Representations of solutions of general nonlinear ODEs in singular regions

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O. Costin, RDC: Invent. 2001, Kyoto 2000, [...]

Most analytic differential equations do have irregular singularities.

The point at infinity: often an irregular singularity (also for the Painlevé equations).

Let $x = \infty$ be an irregular sing. of $\mathbf{y}' = \mathbf{f}(x, \mathbf{y})$ ($x \in \mathbb{C}$, $\mathbf{y}, \mathbf{f} \in \mathbb{C}^d$).

Algorithmic procedures $\rightsquigarrow$ a rank 1 irregular singularity $\rightsquigarrow$

Normal form:

$$\boxed{\mathbf{y}' + \left(\Lambda - \frac{1}{x}A\right)\mathbf{y} = \mathbf{g}(x^{-1}, \mathbf{y})}$$

If Λ, A are diagonalizable, then

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots \lambda_{\mathrm{d}}),\ \mathrm{A} = \mathrm{diag}(\alpha_1, \ldots \alpha_{\mathrm{d}})$$

Also: $\mathbf{g}$ analytic at $(0, \mathbf{0})$, with $\mathbf{g}(x^{-1}, \mathbf{y}) = O(x^{-2}) + O(|\mathbf{y}|^2)$

Solutions $\mathbf{y} \to 0$ as $x \to \infty$ have a unique asymptotic power series:

$$\mathbf{y} \sim \hat{\mathbf{y}}_{\mathbf{0}} \equiv \sum_{n}^{+\infty} x^{-n} \mathbf{y}_{0,n} \qquad \text{(usually divergent)}$$

Example: linear, $d = 1$ (the simplest!) $\boxed{y' + y = x^{-1}}$

Unique formal solution: $y(x) \to 0$ as $x \to +\infty \implies y \sim \hat{y}_0(x) = \sum_{n \geq 0} n!\, x^{-n-1}$

Exact solution: $y = y(x; C) = y_0(x) + C\mathrm{e}^{-x}$ where

$y_0(x) = \mathrm{e}^{-x} Ei(x) \sim \hat{y}_0(x)$ as $x \to +\infty$ and $C =$ parameter.

$\rightsquigarrow$ complete formal solution: $\hat{y} = \hat{y}(x; C) = \sum_{n \geq 0} n!\, x^{-n-1} + C\mathrm{e}^{-x}$.

Complete formal solution: $\hat{y} = \hat{y}(x; C) = \sum_{n\geq 0} n!\, x^{-n-1} + C\mathrm{e}^{-x}$. (*)

It is not a Poincaré asymptotic expansion on $\mathbb{R}_+$
(since $\mathrm{e}^{-x} \ll x^{-n-1}$ for all n: e^{-x} is beyond all orders of the power series.)

An expansion (*) is a (simple example of) a transseries for $x \to \infty$.

Note: a transseries depends on the direction toward ∞ in $\mathbb{C}$.

E.g. (*) is for $x \to +\infty$. In fact for $x \to \infty$ in the sector $|\arg x| < \frac{\pi}{2}$.

For $x \to \infty$ in the sector $\frac{3\pi}{2} < \arg x < \frac{5\pi}{2}$ we write $C\mathrm{e}^{-x} + \sum_{n\geq 0} n!\, x^{-n-1}$

In $d = 1$:

Linear eq. $\boxed{y' + y = x^{-1}}$

formal sol.: $\hat{y} = \hat{y}(x;C) = \hat{y}_0(x) + C\mathrm{e}^{-x}$ where $\hat{y}_0$ is a divergent series.

Nonlinear eq. $\boxed{y' + y = x^{-1} + y^2}$

formal sol.: $\hat{y} = \hat{y}(x;C) = \hat{y}_0(x) + C\mathrm{e}^{-x}\hat{y}_1(x) + C^2\mathrm{e}^{-2x}\hat{y}_2(x) + \dots$

$= \sum_{k\geq 0} C^k\mathrm{e}^{-kx}\hat{y}_k(x)$ where $\hat{y}_k$ are divergent series.

For $d \geq 1$:

Linear equations $\boxed{\mathbf{y}' + \left(\Lambda - \frac{1}{x}A\right)\mathbf{y} = \mathbf{f}(x^{-1})}$ $(\Lambda, A$ diagonal, nonresonant)

Formal solution : $\hat{\mathbf{y}} = \hat{\mathbf{y}}(x; \mathbf{C}) = \hat{\mathbf{y}}_0(x) + \sum_{j=1}^{d} C_j \mathrm{e}^{-\lambda_j x} x^{\alpha_j} \hat{\mathbf{y}}_j(x)$ where $\hat{\mathbf{y}}_j(x)$ are power series (usually divergent).

General nonlinear equations $\boxed{\mathbf{y}' + \left(\Lambda - \frac{1}{x}A\right)\mathbf{y} = \mathbf{g}(x^{-1}, \mathbf{y})}$ (nonresonant)

Formal solution:

$$\hat{\mathbf{y}} = \hat{\mathbf{y}}(x; \mathbf{C}) = \hat{\mathbf{y}}_0(x) + \sum_{\mathbf{k} \in \mathbb{N}^d \setminus \mathbf{0}} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda} \cdot \mathbf{k} x} x^{\boldsymbol{\alpha} \cdot \mathbf{k}} \hat{\mathbf{y}}_{\mathbf{k}}(x)$$

where $\hat{\mathbf{y}}_{\mathbf{k}}(x)$ are power series (divergent), determined algorithmically.

$$\hat{\mathbf{y}} = \hat{\mathbf{y}}(x;\mathbf{C}) = \hat{\mathbf{y}}_0(x) + \sum_{\mathbf{k}\in\mathbb{N}^d\setminus 0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}\,x} x^{\boldsymbol{\alpha}\cdot\mathbf{k}} \hat{\mathbf{y}}_{\mathbf{k}}(x) \qquad (*)$$

formal exponential power series. (If resonant - also logs.)

Note: $(*)$ is a valid asymptotic expansion (transseries) only if it can be well ordered w.r.t. $\gg$.

Therefore (*) is a transseries only in the sector

$$S_{trans} = \{x \in \mathbb{C} \,|\, \Re(\lambda_j x) > 0 \text{ for all } j \text{ with } C_j \neq 0\}$$

Introduced by Fabris (1885). Studied by Cope (1934).
Vastly generalized by Ecalle (1981) to formal expressions closed under all operations.
In logic.

