Generalization of classical summation methods

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Taylor series

- Convergent power series are very convenient **local** representations of functions in solving many types of equations, ODEs, PDEs, difference equations which are regular enough.
- E.g. $y'' = xy$ (Airy). Inserting $y = \sum_{k=0}^{\infty} c_k x^k$ we get

$$c_{k+1} = \frac{c_{k-1}}{(k+1)(k+2)} \Rightarrow c_k = 9^{-k/3}\Gamma(k/3+2/3)^{-1}\Gamma(k/3-1)^{-1}C$$

- Even when everywhere convergent, Taylor representations are useful for x near the center of expansion but less so farther away. If, say, $y = \sum_{k=0}^{\infty} \frac{(-x)^k}{(k/e)^k}$ how does y behave as $x \to \infty$? Does it decay? Oscillate? Grow? Scope: One goal is to obtain global info, often in closed form, for three types of expansions:

$$\sum_{k\geq 0} c_k z^k \ (1), \ \sum_{k\geq 0} c_k \frac{z^k}{k!} \text{ (or combinations of Gamma functions)} \qquad (1)$$

$$(3) \sum_{k\geq 0} c_k k! z^k; \quad \log(c_k)/\log k \to a \neq 0 \quad \text{as } k \to \infty \qquad (2)$$

- $$(1)\ \sum_{k\geq 0} c_k z^k \ e.g.\ \sum_{n\geq 1} \frac{z^k}{k^b + \ln k},\ (2)\ \sum_{k\geq 0} c_k \frac{z^k}{k!}\ e.g. \sum_{k\geq 1} \frac{z^k}{(k/e)^k} \tag{3}$$
  $$(3) \sum_{k\geq 0} c_k k! z^k \ e.g.\ \sum_{k\geq 1} k^k/(\ln k) z^k \tag{4}$$
- Questions (1) does the analytic function extend past disk of convergence? All the way to infinity? If so, what is the behavior at ∞? (2): question (1). (3) Make sense of (3), and then answer similar questions.
- Solve in closed form new type of eqns. s.a.
  $$A\eta^2 f^{(4)} + 2A\eta f''' + \frac{1}{2}\eta f' - (1+a)f = 0 \tag{5}$$
  arising as the scaling pinching profile $h \sim (t_c - t) f(x(t_c - t)^{-1/2})$ of the thin film PDE, $h_t + (h_{xxx} h)_x = 0$ where t_c is the singularity time. W e solved it from explicitly summing Taylor series at zero, *and we know of no other way*.
- Find all sings. in $\mathbb{C}$ of classes of PDEs, e.g. (w. H. Park and Y. Takei)
  $$w_{qq} = z h_{zz}; h(z,0) = (z^2+1)^{-1};\ w_q(z,0) = 0 \tag{6}$$

- In essence the method generalizes the Poisson, Euler-Maclaurin, Abel-Plana, Borel methods.
- There are surprizing dualities between coefficients and generating functions. Will need a short discussion of transseries and generalized Borel summability before. One duality is, essentially, that a series is Borel summable in z **iff** c_k have generalized Borel summable transseries **in k**. Proving this in full generality would be a huge task, and we'll limit the analysis to transseries of level one, almost always enough in applications.
- Transseries were invented/discovered by Écalle to deal with local information at points where classical local expansions don't exist or are divergent. In the Airy case, and generically, the c_k have divergent transseries.
- Transseries are obtained from convergent power series by closing them under a wide class of operations. Convergent series are not closed:

$$z \mapsto 1/z \overset{\int}{\mapsto} \log z \mapsto 1/\log z \overset{\int}{\mapsto} z \sum_{k=0}^{\infty} \frac{k!}{(\ln z)^{k+1}} \text{ (integration by parts)}$$

- **How to close expansions.** Here is a natural but seemingly hopeless way. Take for T_0 the space of convergent Taylor series at 0 and apply to it the differential field operations, $\int$, restricted $\circ$, function inversions,(and others, depending), and for each combination of operations find a unique *asymptotic-like expansion* (allowing for direction dependence) say $z \to 0^+$. This defines T_1. Let T_2 be obtained from T_1 in the same way, etc. Then $T_\infty := \cup_0^\infty T_k$ will be closed under all *these* operations.
- Is this enterprise doable? Is this a reasonable one? What is T_∞?!
- Surprisingly perhaps, T_∞ is quite simple. This is a first major discovery of Ecalle in the 80s. T_∞ is a subspace of transseries: ordered combinations of exponentials, powers and logs. Formal multiseries in monomials built recursively from the **exponential alone**.

