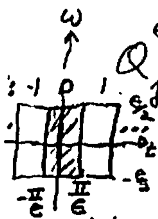
Wavelets and Multiresolution Analysis

Question: What is the most efficient way of representing a signal e.g. $\sim\sim \in L^2(-\infty,\infty)$

Answer: in term of O.N. basis with elements localized in t and in ω-space.

Consider the set of O.N. wavepackets



$$Q^{\epsilon}_{j\ell} = \int_{-\epsilon/2}^{\epsilon/2} \frac{e^{-i\frac{2\pi\ell}{\epsilon}}}{\sqrt{\epsilon}}\frac{e^{i\omega t}}{\sqrt{2\pi}}\,d\omega = \underbrace{\frac{2}{\sqrt{2\pi\epsilon}}\frac{\epsilon}{2}}_{\sqrt{\frac{\epsilon}{2\pi}}}\;\frac{\sin\left(t-\frac{2\pi\ell}{\epsilon}\right)\frac{\epsilon}{2}}{\left(t-\frac{2\pi\ell}{\epsilon}\right)\frac{\epsilon}{2}}$$

$j=0 \qquad \Delta t\,\Delta\omega = 2\pi$

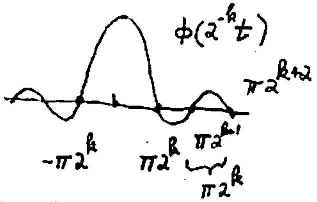
Having so far left the frequency window width unspecified, we now consider families of wave packets. These families are characterized by

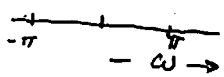
$$\epsilon = 2\pi\, 2^{-k} \qquad k = 0, \pm 1, \pm 2, \ldots$$

We have

$$Q^{\epsilon}_{0\ell} = \int_{-\pi 2^{-k}}^{\pi 2^{k}} \frac{e^{-2\pi\ell\omega/2\pi 2^{-k}}}{\sqrt{2\pi 2^{-k}}}\frac{e^{i\omega t}}{\sqrt{2\pi}}\,d\omega = \sqrt{2^{-k}}\,\frac{\sin\pi(2^{-k}t-\ell)}{\pi(2^{-k}t-\ell)}$$

$$\equiv \sqrt{2^{-k}}\,\phi\left(2^{-k}t-\ell\right) \qquad \text{P113}$$

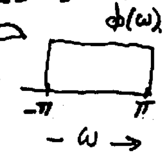




Comment:

We see that the ℓ^{th} member of the k^{th} family is obtained from the scaling function (= "father" wavelet)

$$\phi(t) = \frac{\sin\pi t}{\pi t} \qquad \hat{\phi}(\omega) =$$



by

(i) translating by the amount $\Delta t = \ell$ to obtain

$$T\phi(t) = \phi(t-\ell)$$

(ii) dilating it by the factor 2^k

$$DT\phi(t) = \sqrt{2^{-k}}\,\phi\left(2^{-k}t-\ell\right).$$

We also have renormalized in order to guarantee orthonormality in $L^2(-\infty,\infty)$:

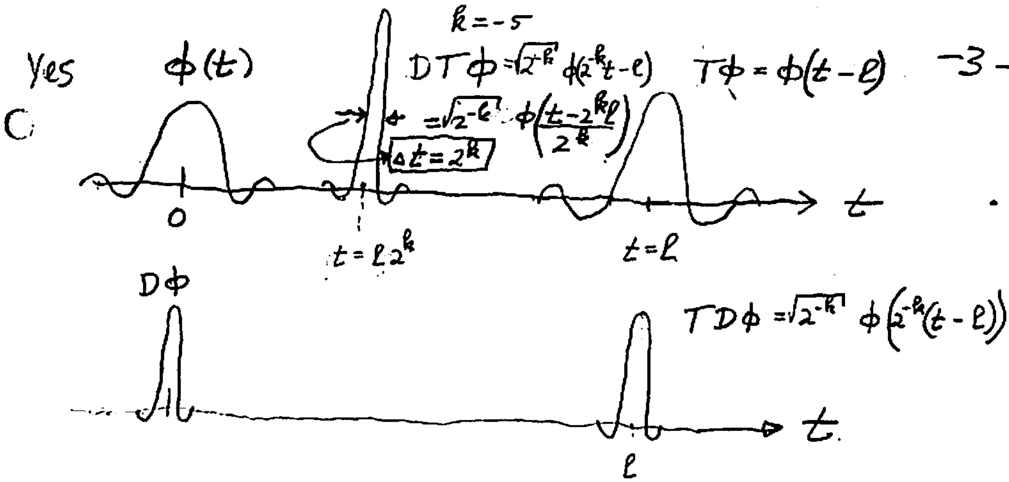
$$\int_{-\infty}^{\infty} \sqrt{2^{-k}}\,\phi\left(2^{-k}t-\ell\right)\sqrt{2^{-k}}\,\phi\left(2^{-k}t-\ell'\right)dt =$$