Correspondence between formal and actual solutions?

Example: linear equation $\boxed{y' + y = x^{-1}}$

Formal solution $\hat{y}(x;C) = \sum_{n\geq 0} n!\, x^{-n-1} + C\mathrm{e}^{-x}$

Borel summation? Looking for $y = \mathcal{L}Y$ take $\mathcal{L}^{-1}$ in the ODE:

$$\Longrightarrow \quad (1-p)Y(p) = 1 \implies Y(p) = \frac{1}{1-p} \text{ so } \quad y(x) = \int_d \mathrm{e}^{-px}\frac{1}{1-p}\,dp.$$

- $\mathbf{d} \neq \mathbb{R}_+$. We can integrate on $d_\pm$ half-lines above/below $\mathbb{R}_+$.
- Furthermore: $\displaystyle\int_{d_+} \neq \int_{d_-}$. **Which one to choose?**
- Note: $\displaystyle\int_{d_+} - \int_{d_-} = 2\pi i e^{-x}$. **Recovers the exponentially small term!**

(*) $\int_{d_+} - \int_{d_-} = 2\pi i e^{-x}$ where $\arg d_+ \in (0, \pi/2)$, $\arg d_- \in (-\pi/2, 0)$

Ecalle:

- The *median average* gives the solution with no exponentially small term ($C = 0$); e.g., in the 1-d linear case:

$$y(x;0) = y_0(x) = \frac{1}{2}\int_{d_+} \mathrm{e}^{-px}\frac{1}{1-p}\,dp + \frac{1}{2}\int_{d_-} \mathrm{e}^{-px}\frac{1}{1-p}\,dp$$

- The difference (*) gives the exponentially small term.

- These generalize to nonlinear equations.

Example: nonlinear equation $\boxed{y' + y = x^{-1} + y^2}$ in $d = 1$.

$$\text{Take } \mathcal{L}^{-1} \implies (1-p)Y(p) = 1 + \int_0^p Y(q)Y(p-q)\,dq$$

- Clearly $\exists!$ solution $Y(p)$ analytic at $p = 0$. It is analytic for $|p| < 1$.

- Clearly $Y(p)$ is singular at $p = 1$.

- Convolution $\rightsquigarrow$ the singularity at $p = 1$ gives rise to singularities at $p = 2, 3, 4, \ldots$ (an array of singularities, in the Borel plane, equally spaced).

$Y(p)$ is the sol. an. at $p = 0$ of the convol. eq. $(1-p)Y(p) = 1 + (Y * Y)(p)$.

Formal solution of the ODE: $\hat{y}(x;C) = \sum_{k\geq 0} C^k e^{-kx} \hat{y}_k(x)$

Each $\hat{y}_k(x)$ is summed using $y_k(x) = \sum_j \alpha_{j,k} \int_{d_{j,k}} e^{-px} Y(p)\, dp$

weighted averages of Laplace transforms along paths winding in prescribed ways among $p = 1, 2, 3, \ldots$.

Finally, the series $y(x;C) = \sum_{k\geq 0} C^k e^{-kx} y_k(x)$ converges to solutions for $x \in S_{an}$

$$S_{an} = \{x \mid -\frac{\pi}{2} + \epsilon < \arg(x) < \frac{\pi}{2} - \epsilon,\ |x| > R\}$$

General nonlinear equations:

their transseries solution

$$\hat{\mathbf{y}}(x;\mathbf{C}) = \hat{\mathbf{y}}_0(x) + \sum_{\mathbf{k}\in\mathbb{N}^d\setminus\mathbf{0}} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\boldsymbol{\alpha}\cdot\mathbf{k}} \hat{\mathbf{y}}_{\mathbf{k}}(x)$$

can be summed similarly, in sectors. Generalized Borel summation:

- each $\hat{\mathbf{y}}_{\mathbf{k}}(x)$ is generalized Borel summable to $\mathbf{y}_{\mathbf{k}}(x)$

(using special averages of Borel summation along special paths), then

- the series $\sum_{\mathbf{k}\in\mathbb{N}^d} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\boldsymbol{\alpha}\cdot\mathbf{k}} \mathbf{y}_{\mathbf{k}}(x)$ converges, and the limit is a solution.

(1957-59) Iwano showed that $\mathbf{y}(x;\mathbf{C}) = \mathbf{y}_0(x) + \sum \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\boldsymbol{\alpha}\cdot\mathbf{k}} \mathbf{y}_{\mathbf{k}}(x)$ with $\mathbf{y}_{\mathbf{k}}(x)$ analytic, and convergent in sectors.

(1981) Ecalle constructed the summation of transseries (formal solutions of most problems), establishing an isomorphism with a class of functions ("analyzable").

(1990) Balser, Braaksma, Ramis, Sibuya proved multisummability of formal power series solutions of linear equations.

(1992) Braaksma proved multisummability of formal power series solutions for nonlinear equations.

(1998) O. Costin proved generalized Borel summation for transseries solutions of rank 1, their 1-to-1 correspondence with solutions $y(x) \to 0$ (in a sector), and compatibility with all operations.

(2001-04) Braaksma proved similar results for solutions of difference equations.

Let $\boxed{\mathbf{y}' + \left(\Lambda - \frac{1}{x}A\right)\mathbf{y} = \mathbf{g}(x^{-1}, \mathbf{y})}$ nonresonant, Λ, A diag., $\mathbf{g}$ analytic at $(0,0)$.

Recall: the antistokes lines are $\pm i\overline{\lambda}_j\mathbb{R}_+$.

Let d a direction in $\mathbb{C}$ which is **not** an antistokes line.

Let S be the open sector bounded by two consecutive antistokes lines, $d \subset S$.

$\forall \mathbf{y}(x)$ sol. with $\mathbf{y}(x) \to 0$ ($x \in d$, $x \to \infty$) then $\mathbf{y}(x) \sim \mathbf{y}_0(x)$ ($x \in S$, $x \to \infty$).