- Closure under "all operations of natural origin" is perhaps the best to chat about this property. Invariably, to make a mathematical statement one needs definitions, e.g. stating the "natural operations", at the expense of losing, invariably too, some important ones.
- By a **theorem** transseries are a differential field, also closed under $\{\circ, \circ^{-1}, \int\}$. This was sketched by Ecalle, and the proof was made rigorous and general about 15 years ago (M. Aschenbrenner, J. van der Hoefen, OC (I added to all these closure under solutions of formally contractive mappings, to try to beat the game of "other operations").
- Transseries are formal multiseries in a finite # of **small** variables called transmonomials, each of each obtained from *only from the* exp, allowing for zero radius of convergence which occurs quickly in the closure. Transseries are not functions, as convergent expansions are not closed under operations above: $z \mapsto 1/z \overset{\int}{\mapsto} \log z \mapsto 1/\log z \overset{\int}{\mapsto} z \sum_{k=0}^{\infty} \frac{k!}{(\ln z)^{k+1}}$ $(*)$ (integration by parts).

- Example of transseries as $z \to 0^+$

$$\sum_{k=0}^{\infty} \frac{k!z}{(\ln z)^{k+1}} \qquad Ae^{1/z^2} \sum_{k=1}^{\infty} k^k z^k + \frac{1}{\ln z} + \sum_{k=0}^{\infty} k! z^{k+1}$$

It is more convenient computationally to place the singularity $z = 0$ at infinity, $z = 1/x$,

$$Ae^{x^2} \sum_{k=1}^{\infty} k^k x^{-k} - \frac{1}{\ln x} + \sum_{k=0}^{\infty} \frac{k!}{x^{k+1}}$$

- More generally (though nothing this complicated arises naturally in problems)

$$\underbrace{\sum \frac{k!^2}{(\ln x)^{k+1}} + e^{-e^x \sum_{k=0}^{\infty} k! x^{-k-1}} + e^{-e^{x^2}} + \cdots}_{\text{terms decrease}}$$

- In a transseries, the requirement is that the terms decrease from left to right.
- For most 1-variable problems s.a. ODEs, transseries are obtained algorithmically.

- Obtaining transseries solutions of $y' = -y + 1/T$ formally:

$$y_0^{[k+1]} = 1/T - Dy_0^{[k]} \Rightarrow y_0 = \sum_{k=0}^{\infty} \frac{k!}{T^{k+1}};$$

$$y = y_0 + \delta \Rightarrow \delta' = -\delta \Rightarrow \delta = Ce^{-T} \Rightarrow y = \sum_{k=0}^{\infty} \frac{k!}{T^{k+1}} + Ce^{-T} \quad (7)$$

- **Closure, on the function side**. As presented, these are not actual solutions. Of course, divergence is a serious issue. (And it comes with complications too.) Of all operations listed ($\{+, -, \times, /, \circ, \circ^{-1}, D, \int, ...\}$) iterated differentiation is responsible of divergence.

Transseries and their sums

- How to find an **extension Σ of summation** on T_∞?
- Σ should coincide with summation in convergent cases, and preserve the properties of usual summation: commutation with all listed transseries operations; i.e., we are looking for an isomorphism between functions and transseries.
- Accept for now Σ exists, and assume, say, an ODE $(\mathrm{P_I})$ has a transseries solution S. Let Σ be the summation operator, assumed to be an isomorphism. Then,

$$\begin{aligned} S'' - 6S^2 = z \Leftrightarrow \Sigma(S'' - 6S^2) &= \Sigma(z) = z \\ &= [\Sigma(S)]'' - 6[\Sigma(S)]^2 = z \Leftrightarrow f'' - 6f^2 = z \end{aligned} \tag{8}$$

That is, a formal solution is transformed into an actual one.

- To deal with divergence, since it originates in iterated D, it is natural to apply the spectral thm. to map D to $F(p) \to pF(p)$. After the transform, instead of iterating on D we iterate on multiplication by symbol, generating geometric behavior. E.g. the previous iteration becomes $Y^{[k+1]} = 1 + pY^{[k]}$ $(p = ik) \Rightarrow \sum_0^\infty p^k = \frac{1}{1-p}$. The transform would be the Fourier transform $\mathcal{F}$, except it requires global info, from all $\mathbb{R}$. We prefer a "far away" half plane–$\mathcal{L}^{-1} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(\cdot)e^{pt}dt$ (Gelfand transf). does this. Note also that $\sum_{k=0}^\infty \mathcal{L}^{-1}\frac{k!}{x^{k+1}} = \sum_0^\infty p^k$.

A candidate for Σ

- $\mathcal{L}\Sigma\mathcal{L}^{-1}$ is a candidate for Σ. Return to the prototypical example before and write, with the spectral measure theorem in mind, $I = \mathcal{L}\mathcal{L}^{-1}$. Formal $\mathcal{L}^{-1} := \mathcal{B}$.

$$I\sum_{k=0}^{\infty}\frac{k!}{x^{k+1}} = \mathcal{L}\mathcal{L}^{-1}\sum_{k=0}^{\infty}\frac{k!}{x^{k+1}} = \mathcal{L}\sum_{k=0}^{\infty}\mathcal{B}\frac{k!}{x^{k+1}} = \mathcal{L}\sum_{k=0}^{\infty}p^k = \mathcal{L}\frac{1}{1-p}, \quad (9)$$

  except that Fubini does not apply and the last integral does not exist...