$$= \boxed{\int_{-\infty}^{\infty} \phi(t-\ell)\,\phi(t-\ell')\,dt = \delta_{\ell\ell'}}$$

see H.W.3 # 4

Comment 1: Reversing the two operation T & D would have yielded

$$TD\,\phi(t) = \sqrt{2^{-k}}\,\phi\left(2^{-k}(t-\ell)\right).$$



$\Delta t = 2^k$ is the scale of the smallest resolvable feature.

Comment 2:
The sinc function is not the only one whose discrete parallel translates form an orthogonal set. According to Stephan <u>Mallat's theorem</u> a function ϕ satisfies

$$\langle \phi(u-\ell), \phi(u-k) \rangle = \delta_{\ell k} \text{ if and only if}$$

$$\sum_{k=-\infty}^{\infty} \left|\hat{\phi}(\omega+2\pi k)\right|^2 = \text{const} \ \forall\, \omega \quad -\infty<\omega<\infty.$$

See HW.3 #4 (const = ? has to be determined.

MULTI SCALE ANALYSIS.

The process of identifying large scale structures and distinguishing them from small scale structures in a mathematically precise and optimal way is achieved by means of a Multi Scale Analysis (MSA)

One arrives at a MSA from the following considerations:

(7) Any function satisfying $(*)$ and the orthonormality condition

$$\langle \phi(u-\ell), \phi(u-\ell') \rangle = \delta_{\ell\ell'}$$

or equivalently,

$$\boxed{\langle \phi(u-\ell), \phi(u) \rangle = \delta_{\ell 0}} \quad (**) \qquad P132$$

is the scaling function for a MSA; in other words, there is a one-to-one correspondence

$$\boxed{MSA \longleftrightarrow \phi(t)}$$

(8) Eqs $(*)$ and $(**)$ have a much simpler form in the Fourier domain:

$$\phi(t) \quad (*) : \quad \phi\left(\frac{t}{2}\right) = \sqrt{2} \sum_\ell h_\ell \, \phi(t-\ell) \quad (*) \qquad P134$$

$$\widehat{(*)} : \quad \boxed{\hat{\phi}(2\omega) = \underbrace{\frac{\sqrt{2}}{2} \sum_\ell h_\ell e^{i\omega\ell}}_{H(\omega)} \hat{\phi}(\omega) \equiv H(\omega)\hat{\phi}(\omega)} \quad \widehat{(*)} \quad P134$$

where

$$H(\omega) = \frac{\sqrt{2}}{2} \sum_{\ell=-\infty}^{\infty} h_\ell e^{i\omega\ell}$$

Eq. $\widehat{(*)}$ is a linear input-output relation. The function $H(\omega)$, which is periodic,

$$H(\omega + 2\pi) = H(\omega), \qquad P134$$

is called a filter function, or simply a filter.

(5) $\phi(2^{-(k+1)}t-\ell) \in V_{k+1}$ and $\phi(2^{-k}t-\ell) \in V_k$ P130 implies

$$\sqrt{2^{-(k+1)}} \, \underbrace{\phi\left(2^{-(k+1)}t-\ell'\right)}_{\in V_{k+1}} = \sum_{\ell=-\infty}^{\infty} \sqrt{2^{-k}} \, \underbrace{\phi(2^{-k}t-\ell)}_{\in V_k}$$

$$\underbrace{\left\langle \sqrt{2^{-k}}\,\phi(2^{-k}u-\ell), \sqrt{2^{-(k+1)}}\,\phi(2^{-(k+1)}u-\ell') \right\rangle}$$