Theorem (O. Costin, 1998)

There exists a 1-to-1 correspondence between :
solutions $\mathbf{y}(x) \to 0$ *(*$x \in d$, $x \to \infty$*) and*
generalized Borel summations of $\hat{\mathbf{y}}(x; C)$ *transseries solutions in* S*.*

These solutions $\mathbf{y}(x; C)$ *are analytic in* S *for* $|x|$ *large.*

Solutions $\mathbf{y}(x;C) \to 0$ for $x \to \infty,\ x \in d$ are analytic in S for $|x|$ large.

Question: what happens to $y(x;C)$ as x approaches the boundary of S?

Example: d=1 $\quad y' + \left(1 - \frac{\alpha}{x}\right) y = g(x^{-1}, y) \quad (\lambda = 1)$. Formal solution:

$$\hat{y}(x;C) = \hat{y}_0(x) + Ce^{-x}x^{\alpha}\hat{y}_1(x) + C^2e^{-2x}x^{2\alpha}\hat{y}_2(x) + C^3e^{-3x}x^{3\alpha}\hat{y}_3(x) + \dots$$

with $\hat{y}_k(x) = \sum_{j=0}^{\infty} \frac{y_{k,j}}{x^j}$

valid in the sector $S_{trans} = \{x; -\frac{\pi}{2} < \arg x < \frac{\pi}{2}\}$

generalized Borel summable to a solution $y(x;C)$ analytic in

$$S_{an} = \{x \mid -\tfrac{\pi}{2} + \epsilon < \arg x < \tfrac{\pi}{2} - \epsilon,\ |x| > R,\ |Ce^{-x}x^{\alpha}| < \delta^{-1}\}$$

What happens to $y(x;C)$ as $\arg x$ approaches $\frac{\pi}{2}$? (Similarly, for $-\frac{\pi}{2}$.)

$$\hat{y}(x;C) = \hat{y}_0(x) + Ce^{-x}x^{\alpha}\hat{y}_1(x) + C^2e^{-2x}x^{2\alpha}\hat{y}_2(x) + C^3e^{-3x}x^{3\alpha}\hat{y}_3(x) + \dots$$

Denote $\boxed{Ce^{-x}x^{\alpha} = \xi}$. The transseries has the form

$$\hat{y} = \left[\frac{y_{0,1}}{x} + \frac{y_{0,2}}{x^2} + \dots\right] + \xi\left[y_{1,0} + \frac{y_{1,1}}{x} + \frac{y_{1,2}}{x^2} + \dots\right] + \xi^2\left[y_{2,0} + \frac{y_{2,1}}{x} + \frac{y_{2,2}}{x^2} + \dots\right] + \dots$$

For $\arg x < \pi/2$: $\xi \ll x^{-k} \mapsto$ the terms are well ordered.

For $\arg x = \pi/2$: $\xi \not\ll x^{-k} \mapsto$ the transseries breaks.

There is an intermediate region: $x^{-k} \ll \xi \ll 1$. **Reorder the transseries:**

$$\hat{y} = \left[\xi y_{1,0} + \xi^2 y_{2,0} + \dots\right] + \frac{1}{x}\left[y_{0,1} + \xi y_{1,1} + \xi^2 y_{2,1} + \dots\right] + \frac{1}{x^2}\left[y_{0,2} + \xi y_{1,2} + \dots\right] + \dots$$

with the form $\hat{y}(x;C) = F_0(\xi) + \frac{1}{x}F_1(\xi) + \frac{1}{x^2}F_2(\xi) + \dots$

Note: $F_0(0) = 0$. *Note:* choose $y_{1,0} = 1$ (to fix C). $\rightsquigarrow F_0'(0) = 1$.

For $d > 1$: Say $d = 2$, and take $\lambda_1 = 1$

$$\mathbf{y}(x;\mathbf{C}) = \hat{\mathbf{y}}_{(0,0)}(x) + C_1 e^{-x} x^{\alpha_1} \hat{\mathbf{y}}_{(1,0)}(x) + C_2 e^{-\lambda_2 x} x^{\alpha_2} \hat{\mathbf{y}}_{(0,1)}(x)$$
$$+ C_1^2 e^{-2x} x^{2\alpha_1} \hat{\mathbf{y}}_{(2,0)}(x) + C_1 C_2 e^{-x-\lambda_2 x} x^{\alpha_1+\alpha_2} \hat{\mathbf{y}}_{(1,1)}(x) + C_2^2 e^{-2\lambda_2 x} x^{2\alpha_2} \hat{\mathbf{y}}_{(0,2)}(x) + \ldots$$

where $\mathbf{y}_{(0,0)} = O(x^{-2})$, $\hat{\mathbf{y}}_{(1,0)}(x) = \mathbf{e}_1 + O(x^{-1})$, $\hat{\mathbf{y}}_{(0,1)}(x) = \mathbf{e}_2 + O(x^{-1})$.

Let $\xi = C_1 e^{-x} x^{\alpha_1}$. Reorder for $e^{-\lambda_2 x} \ll x^{-k} \ll \xi \ll 1$:

$$\mathbf{y} \sim \left[\xi \mathbf{y}_{(1,0),0} + \xi^2 \mathbf{y}_{(2,0),0} + \ldots\right] + \frac{1}{x}\left[\mathbf{y}_{(0,0),1} + \xi \mathbf{y}_{(1,0),1} + \ldots\right] + \ldots$$

therefore $\mathbf{y} \sim \mathbf{F}_0(\xi) + \dfrac{1}{x}\mathbf{F}_1(\xi) + \dfrac{1}{x^2}\mathbf{F}_2(\xi) + \ldots$ where $\mathbf{F}_0(0) = 0,\ \mathbf{F}_1(0) = \mathbf{e}_1$.

Note that $e^{-\lambda_2 x}$ is beyond all orders.

In fact $\mathbf{F}_0(\xi),\ \mathbf{F}_1(\xi),\ \mathbf{F}_2(\xi)$ are functions that can be calculated from the ODE!