- If it were not for the integral, "Fubini" could be part of the *definition* of summation (which would be Borel summation, actually). In the problem above, we can resort to the Cauchy's PV: $f(x) := PV\int_0^\infty e^{-px}(1-p)^{-1}dp$, also $=$ half sum of $\int_{\mathbb{R}^+ e^{\pm i\varepsilon}}$.

- Nonlinear ODEs have infinitely many singularities, and of a worse kind. the average of two solutions is not a solution. Can we extend PV in any generality?

Generalized Borel Summation (GB)

- The second major discovery of Ecalle is that there is a *universal* sequence of weights s.t. analytically continuing the integrand around singularities and the averaged $\mathcal{L}^{-1} := \mathcal{B}$ (*Borel transform*) commutes with the transseries operations, should *always work* on T_∞.
- A deep form of regularity called analyzability, implying in particular unique continuation, withstands all classical operations we can think of.
- In this utmost generality, the statement is still however conjectural. That's not because we don't have a good plan on how to prove this, but the proof would be huge, and the game of "find another natural operation" may continue forever, or end without a hope to prove it ended.

Why $\mathcal{B} \neq \mathcal{L}^{-1}$ and other quirks

- Note that $\sum_{k=0}^{\infty}(2k!)/x^k$ is not (directly) summable in x but it is in x', $x' = \sqrt{x}$. Neither is $\sum_{k=0}^{\infty}(k/2)!/x^k$ (!!), but it is in $x = \sqrt{x'}$. There is however no change of variables to deal with $S = \sum_{k=0}^{\infty}(2k! + (k/2)!)/x^k$. And entangled factorial growth can come in many shapes.
- Here comes a third major invention of Ecalle, *acceleration*. After a sequence of acceleration operators S becomes GB-summable.
- With these tools, summable transseries *seem* closed under all operations. Proved in restricted, nonetheless useful particular cases: nonlinear (or linear) meromorphic systems of ODEs (OC, Duke 98), quite general classes of nonlinear PDEs (OC, S. Tanveer 2007) and *difference equations* (Braaksma 2000) very important for us here.

Borel summation as a regularizing procedure.

I will still denote by $\mathcal{B}(=\mathcal{B}_x)$ (x is large) the combination of $\mathcal{B}$ with acceleration operators and averaging, if needed, and $\mathcal{L}$ will be the Laplace transform.
The first step, $\mathcal{B}$ transforms divergent series solutions into convergent ones, and is thus smoothening.
It must then smoothen equations too. $\mathcal{B}[y' - y = 1/x] = (p+1)Y = 1$.
Another simple example: Modified Bessel f. equation:

$$t^2 y'' + ty' - (\nu^2 + t^2)y = 0 \tag{10}$$

- We take $y(t) = t^{-\nu} e^{-t} h(t)$ (to eliminate expo behavior and be able to take $\mathcal{L}^{-1}$) we get

$$h'' - \left(2 - \frac{2\nu+1}{t}\right) h' - \frac{2\nu+1}{t} h = 0 \tag{11}$$

Applying $\mathcal{B}$ we get a first order ODE

$$p(p+2)H' + (1 - 2\nu + p(1-2\nu))H = 0 \Rightarrow \; H = Cp^{\nu-\frac{1}{2}}(2+p)^{\nu-\frac{1}{2}} \tag{12}$$

- $$p(p+2)H' + (1-2\nu + p(1-2\nu))H = 0 \Rightarrow \ H = Cp^{\nu-\frac{1}{2}}(2+p)^{\nu-\frac{1}{2}} \quad (13)$$

  only regular singularities, and only 2 since Bessel is second order. Thus elementary solution.
- Thus $K_\nu(t) = Ce^{-t}t^\nu \int_0^\infty p^{\nu-\frac{1}{2}}(2+p)^{\nu-\frac{1}{2}}e^{-tp}dp$ ($C = \frac{\sqrt{\pi}}{2^\nu\Gamma(\nu+\frac{1}{2})}$ by comparing with the definition of K).
- Same can be done on **all** classical special functions, to obtain the integral representations, and more generally to n-systems of ODEs or PDEs to regularize them.
- Same can be done from the Taylor series by the summation I'll talk about, but in these simple cases it's not worth it.