verify this! → $\langle \phi(u-\ell), \phi(2^{-1}u-\ell') \rangle = \langle \phi(u-(\ell-2\ell')), \phi(2^{-1}u) \rangle$

$$\equiv \sqrt{2}\, h_{\ell-2\ell'},$$

$\overset{\ell'=0}{=} \langle \phi(u-\ell), \phi\, 2^{-1}(u) \rangle = \int \frac{\sin\pi(u-\ell)}{\pi(u-\ell)} \frac{\sin(\pi\frac{u}{2})}{\pi\frac{u}{2}}\, du$; $\bar{u} = \frac{u}{2}$, $du = 2\,d\bar{u}$; $= 2\langle \phi(2\bar{u}-\ell), \phi\bar{u} \rangle = \sqrt{2}\, h_\ell$

which is independent of k!

(6) For $k=-1$ and $\ell' \equiv 0$ one has

$$\boxed{\phi(t) = \sqrt{2} \sum_{\ell=-\infty}^{\infty} h_\ell \, \phi(2t-\ell)} \quad (*) \qquad P132$$

where

$$h_\ell = \langle \phi(2u-\ell), \phi(u) \rangle$$

The boxed equation $(*)$ is called the scaling eq'n for $\phi(t)$.

$$\phi(t) \rightleftarrows \{h_\ell\}_{\ell=-\infty}^{\ell=\infty}$$

Thus, specifying a "father" wavelet (= scaling function) is equivalent to specifying a set of coefficients $\{h_\ell\}$.

Problem: Find $\{h_\ell\}$ for the Shannon scaling function.

$$\phi(t) = \frac{\sin \pi t}{\pi t}$$

$\widehat{(**)}$: $\boxed{\sum_{m=-\infty}^{\infty} \left|\hat{\phi}(\omega+2\pi m)\right|^2 = \frac{1}{2\pi} \quad \forall \omega}$ $\widehat{(**)}$ -9-

P134

comment:

By combining $\widehat{(*)}$ with $\widehat{(**)}$ one obtains the following constraint on the filter function:

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 1$$

P134

(9) Question:

In item (6) What is the relation between the representation $\{c_\ell^{k+1}\}_{\ell=-\infty}^{\infty}$ of f in the $(k+1)$st resolution space V_{k+1} and the representation $\{c_\ell^k\}_{\ell=-\infty}^{\infty}$ of f in the k th resolution space $\underset{(k^{th}\text{ scale})}{V_k} \supset \underset{(k+1)\text{st scale}}{V_{k+1}}$

Answer: (Pyramid algorithm)

$$\boxed{c_{\ell'}^{k+1} = \sum_{\ell} \tilde{h}_{2\ell'-\ell}\, c_\ell^k}$$

P131

where

$$\tilde{h}_{2\ell'-\ell} = \overline{h_{\ell-2\ell'}} \quad \text{(complex conjugate)}$$

Comment: $c_{\ell'}^{k+1}$ has half as many samples as c_ℓ^k does.

(10) Given the k^{th} resolution space

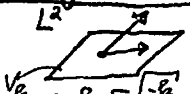
$$V_k = \text{span}\{\phi(2^{-k}t-\ell)\}_{\ell=-\infty}^{\infty}$$

and the $(k+1)^{st}$ resolution space

$$V_{k+1} = \text{span}\{\phi(2^{-(k+1)}t-\ell)\} \subset V_k$$

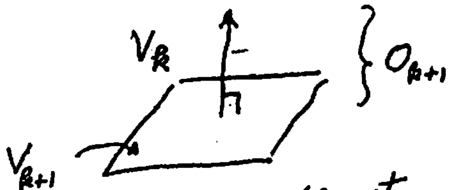
one can find its orthogonal complement

$O_{k+1} \perp V_{k+1}$ in V_k:



$P_k f = \sum_\ell \sqrt{2^{-k}}\, \phi(2^{-k}t-\ell)\underbrace{\langle \sqrt{2^{-k}}\,\phi(2^{-k}u-\ell), f\rangle}_{= c_\ell^k}$

$$V_k = V_{k+1} \oplus O_{k+1} \qquad V_{k+1} \subset V_k$$



O_{k+1} is called the $(k+1)^{st}$ space of details.