Substitute $\mathbf{y}(x) \sim \mathbf{F}_0(\xi) + \frac{1}{x}\mathbf{F}_1(\xi) + \frac{1}{x^2}\mathbf{F}_2(\xi) + \dots$

in $\quad \mathbf{y}' + \left(\Lambda - \frac{1}{x}A\right)\mathbf{y} = \mathbf{g}(x^{-1}, \mathbf{y})$ (with $\mathbf{g}(x^{-1}, \mathbf{y}) = O(x^{-2}) + O(|\mathbf{y}|^2)$)

Use $x^{-k} \ll \xi = C_1 e^{-x} x^{\alpha_1} \ll 1 \rightsquigarrow \mathbf{F}_m$ recursively,

$\mathbf{F}_m$ are the unique sol. analytic at $\xi = 0$, of

$$\xi \mathbf{F}_0' = \Lambda \mathbf{F}_0 - \mathbf{g}(0, \mathbf{F}_0), \qquad \mathbf{F}_0'(0) = \mathbf{e}_1$$

$$\xi \mathbf{F}_m' = \left[\Lambda - \partial_{\mathbf{y}} \mathbf{g}(0, \mathbf{F}_0)\right] \mathbf{F}_m + \alpha_1 \mathbf{F}_{m-1}' + \mathbf{R}_m(\mathbf{F}_0, \dots, \mathbf{F}_{m-1})$$

Representation for x near $i\mathbb{R}_+$ (recall $\lambda_1 = 1$). Denote

$$\mathcal{E}_+ = \{x\,;\, -\frac{\pi}{2} + \delta < \arg x < \frac{\pi}{2} + \delta,\ \Re(\lambda_j x/|x|) > c,\ j = 2, \ldots .\}$$

$$\mathcal{S}_{\delta_1} = \{x \in \mathcal{E}_+\,;\, |\xi(x)| < \delta_1\}$$

Theorem 1. [Inv. Math., 2001]
There exists $\delta_1 > 0$ so that all $\mathbf{F}_m$ are analytic for $|\xi| < \delta_1$ and

$$\mathbf{y}(x) \sim \mathbf{F}_0(\xi) + \frac{1}{x}\mathbf{F}_1(\xi) + \frac{1}{x^2}\mathbf{F}_2(\xi) + \ldots \text{ uniformly for } x \in \mathcal{S}_{\delta_1},\ x \to \infty.$$

The series is differentiable and satisfies Gevrey estimates.

It turns out that **the series remains asymptotic** in part of $\mathcal{E}_+ \setminus \mathcal{S}_{\delta_1}$ near $\xi = \xi_s$ **singularity** of $\mathbf{F}_0$.

$$\mathbf{y}(x;\mathbf{C}) \sim \mathbf{F}_0(\xi) + \frac{1}{x}\mathbf{F}_1(\xi) + \frac{1}{x^2}\mathbf{F}_2(\xi) + \ldots$$

The picture: If ξ_s is an isolated singularity of $\mathbf{F}_0$, calculate $x = \tilde{x}_n$ solutions of $\xi(x) = C_1 e^{-x} x^{\alpha_1} = \xi_s \implies$
$x = \tilde{x}_n = 2n\pi i + \alpha_1 \ln(2n\pi i) + \ln C_1 - \ln \xi_s + o(1), \qquad (n \to \infty)$

Then each solution $\mathbf{y}(x;\mathbf{C})$ (specified by $\mathbf{C}$) has an array of singularities at:

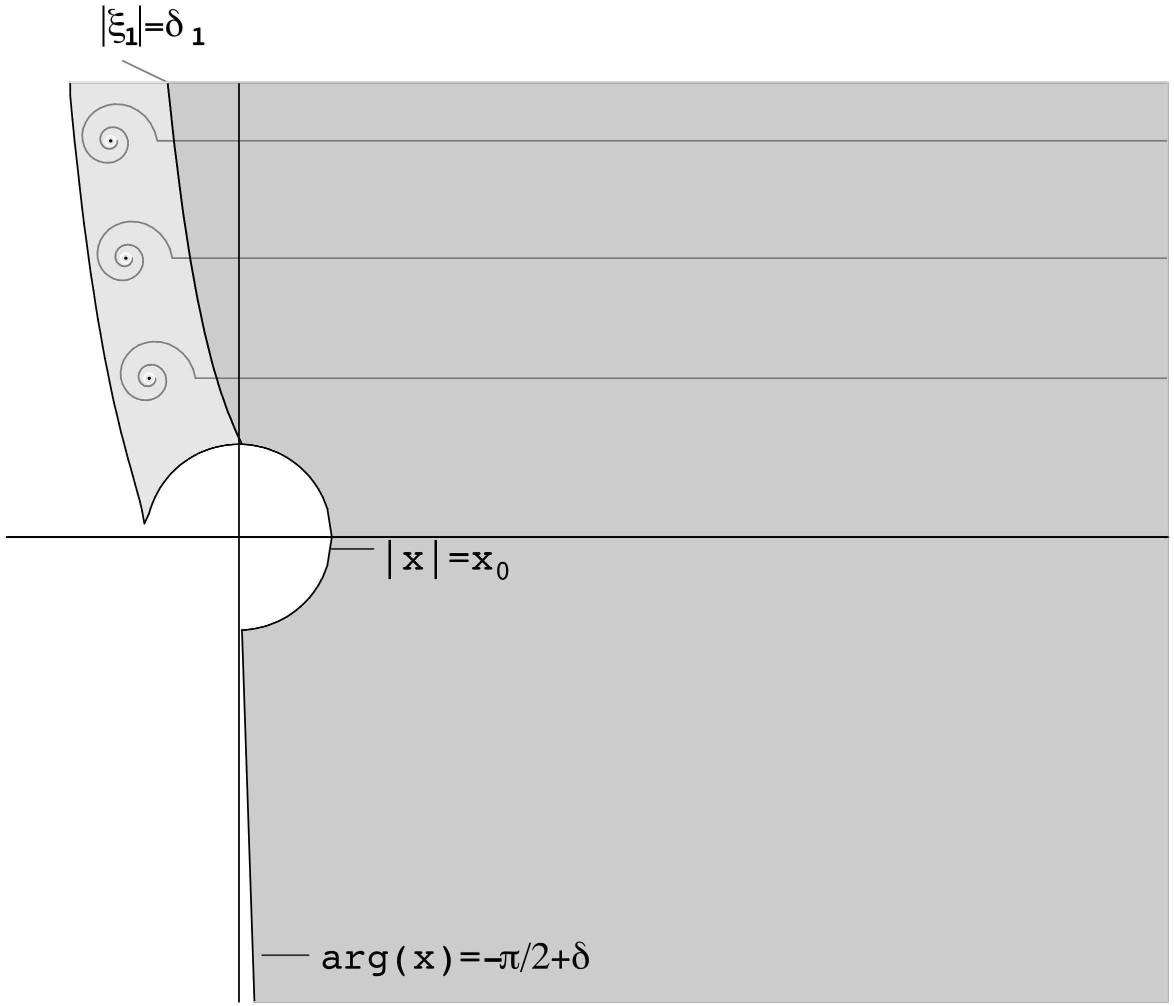
$$x_n = \tilde{x}_n + o(1) = 2n\pi i + \alpha_1 \ln(2n\pi i) + \ln C_1 - \ln \xi_s + o(1), \qquad (n \to \infty).$$

(almost periodic).

