-ix+y})$$

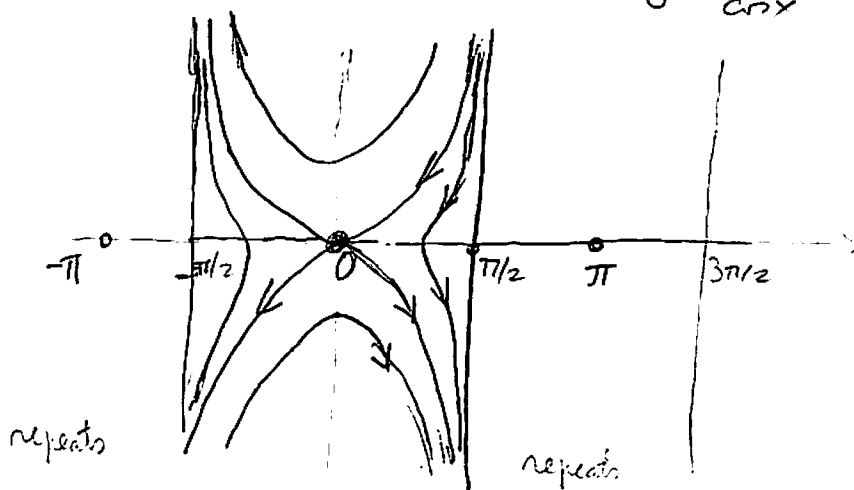
$$= \frac{i}{2} e^{-y}(\cos x + i \sin x) + \frac{i}{2} e^y(\cos x - i \sin x)$$

$$= \frac{i}{2} [e^{-y}(i \cos x - \sin x) + e^y(i \cos x + \sin x)]$$

$$= \sin x \frac{(e^y - e^{-y})}{2} + i \cos x \frac{e^y + e^{-y}}{2} = \sin x \cosh y + i \cos x \cosh y$$

So:  $\cos x \cosh y = c$ . The curves are  $x = \frac{(2k+1)\pi}{2}$  (and  $c=0$ )

$$\text{and } \cosh y = \frac{c}{\cos x}$$

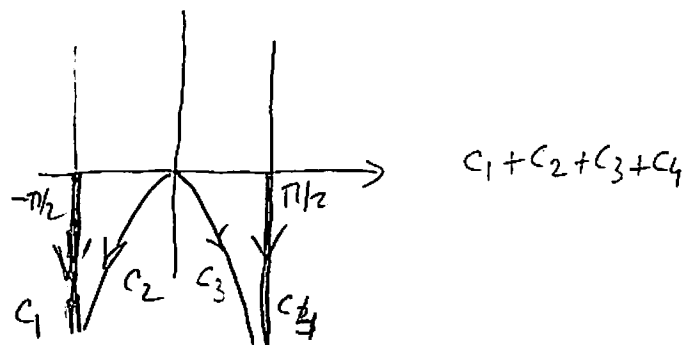


The paths of steepest descent through the saddle point 0 is given by  $\cos x \cosh y = 1$ .

If  $\cos x \cosh y = k > 0$ ,  $x \in [0, \pi/2]$ ,  $y > 0$

then  $x = \cos^{-1} \frac{k}{\cosh y}$  so  $\operatorname{Re} f = \sin x \sinh y$  increases  
 $x(y)$  increases

Hence, using steepest descent,  $[-\frac{\pi}{2}, \frac{\pi}{2}]$  can float down to



where the path  $C_2 \cup C_3$  corresponds to  $k=1$ :  $\cos x \cosh y = 1$ ,  $y < 0$

On each of the 4 paths the integral  $\int_{\text{can be}}$  calculated using Watson's Lemma to any order of precision.

Alternatively, the dominant order can be calculated using stationary phase method as follows.

We need to split  $[A, B]$  - Say  $[A, B] = [-\frac{\pi}{2}, \frac{3\pi}{2}]$

Write  $[-\frac{\pi}{2}, \frac{3\pi}{2}] = [-\frac{\pi}{2}, \frac{\pi}{2}] \cup [\frac{\pi}{2}, \frac{3\pi}{2}]$  each interval containing one stationary point.

$$I_1(p) = \int_{-\pi/2}^{\pi/2} e^{ip \cos z} e^{i0z} dz$$

Write Taylor approx of  $\cos z$  at  $z=0$  :  $\cos z = 1 - \frac{1}{2}z^2 + O(z^4)$

$$\begin{aligned} \approx I_1(p) &= \int_{-\pi/2}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\pi/2} \sim \int_{-\delta}^{\delta} e^{ip(1-\frac{1}{2}z^2)} (1+O(\delta)) + O\left(\frac{1}{p}\right) \\ &\quad \underbrace{\qquad\qquad\qquad}_{=O\left(\frac{1}{p}\right)} \qquad \underbrace{\qquad\qquad\qquad}_{=O\left(\frac{1}{p}\right) \text{ see SP p.2}} \end{aligned}$$

$$\sim e^{ip} \int_{-\infty}^{\infty} e^{-ipz^2/2} dz = \text{see SP, 3.4}$$

Etc.