Lecture 1 (Summarized Lectures 5, 8, 9 in the typeset Notes)

Generalized Fourier rep'n
Parceval's rel'n
L^2 & ℓ^2 are isometric
Completeness relation
Isomorphic isometric Hilbert spaces

The difference between Rietz and Fischer as to how they validated the "onto" property of $\mathcal{F}: L^2 \to \ell^2$.

The framework for Fourier Series Theory is the relation between two Hilbert (=Cauchy complete inner product) spaces.

1. $L^2(a,b) = \{f : \int_a^b \bar{f}(x) f(x) \rho(x)\, dx \equiv \langle f, f\rangle \equiv \|f\|^2 < \infty\}$ (*)

the space of "square integrable" functions, which is engineering is called / the space of finite energy functions /

Let $\{u_k\}$ be a complete O.N. basis for L^2:

This means

(a) $\langle u_k, u_\ell\rangle \equiv \int_a^b \bar{u}_k(x)\, u_\ell(x)\, \rho(x)\, dx = \delta_{k\ell}$

the orthonormality condition.

(b) For any $g \in L^2(a,b)$ one has

$$\lim_{N\to\infty} \left\| \sum_{k=1}^{N} u_k \langle u_k, g\rangle - g \right\|^2 = 0$$

which is equivalent to

$$\lim_{N\to\infty} \|\langle u_k, g\rangle\|^2 = \|g\|^2.$$

2. The second Hilbert space is

$$\ell^2 = \{c_n : \sum_{n=1}^{\infty} |c_n|^2 < \infty\}, \qquad (**)$$

the space of square summable sequences.

3. These two Hilbert spaces are obviously very different linear spaces; the set of functions on $[a,b]$ in one case, the set of complex sequences in the other.

Nevertheless, they are related; in fact, they are isomorphic. This relationship is a consequence of their essential (= defining) properties.

The mathematically precise relationship is induced by the basis $\{u_k(x)\}$ and is defined by

$$\mathcal{F}: L^2(a,b) \longrightarrow \ell^2$$

$$f \rightsquigarrow \mathcal{F}[f] = \{\langle u_k, f\rangle\} \equiv \{c_k\}$$

This relationship is a linear transformation (= "mapping"). Its <u>domain</u> is the linear space $L^2(a,b)$. Its <u>target</u> space is ℓ^2. <u>Its range</u> space, $\mathcal{R}(\mathcal{F})$, is the set of all sequences $\{\langle u_k, f\rangle\}_1^{\infty}$.

Being a linear transformation, it must answer at least two questions:

(1) Is $\mathcal{F}$ one-to-one?

(2) Is $\mathcal{F}$ onto, i.e. $\mathcal{R}(\mathcal{F}) = \ell^2$

Surprisingly the answer to both questions is "YES"!

(1) $\mathcal{F}$ is one-to-one:

Let $\{c_k = \langle u_k, f\rangle\} = \{d_k = \langle u_k, g\rangle\}$ two elements in $\mathcal{R}(\mathcal{F})$ which are the same. Thus

$$0 = \lim_{N\to\infty} \left\| \sum_1^N (c_k - d_k) u_k \right\|^2 = \lim_{N\to\infty} \left\| \sum_1^N c_k u_k - \sum_1^N d_k u_k \right\|^2 =$$

$$= \lim_{N\to\infty} \left\| \sum^N c_k u_k - f + f - g + g - \sum d_k u_k \right\|^2$$

"triangle inequality"

$$\le \lim_{N\to\infty} \left\{ \left\| \sum^N c_k u_k - f \right\|^2 + \| f - g \|^2 + \left\| g - \sum^N d_k u_k \right\|^2 \right\}$$

$$= \quad 0 + \| f - g \|^2 + 0$$

Hence $\boxed{f \doteq g}$ are the same element ("$\doteq$") in the space $L^2(a,b)$ of square int'ble fns.

(2) Is $\mathcal{F}$ onto:

This means that given any

$$\{c_k\} \in \ell^2,$$

can one find

$f \in L^2(a,b)$ such that $\underbrace{\mathcal{F}[f]}_{\in \mathcal{R}(\mathcal{F})} = \underbrace{\{c_k\}}_{\in \ell^2}$.

If there exists such an f, then one has

$$\left(\text{"range space"} =\right) \mathcal{R}(\mathcal{F}) = \ell^2 \left(= \text{"target space"}\right)$$

The validation of this onto is a highly non-trivial task, but it was done by two mathematicians

Frederic Riesz in March 1907

and Ernst S. Fischer in May 1907

in two different ways in the same Journal. Their result is the Rietz-Fisher Theorem

Theorem (Riesz-Fischer)

Given an orthonormal system $\{u_k\}$ in the complete $L^2(a,b)$.