Moreover:
$\mathbf{y}(x;C) \sim \mathbf{F}_0(\xi(x)) + \frac{1}{x}\mathbf{F}_1(\xi(x)) + \frac{1}{x^2}\mathbf{F}_2(\xi(x)) + \ldots$ for $x \to \infty,\ x \in D_x$

where D_x is a connected domain surrounding all x_n with $n > N$.

(An asymptotic series valid near infinitely many singularities!)



Singularities at one side of S_{trans} (for $\lambda_1 > 0,\ C_1 \neq 0,\ C_{2,3,\ldots} = 0$).

Theorem 2. Let ξ_s be an isolated singularity of $\mathbf{F}_0$, so that
$\exists\, D \subset$ Riemann surface above $\mathbb{C} \setminus \xi_s$ with
- D open, rel. compact, connected, and $\{|\xi| < \delta_1\} \subset D$,
- $\mathbf{F}_0$ analytic in an ϵ-neighborhood of D,
- $\sup_D |\mathbf{F}_0| = \rho_3$ not too large (so that $\mathbf{g}(x^{-1}, \mathbf{y})$ is analytic near D).

Let $\xi(x_n) = \xi_s \mapsto x_n = 2n\pi i + \alpha_1 \ln(2n\pi i) + \ln C_1 - \ln \xi_s + o(1), \qquad (n \to \infty)$.

Let $D_x = \{x \,|\, \xi(x) \in D,\ |x| > R\}$: connected Riemann surf. above $\mathbb{C} \setminus \{x_n\}_{n>N}$.

Then for $C_1 \neq 0$: (a) $\mathbf{F}_m$ analytic on D, (b) $\mathbf{y}(x; C)$ analytic on D_x, and

$$\mathbf{y}(x) \sim \mathbf{F}_0(\xi(x)) + \frac{1}{x}\mathbf{F}_1(\xi(x)) + \frac{1}{x^2}\mathbf{F}_2(\xi(x)) + \ldots \text{ for } x \to \infty,\ x \in D_x$$

(c) with Gevrey estimates $|\mathbf{y}(x) - \sum_0^{m-1} x^{-j}\mathbf{F}_j| < K m! B^m |x|^{-m}$.

(d) $\mathbf{y}(x)$ is singular at a distance $o(1)$ of x_n.

A very simple **Example:** the integrable equation $\boxed{y' + y = x^{-2} + y^2}$

$$\hat{y} = (\frac{1}{x^2} + \frac{2}{x^3} + \frac{5}{x^4} + \ldots) + Ce^{-x}(1 - \frac{2}{x^2} - \frac{8}{x^3} + \ldots) + C^2 e^{-2x}(1 - \frac{2}{x^2} + \ldots) + \ldots$$

On the other hand, $y(x) \sim F_0(\xi) + \frac{1}{x}F_1(\xi) + \ldots$ for $x \ll Ce^{-x} = \xi$, where $-\xi F_0' + F_0 = F_0^2, \quad F_0 = \xi + O(\xi^2) \quad \Longrightarrow \; F_0(\xi) = \xi/(1-\xi)$.

F_0 analytic for $|\xi| < 1$ (so all F_m are). F_0 has a pole at $\xi_s = 1 = Ce^{-x}$. Therefore any solution $y(x;C)$ is singular at the array $x_n = 2n\pi i + \ln C + o(1)$.

Check: $y = \dfrac{v'}{v}$ where $v'' + v' - \dfrac{v}{x^2} = 0$ (Bessel). Hence: each y has a pole at each zero of v.

Substitute: $v = e^{-x/2}u \rightsquigarrow u'' - (\frac{1}{4} + \frac{1}{x^2})u = 0$. For large x: $u'' - \frac{1}{4}u \sim 0$ hence $u \sim \sin(\frac{x}{2} + B)$ with zeroes at $2n\pi i + 2B$. It checks!

Note: *The singularities of y have the same type as ξ_s of $\mathbf{F}_0$:* first order poles.

Example $\boxed{u' = u^3 + z}$ *Nonintegrable*: Kruskal, Clarkson (poly-Painlevé test).

Needs **normalization**: algorithmic, find the formal solutions (transseries), and use the transformation which bring them to the rank 1 form. Formal sol:

$$\hat{u} = \hat{u}_0 + Ce^{\frac{9A^2}{5}z^{5/3}}\, z^{2/3}\, \hat{u}_1 + \ldots, \text{ with } \hat{u}_0 = Az^{1/3}(1 + \sum_{k=1}^{\infty} \hat{u}_{0,k} z^{-5k/3}), \quad (A^3 = 1)$$

$x = -\frac{9A^2}{5}z^{5/3}$, $u(z) = Kz^{1/3}h(x)$, $K = A^{3/5}(-135)^{1/5}$, $h = y + \frac{1}{3} - \frac{1}{15x} \rightsquigarrow$

$$y' + (1 - \frac{1}{5x})y = g(x^{-1}, y) \text{ with } g = O(x^{-2}) + O(y^2). \text{ Thus } \xi = Cx^{1/5}e^{-x}$$

$$\xi F_0' = F_0(1 + 3F_0 + 3F_0^2),\ F_0 = \xi + O(\xi^2) \rightsquigarrow \xi \ = \ \xi_0\, F_0\, (F_0 + \omega_0)^{-\theta}\, (F_0 + \overline{\omega_0})^{-\overline{\theta}}$$

with singularities of type $(\xi - \xi_p)^{-1/2}$ at $\xi_p = (-1)^{p_{1,2}}\xi_0 e^{p_2 \pi \sqrt{3}}$, $p_{1,2} \in \mathbb{Z}$.