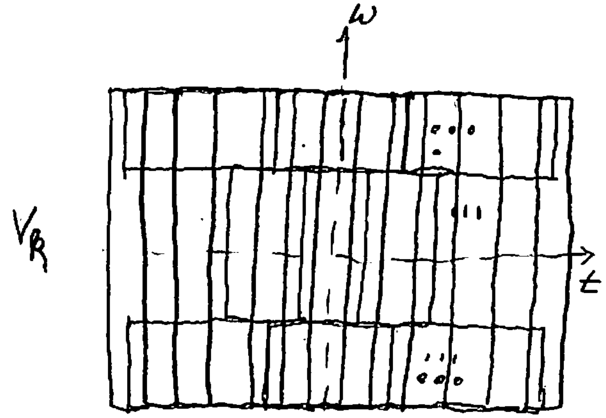
We start with $f \in L^2(-\infty,\infty)$ and then use the basis-induced projection operators to obtain

$P_{V_k} f \in V_k$ — function at resolution 2^{-k}

$P_{V_{k+1}} f \in V_{k+1}$ — " " " $2^{-(k+1)}$

$P_{O_{k+1}} f \in V_{k+1}$ — detail of f relative to res. $2^{-(k+1)}$

$P_{V_k} f = P_{V_{k+1}} f + P_{O_{k+1}} f = [\text{fn at res'n } 2^{-(k+1)}] + [\text{detail of fn rel've to res'n } 2^{-(k+1)}]$



$O^+_{k+1} = \text{span}\{\psi^+(2^{-(k+2)}t-\ell)\}$

$V_{k+1} = \text{span}\{\phi(2^{-(k+1)}t-\ell)\}$

$O^-_{k+1} = \text{span}\{\psi^-(2^{-(k+2)}t-\ell)\}$

---

correction on P119 (March) P118 (February) in section 2.6.4

$V_{k+1} = \{ \quad \cdots \quad \cancel{k}, \ell = 0, \pm1, \pm2 \ldots \}$

$O^+_{k+1} = \{ \quad " \quad \cancel{k}, \ell = \ " \ \}$

$O^-_{k+1} = \{ \quad \cancel{k}, \ell = \ " \ \}$

$$V_k = \text{span}\{\psi^+(2^{-(k+2)}t-\ell)\} \cup \{\phi(2^{-(k+1)}t-\ell)\} \cup \{\psi^-(2^{-(k+2)}t-\ell)\}$$

Consequently the MRA has the following more precise hierarchical structure P123

$$\begin{tikzcd}
\text{large scale context} \quad \cdots & V_{k+1} & V_k \arrow[l, "P"'] \arrow[dl, "P"] & V_{k-1} \arrow[l, "P"'] \arrow[dl, "P"] & \cdots \quad L^2(-\infty,\infty) \\
& \oplus & \oplus & & \\
\text{detail} & O_{k+1} = O^+_{k+1} \oplus O^-_{k+1} & O_k = O^+_k \oplus O^-_k & &
\end{tikzcd}$$

See section 2.6.8 on P129 f.f.

Let $\phi(t) \in L^2$ such that $0 < \int_{-\infty}^{\infty} \phi(t)\,dt = \sqrt{2\pi}\,\hat{\phi}(0) < \infty$ (13)

start here:

Let $\{h_\ell\} \in \ell^2$

so that $H(\omega) = \frac{\sqrt{2}}{2} \sum_{\ell=-\infty}^{\infty} h_\ell e^{i\omega\ell}$: (1) $|H(\omega)|^2 + |H(\omega+\pi)|^2 = 1$

and (2) $\hat{\phi}(\omega) = \hat{\phi}(0) \prod_{k=1}^{\infty} H\left(\frac{\omega}{2^k}\right) \in L^2$

Then $\phi(t) = \mathcal{F}^{-1}\left[\hat{\phi}\right](t)$ will

satisfy

$$\phi\left(\tfrac{t}{2}\right) = \sqrt{2} \sum_\ell h_\ell\, \phi(t-\ell) \quad \text{scaling eq'n}$$

