Watson's lemma

<u>Problem</u> Find the asymptotic behavior (i.e. does it grow, decrease, at what rate, find an approximation) of integrals of the type $\int_{\ell} e^{W f(z)} g(z)\, dz$

as $W \to \infty$, W on a path in the complex plane where ℓ is a path in $\mathbb{C}$ (finite or infinite).

A lot can be said if $f(z), g(z)$ are analytic.

<u>Remark</u> If W goes to infinity along a half line, say $\begin{cases} W = |W| e^{i\theta} \\ |W| \to +\infty \end{cases}$ then we can absorb $e^{i\theta}$ in $f(z)$: $\tilde{f}(z) = e^{i\theta} f(z)$

and study $\int_{\ell} e^{|W| \tilde{f}(z)} g(z)\, dz$ as $|W| \to +\infty$ (*)

Or we could study $\int_{\ell} e^{-|W| \tilde{\tilde{f}}(z)} g(z)\, dz$ as $|W| \to +\infty$ where $\tilde{\tilde{f}}(z) = -\tilde{f}(z)$ (**)

etc.

I. Real-valued $f(z)$. Watson's lemma.

<u>Example</u> (the archetype of such problems)

$$I(\rho) = \int_0^N p^{\beta-1} e^{-p\rho}\, dp \sim \frac{\Gamma(\beta)}{\rho^{\beta}} = \int_0^\infty p^{\beta-1} e^{-p\rho}\, dp \quad \begin{pmatrix} \text{for } \operatorname{Re}\rho > 0 \\ \operatorname{Re} N > 0 \\ \operatorname{Re}\beta > 0 \end{pmatrix}$$

<u>Intuition</u>: the integrand, $p^{\beta-1} e^{-p\rho}$ is exponentially small if $p \neq 0$, so the main contribution is collected from the end-point $p = 0$.

In example $\displaystyle\int_0^N e^{-p\rho}\,d\rho = -\frac{1}{\rho} e^{-p\rho}\Big|_0^N$

$$= \underbrace{-\frac{1}{\rho} e^{-\rho N}}_{\text{exponentially smaller than } \frac{1}{\rho}} + \frac{1}{\rho} \sim \frac{1}{\rho}$$

or $\displaystyle\int_0^N p\, e^{-p\rho}\,d\rho = -\frac{1}{\rho^2}(1+p\rho)\,e^{-p\rho}\Big|_0^N$

$$= \underbrace{-\frac{1}{\rho^2}(1+N\rho)\,e^{-N\rho}}_{\substack{\text{exponentially}\\ \text{small as } \rho\to+\infty}} + \frac{1}{\rho^2} \sim \frac{1}{\rho^2}$$

or $\displaystyle\int_0^N p^2 e^{-p\rho}\,d\rho = -\frac{1}{\rho^2}(2+2p\rho+p^2\rho^2)\,e^{-p\rho}$

$$= -\frac{1}{\rho^2}(2+2pN+p^2N^2)\,e^{-pN} + \frac{1}{\rho^2}\cdot 2 \sim \frac{2}{\rho^2}$$

For the general $I(\rho)$ substitute $p\rho = t$ and

$$I(\rho) = \int_0^\infty \frac{t^{\beta-1}}{\rho^\beta} e^{-t}\,dt = \frac{1}{\rho^\beta}\int_0^\infty t^{\beta-1} e^{-t}\,dt = \frac{1}{\rho^\beta}\Gamma(\beta)$$

(by the def of the Gamma function)

Example More generally, if $\rho = |\rho| e^{i\alpha}$, $|\alpha| < \frac{\pi}{2}$ then the same asymptotic behavior happens.

Note: but for $\alpha = \pm\frac{\pi}{2}$ then $e^{-p\rho}$ is no longer exponentially small, and for $|\alpha| > \frac{\pi}{2}$ it is even exponentially large, so the result is false.

More generally:

Watson's Lemma

$$I(\rho) = \int_0^\infty F(p)\, e^{-p\rho}\, dp \qquad \text{with } |\arg\rho| < \frac{\pi}{2}$$

and $F(p)$ has asymptotic power series for $p \to 0+$ (**)

$$F(p) \sim c_0 p^{\beta_0 - 1} + c_1 p^{\beta_1 - 1} + c_2 p^{\beta_2 - 1} + \dots \text{ as } p \to 0+$$

$$(\beta_0 < \beta_1 < \beta_2 < \dots)$$

(and, of course, $\int_0^\infty |F(p)|\, dp < \infty$)

Then

$$I(\rho) \sim c_0 \frac{\Gamma(\beta_0)}{\rho^{\beta_0}} + c_1 \frac{\Gamma(\beta_1)}{\rho^{\beta_1}} + c_2 \frac{\Gamma(\beta_2)}{\rho^{\beta_2}} + \dots \quad \text{as } |\rho| \to +\infty.$$

Idea of the proof:

the integrand $F(p)\, e^{-p\rho}$ is exponentially small as $|\rho| \to +\infty$ unless $p = 0$. The asymptotic behavior has the main contribution from the endpoint $p=0$ (**) and use the example before to integrate term-by-term.

<u>Example</u>

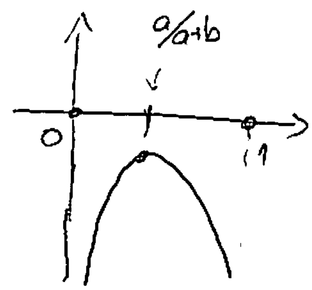
$$I(\vartheta) = \int_0^1 x^{\vartheta a-1}(1-x)^{\vartheta b-1}\,dx \qquad \text{where } a, b > 0, \quad \vartheta \to +\infty.$$

We first rewrite

$$I(\vartheta) = \int_0^1 e^{\vartheta[a \ln x + b \ln(1-x)]}\; x(1-x)\,dx$$

$$f(x) = a \ln x + b \ln(1-x)$$

$$f'(x) = \frac{a}{x} + \frac{b}{x-1} = \frac{(a+b)x - a}{x(x-1)}$$



the asymptotic behavior is collected near $x = x_s = \frac{a}{a+b}$.

Write $f(x) \approx f(x_s) + \frac{1}{2} f''(x_s)(x-x_s)^2$

and $I(\vartheta) \sim x_s^{\vartheta a-1}(1-x_s)^{\vartheta b-1} \int_{-\infty}^{\infty} e^{\vartheta \frac{1}{2} f''(x_s)(x-x_s)^2}\,dx$

(recall $\int_{-\infty}^{\infty} e^{-ct^2}\,dt = \frac{\sqrt{\pi}}{\sqrt{c}}$ for $c>0$)

$$= \left(\frac{a}{a+b}\right)^{\vartheta a-1}\left(\frac{b}{a+b}\right)^{\vartheta b-1} \frac{\sqrt{\pi}}{\sqrt{\frac{1}{2}\vartheta \frac{(a+b)^3}{ab}}} = \left(\frac{ab}{a+b}\right)^{\vartheta} \frac{1}{\sqrt{\vartheta}} \; \frac{\sqrt{\pi}\,(a+b)^2}{ab} \; \frac{\sqrt{2}\sqrt{ab}}{(a+b)^{3/2}}$$