In fact this is quite general:

Theorem 3. Equation weakly nonlinear (i.e. $\mathbf{g}(x^{-1}, \mathbf{y})$ is not too big) have (generically) F_0 with square root singularities, and then $y(x; C)$ have array of singularities of the same type.

* * *

Solutions of $u' = u^3 + z$ have a uniform asymptotic series

$$u(z) \sim z^{1/3} \left(1 + \frac{1}{9z^{5/3}} + \sum_{k=0}^{\infty} \frac{F_k\,(C\xi(z))}{z^{5k/3}} \right) \text{(as } z \to \infty; \quad z \in \mathcal{R}_{z;K,\epsilon})$$

on a Riemann surface surrounding at $o(1)$ distance infinitely many $\sqrt{\ }$- type sing. (3 similar arrays).

The Painlevé equation $\mathrm{P_I}$

$$\boxed{\frac{d^2u}{dz^2} = 6u^2 + z}$$

Solutions of the $\mathrm{P_I}$ equation have arrays of poles, asymptotically represented by elliptic functions (Boutroux, Joshi and Kruskal - expansions for generic solutions and connection problem).

However: **the truncated solutions** (free of poles in some sectors) have the same classical asymptotic expansion in the pole free sector:
they differ by a constant C *beyond all orders*.

Consider the one parameter family of solutions with $u(z) \sim +\sqrt{\frac{-z}{6}}$ for $z \to -\infty$.

(The family $u(z) \sim -\sqrt{\frac{-z}{6}}$ is similar.)

Normalization: Find the general formal solution (transseries):

$$\hat{u} = \sqrt{\frac{-z}{6}} \sum_{k=0}^{\infty} \xi^k \ \hat{u}_k, \qquad \text{where } \xi = C x^{-1/2} \mathrm{e}^{-x}, \quad x = \frac{(-24z)^{5/4}}{30}$$

where $\hat{u}_0 = 1 - \frac{1}{8\sqrt{6}(-z)^{5/2}} - \frac{7^2}{2^8 \cdot 3} \frac{1}{z^5} - \dots - \frac{\tilde{u}_{0;k}}{(-z)^{5k/2}} - \dots$

Normalizing substitution: $x = \frac{(-24z)^{5/4}}{30}$; $u(z) = \sqrt{\frac{-z}{6}} Y(x) \rightsquigarrow$

$Y''(x) - \frac{1}{2} Y^2(x) + \frac{1}{2} = -\frac{1}{x} Y'(x) + \frac{4}{25} \frac{1}{x^2} Y(x)$ Boutroux form!

We need $Y(x) = O(x^{-2})$ so substitute $Y(x) = 1 - \frac{4}{25x^2} + h(x) \rightsquigarrow$

P_I *normalized* $\qquad h'' + \frac{1}{x} h' - h - \frac{1}{2} h^2 - \frac{392}{625} \frac{1}{x^4} = 0$

$$\mathrm{P_I}: \qquad h'' + \frac{1}{x}h' - h - \frac{1}{2}h^2 - \frac{392}{625}\frac{1}{x^4} = 0$$

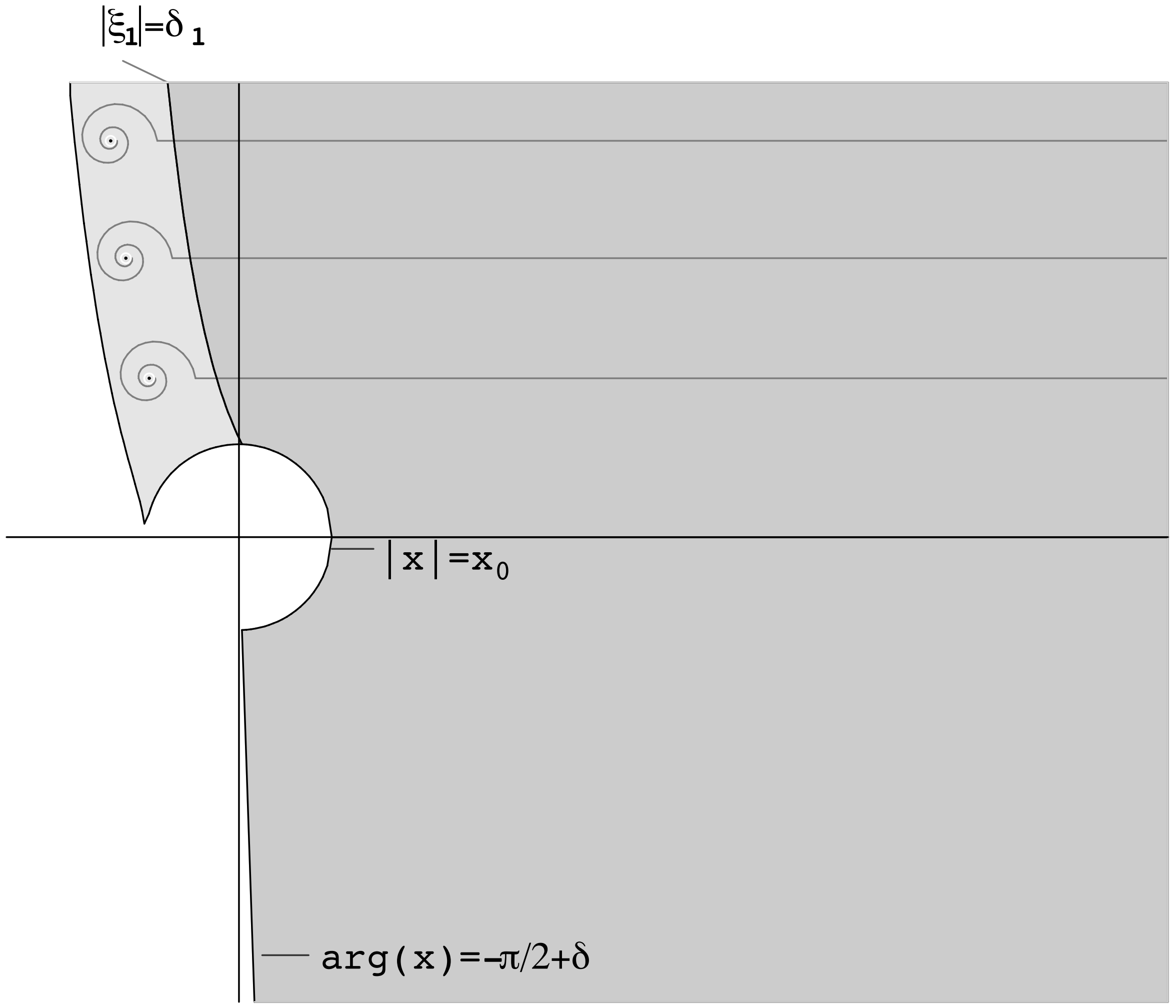
For $\mathbf{y} = (h, h')$ we have $\lambda_{1,2} = \pm 1,\ \alpha_{1,2} = -1/2$. Let $\xi = Ce^{-x}x^{-1/2}$.

