Lecture 1 (Summarized Lectures 5, 8, 9 in the text and Notes)

Generalized Fourier rep
Poisson's rep
$L^2$ and $L^2$ are isometric
Completeness relation
Isomorphic isometric Hilbert spaces

The difference between Riesz and Fischer as to how they validated the "onto" property of $F: L^2 \rightarrow L^2$.

The framework for Fourier Series

Theory is the relation between two Hilbert (Cauchy complete inner product) spaces.

1. $L^2(a, b) = \{ f: \int_a^b f(x) \overline{g(x)} \, dx \leq \, \|f\|^2 < \infty \}$

the space of "square integrable" functions

which is engineering is called

the space of finite energy functions

Let $\{ u_n \}$ be a complete ON basis for $L^2$:

This means

\( (a) \quad \langle u_n, u_k \rangle = \int_a^b \overline{u_n(x)} u_k(x) \, dx = \delta_{nk} \)

the orthonormality condition

\( (b) \quad \text{For any } g \in L^2(a, b) \text{ one has} \)

\[ \lim_{N \to \infty} \left| \sum_{k=1}^{N} u_k \langle u_k, g \rangle - g \right| = 0 \]
which is equivalent to
\[ \lim_{N \to \infty} \| \langle u_k, g \rangle \|^2 = \| g \|^2. \]

2. The second Hilbert space is
\[ L^2 = \{ c_n: \sum_{n=1}^{\infty} |c_n|^2 < \infty \}. \] (**) the space of square summable sequences.

3. These two Hilbert spaces are obviously different, the set of functions on [a, b] in one case, complex sequences on the other.

Nevertheless, they are related; in fact, they are isomorphic. This relationship is a consequence of their essential properties.

The mathematically precise relationship is induced by the basis \{u_k\}, and is defined by
\[ F: L^2(a, b) \to L^2 \]
\[ f \mapsto F[f] = \{ \langle u_k, f \rangle \} = \{ c_k \}. \]
This relationship is a linear transformation ("mapping"). Its domain is the linear space, \(L^2(a, b)\). Its target space is \(L^2\). Its range space, \(R(F)\), is the set of all sequences \(\{ \langle u_k, f \rangle \} \). Being a linear transformation, it must answer at least two questions:
1. Is \(F\) one-to-one?
2. Is \(F\) onto, i.e., \(R(F) = L^2\)?
Surprisingly the answer to both questions is "YES"!

(1) \( F \) is one-to-one:

Let \( \{ c_k \} = \{ u_k, f \} \) two elements in \( \mathbb{R}(\ell^2) \) which are the same. Thus,

\[
0 = \lim_{N \to \infty} \left\| \sum_{k=1}^{N} (c_k - \delta_k) u_k \right\|^2 = \lim_{N \to \infty} \left\| \sum_{k=1}^{N} c_k u_k - \sum_{k=1}^{N} \delta_k u_k \right\|^2
\]

\[
= \lim_{N \to \infty} \left\| \sum_{k=1}^{N} c_k u_k - f + f - g + g - \sum_{k=1}^{N} \delta_k u_k \right\|^2
\]

\[
\leq \lim_{N \to \infty} \left\{ \left( \sum_{k=1}^{N} c_k u_k - f \right)^2 + \left( f - g \right)^2 + \left( g - \sum_{k=1}^{N} \delta_k u_k \right)^2 \right\}
\]

\[
= 0 + \| f - g \|^2 + 0
\]

Hence \( f = g \) are the same element in the space \( \mathbb{L}^2(\alpha, \beta) \) of square integrable functions.

(2) \( F \) is onto.

This means that given any \( \{ c_k \} \in \ell^2 \),

\[
\text{can one find}
\]

\( f \in \mathbb{L}^2(\alpha, \beta) \) such that \( F[f] = \{ c_k \} \), \( \forall c_k \in \ell^2 \).

If there exists such an \( f \), then one has

\( \text{"range"} = \mathbb{R}(F) = \ell^2 \) ("target space")

The validation of this onto is a highly non-trivial task, but it was done by two mathematicians, Fredorf Riesz in March 1907 and Ernst S. Fischer in May 1907 in two different ways in the same Journal. Their result is the Riesz-Fisher Theorem.
**Theorem (Riesz–Fischer)**

Given an orthonormal system \( \{ u_k \} \) in the complete \( L^2(a,b) \).

Given any sequence

\[
\{ c_1, c_2, \ldots, c_k, \ldots \} = \{ c_k \}
\]

such that

\[
\sum_{k=1}^{\infty} |c_k|^2 = \| f \|^2
\]

converges, i.e., \( \{ c_k \} \) is any given element in \( L^2 \).

Then there exists an element \( f \in L^2(a,b) \) with \( c_1, c_2, \ldots, c_k, \ldots \) as its generalized Fourier coefficients, i.e., there exists an \( f \) such that

\[
\sum_{k=1}^{\infty} |c_k|^2 = \| f \|^2 \quad (\Longleftrightarrow \lim_{N \to \infty} \| f - \sum_{k=1}^{N} c_k u_k \|^2 = 0)
\]

where

\[
c_k = \langle u_k, f \rangle, \quad k = 1, 2, \ldots
\]
2. F. Riesz's proof (in Comptes Rendus, March 1907) was different. He showed that
\[ \sum_{i=1}^{\infty} |c_k|^2 < \infty \iff \exists f \in L^2 \text{ such that} \]
\[ \int_{a}^{b} \overline{u_k(x)} f(x) g(x) \, dx = c_k \quad \forall k \]
In other words, Riesz showed that there exists a unique solution \( f \in L^2 \) to the system of integral equations
\[ \begin{align*}
\int_{a}^{b} u_1(x) f(x) g(x) \, dx &= c_1 \\
\int_{a}^{b} u_2(x) f(x) g(x) \, dx &= c_2 \\
\int_{a}^{b} u_k(x) f(x) g(x) \, dx &= c_k
\end{align*} \]
By this method, Riesz showed that \( \{c_k\} \in \ell^2 \) as well.

The easy part is to show that \( f \) yields a unique \( \{c_k\} \), indeed its elements are simply the generalized Fourier coefficient \( \{\langle u_k, f \rangle = c_k \}_{k=1}^{\infty} \).

The more challenging part is to show that \( F \) yields a unique solution \( f \) \((\text{in } L^2, b)\) to the system of integral equations (\( \star \)) on \( P \& \).

3. The uniqueness of \( f \) in Fischer's approach guarantees that it coincides with the unique \( f \) in Riesz's approach, and vice versa.
Comment about the linearity of $\mathcal{F}$ and $\mathcal{F}^{-1}$: The linearity of $\mathcal{F}$, 
$\mathcal{F}[f+g] = \mathcal{F}[f] + \mathcal{F}[g]$ 
is a mathematically precise way of expressing the fact that $\mathcal{F}$ maps closed triangles in $L^2(a,b)$ into closed triangles in $L^2$;

\[ \begin{array}{c}
\begin{array}{c}
\mathcal{F}[f+g] \\
\mathcal{F}[f] + \mathcal{F}[g]
\end{array}
\end{array} \]

Image triangle is closed:
$\mathcal{F}[f+g] = \mathcal{F}[f] + \mathcal{F}[g]$ 
i.e. $\{c_R + d_R\} = \{c_R\} + \{d_R\}$