Given any sequence

$$\{c_1, c_2, \cdots, c_k, \cdots\} \equiv \{c_k\}$$

such that
$$\sum_1^\infty |c_k|^2$$

converges, i.e. $\{c_k\}$ is any given element in ℓ^2.

Then there exists an element $f \in L^2(a,b)$ with $c_1, c_2, \cdots, c_k, \cdots$ as its generalized Fourier coefficients, i.e there exists an f such that

$$\sum_1^\infty |c_k|^2 = \|f\|^2 \quad \left(\Leftrightarrow \lim_{N\to\infty} \|f - \sum_1^N c_k u_k\|^2 = 0\right)$$

where
$$c_k = \langle u_k, f\rangle, \quad k = 1, 2, \cdots$$

Comments about the R-F Theorem:

1. E. Fischer's proof (in Comptus Rendus, May 1907) hinges on introducing the sequence of L^2 function

$$\sum_1^N c_k u_k(x) = f_N(x),$$

and then showing that

$$\lim_{N,M\to\infty} \|f_N - f_M\| = 0 \Leftrightarrow \exists f \in L^2 \text{ such that } \lim_{N\to\infty} \|f_N - f\|^2 = 0$$

i.e. that a Cauchy sequence in $L^2(a,b)$ converges if and only if it converges to a function $f \in L^2(a,b)$. The $\Leftarrow$ part of this proof is easy. One merely has to use the triangle inequality. The $\Rightarrow$ part was the more challenging part.

2. F. Riesz's proof (in Comptes Rendus, March 1907) was different. He showed that

$$\sum_1^\infty |c_k|^2 < \infty \Leftrightarrow \exists\, f \in L^2 \text{ such that}$$

$$\int_a^b \bar{u}_k(x) f(x) \rho(x)\,dx = c_k \quad \forall k$$

In other words, Riesz's showed that there exists a <u>unique</u> solution $f \in L^2$ to the system of <u>integral equations</u>

$$\left.\begin{aligned} \int_a^b \bar{u}_1\, f(x)\, \rho(x)\,dx &= c_1 \\ \int_a^b \bar{u}_2\, f(x)\, \rho(x)\,dx &= c_2 \\ &\vdots \\ \int_a^b \bar{u}_k\, f(x)\, \rho(x)\,dx &= c_k \\ &\vdots \end{aligned}\right\} (\star)$$

By this method Riesz showed that

$$\{c_k\} \in \ell^2 \quad \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} \quad f \in L^2$$

The easy part is to show $\xrightarrow{\mathcal{F}}$ yields a unique $\{c_k\}$. Indeed its elements

are simply the generalized Fourier coefficient $\{\langle u_k, f\rangle = c_k\}_{k=1}^\infty$.

The more challenging part is to show that $\xrightarrow{\mathcal{F}^{-1}}$ yields a unique solution f (in $L^2(a,b)$) to the system of integral equations $(\star)$ on P 8.

3. The uniqueness of f in Fischer's approach guarantee that it <u>coincides</u> with the unique f in Riesz's approach, and <u>vice versa</u>.

Comment about the linearity of $\mathcal{F}$ and $\mathcal{F}^{-1}$: The <u>linearity</u> of $\mathcal{F}$,

$$\mathcal{F}[f+g] = \mathcal{F}[f] + \mathcal{F}[g]$$

is a mathematically precise way of expressing the fact that $\mathcal{F}$ maps closed triangles in $L^2(a,b)$ into closed triangles in ℓ^2;

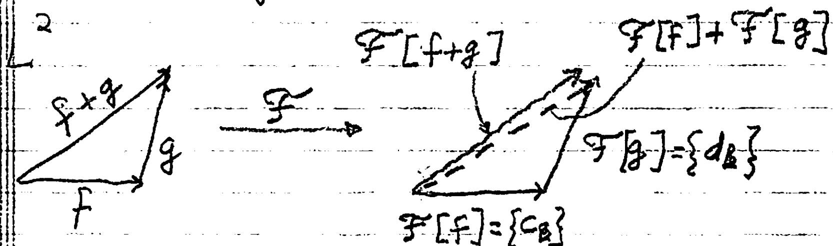

Image triangle is closed:

$$\mathcal{F}[f+g] = \mathcal{F}[f] + \mathcal{F}[g]$$

i.e. $$\{c_k + d_k\} = \{c_k\} + \{d_k\}$$