$$(*) \qquad h(x) \sim \sum_{k=0}^{\infty} x^{-k} H_k(\xi(x)) \quad \rightsquigarrow \quad \xi^2 H_0'' + \xi H_0' = H_0 + \tfrac{1}{2}H_0^2$$

with the initial condition $H_0(\xi) = \xi + O(\xi^2)$ $\rightsquigarrow$ $H_0(\xi) = \dfrac{\xi}{(\xi/12 - 1)^2}$

$\xi_s = 12$ is a 2^{nd} ord. pole, and (it is shown that) so are x_n

Therefore (*) is asymptotic on the grey domain in the complex plane:



Small neighbohoods of the poles in the array are removed.

Returning to the original variables $u(z)$ we obtain

Proposition. Let u be solution of P$_I$: $\dfrac{d^2u}{dz^2} = 6u^2 + z$ such that $u(z) \sim \sqrt{-z/6}$ for $z \to \infty,\ \arg(z) = \pi$.

Let $\epsilon > 0$ and $\mathcal{Z} = \{z \,|\, \arg(z) > \frac{3}{5}\pi; |\xi(z)| < \epsilon^{-1};\ |\xi(z) - 12| > \epsilon\}$.
(Note: $\mathcal{Z}$ surrounds infinitely many poles of u, it starts at the antistokes line $\arg(z) = \frac{3}{5}\pi$ and extends slightly beyond the next antistokes line $\arg(z) = \frac{7}{5}\pi$.)

Then $u \sim \sqrt{\dfrac{-z}{6}}\left(1 - \dfrac{1}{8\sqrt{6}(-z)^{5/2}} + \displaystyle\sum_{k=0}^{\infty} \dfrac{30^k H_k(\xi)}{(-24z)^{5k/4}}\right)$ $(|z| \to \infty,\ z \in \mathcal{Z})$

The functions H_k are rational, and $H_0(\xi) = \xi(\xi/12 - 1)^{-2}$.
The expansion holds uniformly in the sector $3\pi/5 < \arg(z) < 7\pi/5$ and also for $\arg z \approx 7\pi/5$, (where H_0 becomes dominant), down to an $o(1)$ distance of the actual poles of u if z is large.

(*) $$h \sim \sum_{k=0}^{\infty} x^{-k} H_k(\xi(x)), \quad H_0(\xi) = \frac{\xi}{(\xi/12-1)^2}$$

Next terms: (**) $$H_1 = \left(216\,\xi + 210\,\xi^2 + 3\,\xi^3 - \frac{1}{60}\,\xi^4\right)(\xi-12)^{-3}$$

$$H_2 = \left(1458\xi + 5238\xi^2 - \frac{99}{8}\xi^3 - \frac{211}{30}\xi^4 + \frac{13}{288}\xi^5 + \frac{\xi^6}{21600}\right)(\xi-12)^{-4}$$

...

By induction: all H_m are rational functions and

$\frac{1}{h}$ has an asymptotic expansion with terms analytic near its singularities!

Find the zeroes of $1/h$: substitute $\xi_s = 12 + \frac{A}{x} + O(x^{-2})$ in (*), (**)

$\rightsquigarrow A = \frac{109}{10}$. Repeat to all orders $\rightsquigarrow \xi_s \sim 12 + \frac{109}{10x} + \frac{A_2}{x^2} + \ldots$

(an asymptotic series, but sharp estimated *could & should* be given).

Remarks. General features versus special features of P_I:

1. $\xi^2 H_0'' + \xi H_0' = H_0 + \frac{1}{2} H_0^2$ has gen. sol.:Weierstrass elliptic functions of $\ln \xi$ (as expected).
But our initial condition: $H_0(\xi) = \xi + O(\xi^2) \rightsquigarrow$ rational function (degenerate elliptic).

2. To determine H_1 we need 2 constants (H_m solve ODEs of order 2).
- The condition that H_1 analytic at 0 determines one constant.
- The other constant is determined in the *next* step, when solving for H_2.

This continues for each m: typical for generic equations.

3. The last potential obstruction to H_n rational is successfully overcome at $k = 6$. This is the special feature of integrability of P_I.

Sol. $h'' + \frac{1}{x}h' - h - \frac{1}{2}h^2 - \frac{392}{625x^4} = 0$ with $h \sim \sum_{k=0}^{\infty} x^{-k} H_k(\xi)$, ($\xi = Ce^{-x}x^{-1/2}$) are distinguished by **the parameter C** *beyond all orders*.

How is C is linked to $h(x)$?

Using $h(x)$ and **least term truncation** of its series (O. Costin, Kruskal, 1999):

$$C = \lim_{\substack{x\to\infty \\ \arg(x)=\phi}} e^x\, x^{1/2} \left(h(x) - \sum_{k\le|x|} \frac{\tilde{h}_{0,k}}{x^k} \right)$$

Going back to the original variables of the truncated solutions of P$_I$ we have:

Proposition. Let y by solution of P_I: $\quad \dfrac{d^2y}{dz^2} = 6y^2 + z$

such that $y(z) \sim \sqrt{-z/6}$ for $z \to \infty,\ \arg(z) = \pi$.