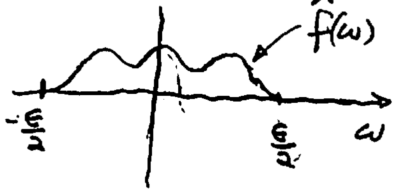
$$\psi\left(\tfrac{t}{2}\right) = \sqrt{2} \sum_\ell g_\ell\, \phi(t-\ell)$$

where $g_\ell = (-1)\, h_{1-\ell}$

Let $f \in L^2(-\infty,\infty)$ be a band-limited function: 114,115

$$\hat{f}(\omega) = 0 \quad \text{for} \quad \omega \notin \left[\frac{\epsilon}{2}, \frac{\epsilon}{2}\right] = \left[-\pi 2^{-k}, \pi 2^{k}\right]$$

Comment: Each function $\phi(2^{-k}t-\ell)$ have this property.

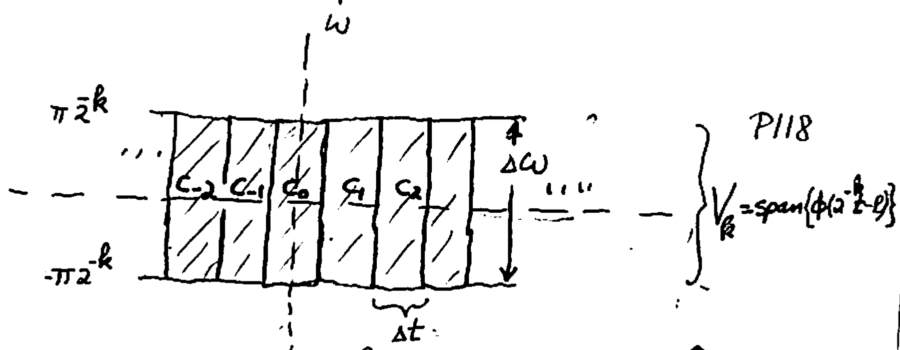


$\epsilon = 2\pi 2^{-k}$

Thus

$$f(t) = \sum_{\ell=-\infty}^{\infty} C_\ell^k \sqrt{2^{-k}}\, \phi(2^{-k}t-\ell)$$

Its phase space representation is



$\Delta\omega = \epsilon = 2\pi 2^{-k}$ = bandwidth of $\widehat{DT\phi}$

$\Delta t = \frac{2\pi}{\epsilon} = 2^k$ = the width of $\phi(2^{-k}t-\ell) = DT\phi(t)$

$\Delta t \Delta\omega = 2\pi$

$\therefore \boxed{f \in \text{span}\{\phi(2^{-k}t-\ell)\}_{\ell=-\infty}^{\infty} = V_k}$ $\begin{pmatrix}\Delta\omega = 2\pi 2^{-k} \\ \Delta t = 2^k\end{pmatrix}$

10 Aspects of the MSA.

Comment: The following 10 observations apply to any function satisfying Mallat's theorem on p 3.

(1) V_k is called the k^{th} resolution space

(2)

$$f(t) = \sum_{\ell=-\infty}^{\infty} C_\ell^k \sqrt{2^{-k}}\, \phi(2^{-k}t-\ell) \qquad \begin{pmatrix}\Delta t = 2^k \\ \Delta\omega = 2\pi 2^{-k}\end{pmatrix}$$

$\Rightarrow \exists$ coefficients C_ℓ^{k-1} s.t. $f(t) = \sum_{\ell=-\infty}^{\infty} C_\ell^{(k-1)} \sqrt{2^{-k+1}}\, \underbrace{\phi(2^{-k+1}t-\ell)}_{\text{also ON.}}$

$\begin{pmatrix}\Delta\omega = 2\pi 2^{-(k-1)} \\ \Delta t = 2^{k-1}\end{pmatrix}$

i.e. $f \in V_k \Rightarrow f \in V_{k-1}$

support $\hat{f}(\omega) = [-\frac{\epsilon}{2}, \frac{\epsilon}{2}] \Rightarrow$ supp $\hat{f}(\omega) \subset [-\epsilon, \epsilon]$

(3) V_k is a subspace of V_{k-1}: $V_k \subset V_{k-1}$

(4) The hierarchy of vector spaces P115-116

$$\{0\} \subset \cdots \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \cdots \subset L^2$$

is called a multi resolution analysis (MRA) or multi scale analysis (MSA)

$k\downarrow$ implies smaller scale (= higher resolution)

$k\uparrow$ " larger scale.