(a) For any $\phi \in (\pi, \pi + \frac{2}{5}\pi)$ the constant C in the transseries $\tilde{y}$ of y is

$$\lim_{\substack{|z|\to\infty \\ \arg(z)=\phi}} \xi(z)^{-1} \left(\sqrt{\frac{6}{-z}}\, y(z) - \sum_{k \le |x(z)|} \frac{\tilde{y}_{0;k}}{z^{5k/2}} \right) = C$$

(b) If $C \neq 0$, $y = y(x; C)$ has poles near the antistokes line $\arg(z) = \dfrac{7\pi}{5}$ at the points $z = z_n$ solutions of

$$\xi(z_n) = 12 + \frac{327}{(-24 z_n)^{5/4}} + O(z_n^{-5/2}) \quad (z_n \to \infty)$$

More precisely, the poles z_n are located at

$$z_n = -\frac{(60\pi i)^{4/5}}{24}\left(n^{\frac{4}{5}} + iL_n n^{-\frac{1}{5}} + \left(\frac{L_n^2}{8} - \frac{L_n}{4\pi} + \frac{109}{600\pi^2}\right) n^{-\frac{6}{5}}\right) + O\left(\frac{(\ln n)^3}{n^{\frac{11}{5}}}\right) \qquad (n \to \infty)$$

where $L_n = \frac{1}{5\pi}\ln\left(\frac{\pi i C^2}{72}n\right)$.

We obtained $h(x) \sim \sum_{k=0}^{\infty} x^{-k} H_k(\xi)$, $\quad (\xi = Ce^{-x}x^{-1/2}) \qquad$ (*) whith

$$H_0(\xi) = \frac{\xi}{(\xi/12 - 1)^2}, \quad H_1 = \left(216\,\xi + 210\,\xi^2 + 3\,\xi^3 - \frac{1}{60}\,\xi^4\right)(\xi - 12)^{-3}$$

$$H_2 = \left(1458\xi + 5238\xi^2 - \frac{99}{8}\xi^3 - \frac{211}{30}\xi^4 + \frac{13}{288}\xi^5 + \frac{\xi^6}{21600}\right)(\xi - 12)^{-4}, \ldots$$

in a domain around an infinite array of poles.

The next array of poles. By induction: $H_n \sim \mathrm{Const.}_n\,\xi^n$ $(\xi \to \infty)$, suggesting a reexpansion

$$h \sim \sum_{k=0}^{\infty} \frac{H_k^{[2]}(\xi_2)}{x^k}, \qquad \text{with} \quad \xi_2 = C^{[2]}\xi x^{-1} = C\,C^{[2]}x^{-3/2}\mathrm{e}^{-x}$$

Matching with (*) at $\xi_2 \sim x^{-2/3}$, we get $H_0^{[2]} = H_0$ with $C^{[2]} = -1/60$.

$$\xi_2 = C^{[2]}\xi x^{-1} = C^{[2]}\,C\,x^{-3/2}\,\mathrm{e}^{-x},\;\; H_0^{[2]} = H_0 \text{ with } C^{[2]} = -1/60.$$

If $x_n^{[1]}$ belongs to the first line of poles, i.e. $x = x_n^{[1]}$ solves $Ce^{-x}x^{-1/2} = \xi_s$, then second line of poles $x_n^{[2]}$ is given by the condition

$$x = x_n^{[2]} \quad \text{solves} \quad -\frac{1}{60}\,C\,x^{-3/2}\,\mathrm{e}^{-x} = \xi_s$$

so the second array it is situated at a logarithmic distance of the first one:

$$x_n^{[2]} - x_n^{[1]} = -\ln x_n^{[1]} + \pi i - \ln(60) + o(1)$$

Similarly, on finds the third array of poles $x_n^{[3]}$ using the scale $\xi_3 = C^{[3]}\xi_2 x^{-1}$, and ... in general $x_n^{[k]}$ ...

Alternatively: the expansion can be matched directly to an *adiabatic invariant*-like expansion valid throughout the sector where h has poles.

... detailed in Ovidiu's talk.

The Painlevé equation P2 $\quad u'' = 2u^3 + xu + \alpha$

Distinct solution manifolds (& asymptotics) $\rightsquigarrow$ different normalizations.

1. For $u \sim -\dfrac{\alpha}{x}$ set $\quad x = (3t/2)^{2/3}; \quad u(x) = x^{-1}(t\,h(t) - \alpha)$

giving the normal form

$$h'' + \frac{h'}{t} - \left(1 + \frac{24\alpha^2 + 1}{9t^2}\right) h - \frac{8}{9}h^3 + \frac{8\alpha}{3t}h^2 + \frac{8(\alpha^3 - \alpha)}{9t^3} = 0$$

with $\lambda_1 = 1, \quad \alpha_1 = -1/2;\ \xi = \frac{e^{-t}}{\sqrt{t}}$.

Then $\quad \xi^2 F_0'' + \xi F_0' = F_0 + \frac{8}{9}F_0^3 \quad$ with solution $\quad F_0(\xi) = \dfrac{\xi}{1 - \xi^2/9}$

$\xi_s = \pm 3 \rightsquigarrow$ two arrays of poles of order one.

2. For $u \sim -B - \frac{\alpha}{2} x^{-3/2}$ set $x = (At)^{2/3}, \quad y(x) = (At)^{1/3}\left(w(t) - B + \frac{\alpha}{2At}\right)$

where $A^2 = -9/8, B^2 = -1/2$. The normal form here is

$$w'' + \frac{w'}{t} + w\left(1 + \frac{3B\alpha}{tA} - \frac{1-6\alpha^2}{9t^2}\right) w$$

$$- \left(3B - \frac{3\alpha}{2tA}\right) w^2 + w^3 + \frac{1}{9t^2}\left(B(1+6\alpha^2) - t^{-1}\alpha(\alpha^2 - 4)\right)$$

$$\lambda_1 = 1, \alpha_1 = -\frac{1}{2} - \frac{3}{2}\frac{B\alpha}{A} \rightsquigarrow \xi^2 F_0'' + \xi F_0' - F_0 = 3BF_0^2 - F_0^3 \rightsquigarrow F_0 = \frac{2\xi(1+B\xi)}{\xi^2+2}$$

$\xi_s = \pm\sqrt{2}i \rightsquigarrow$ two arrays of poles of order one.

Uniform asymptotic series are found in regions surrounding infinite arrays of poles.

Conclusions.

1. Solutions which are analytic in sectors towards an irregular singularity develop, on the boundary of this sector, arrays of singularities, almost periodically spaced.

2. Uniform asymptotic series can be found in regions surrounding each of these arrays, close to $o(1)$ of these singularities.