- Poisson summation serves as an inspiration. Will do roughly the same, but more generally, and discover interesting identities. Wikipedia:

$$\sum_{k=-\infty}^{\infty} \hat{f}(k) = \underbrace{\sum_{k=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x)\, e^{-i2\pi kx} dx \right)}_{\text{"Poisson transform" of } \hat{f}} = \int_{-\infty}^{\infty} f(x) \underbrace{\left(\sum_{k=-\infty}^{\infty} e^{-i2\pi kx} \right)}_{\sum_{n=-\infty}^{\infty} \delta(x-n)} dx$$

$$= \sum_{n=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x)\, \delta(x-n)\, dx \right) = \sum_{n=-\infty}^{\infty} f(n).$$

(rigorous in distributions). Here, key to re-presentation of the sum is the "Poisson transform" $\hat{f}(n) \mapsto T_f(e^{n(2\pi ix)})$. A similar transform is possible in much wider generality using Borel summation.

- Poisson summation is a special case of trace formulas, here for $Tu = \int_{\mathbb{R}} g(s-t)u(t)dt$ leading to deep results s.a. the Riemann-von Mangoldt formula and the Selberg trace formula. We are not going in that direction.

- Back to sums. Braaksma's class of normalized difference equations:

$$\mathbf{x}(n+1) = \hat{\Lambda}\left(I + \frac{1}{n}\hat{A}\right)\mathbf{x}(n) + \mathbf{g}(n, \mathbf{x}(n)) \tag{14}$$

  where $\hat{\Lambda}$ and $\hat{A}$ are constant coefficient matrices, $\mathbf{g}$ is analytic in n at infinity and $\mathbf{g} = O(n^{-2}, \mathbf{x}^2)$ as $n \to \infty$, if $\sum_{j=1}^{m} k_j \leq 1$, Λ non-resonant.

- Theory of DEs is remarkably similar to that of ODEs. Braaksma: recurrences (25) possess generalized Borel summable transseries solutions:

$$\tilde{\mathbf{x}}(n) := \sum_{\mathbf{k}\in\mathbb{N}^m} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\mu} n} \tilde{\mathbf{x}}_{\mathbf{k}}(n) \tag{15}$$

- where $\tilde{\mathbf{x}}_{\mathbf{k}}(n)$ are formal power series in n^{-1} and the small solutions (or all in linear DEs) are Borel sums of transseries of this type.

$$\mathbf{x}(n) := \sum_{\mathbf{k}\in\mathbb{N}^m} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\mu} n} \int_0^\infty e^{-np} \mathbf{X}_{\mathbf{k}}(p)\,dp \tag{16}$$

- $\mathbf{k}$ takes only finitely many values in linear case. We see the "Poisson transform" appearing, in this generality.

- The Γ function: $\Gamma(x+1) = x\Gamma(x)$ (Γ grows faster than exponentially). Look then at $f = \ln\Gamma$: $f(x+1)-f(x) = \log x$. f still grows too fast. Approximating sums by integrals (Euler-Maclaurin summation) we get $f(x) = (x\log x - x - \frac{1}{2}\log x) + g(x)$, g small. g satisfies $g(x+1)-g(x) = 1-\left(\frac{1}{2}+x\right)\ln\left(1+\frac{1}{x}\right)$.
- By taking $G := \mathcal{L}^{-1}g$, we get $g(x) = \int_0^\infty e^{-xp}G(p)dp$, where

$$(e^{-p}-1)G = \frac{1-p/2-(1+p/2)e^{-p}}{p^2} \Rightarrow \sum_{k=n}^{\infty}\int_0^\infty e^{-kp}G(p) = \int_0^\infty \frac{G(p)e^{-np}}{1-e^{-p}}$$

$$f(x) = \frac{1}{2}\ln(2\pi) + x(\log x - 1) - \frac{1}{2}\log x + \int_0^\infty \frac{1-\frac{p}{2}-\left(\frac{p}{2}+1\right)e^{-p}}{p^2(e^{-p}-1)}e^{-xp}dp$$

The constant $C = \frac{1}{2}\ln(2\pi)$ is gotten e.g. by calculating the integral for $x = 1$. Global info: analytic continuation to $x = -z$ (rotating the contour and collecting residues) we get

$$\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin(\pi x)} \tag{17}$$

- $$(1)\ \sum_{k\geq 0} c_k z^k \ \ e.g.\ \sum_{n\geq 1} \frac{z^k}{k^b + \ln k},\ (2)\ \sum_{k\geq 0} c_k \frac{z^k}{k!}\ \ e.g. \sum_{k\geq 1} \frac{z^k}{(k/e)^k} \tag{18}$$

  $$(3) \sum_{k\geq 0} c_k k! z^k \ \ e.g.\ \sum_{k\geq 1} k^k/(\ln k) z^k \tag{19}$$

- Questions (1) does the analytic function extend past disk of convergence? All the way to infinity? If so, what is the behavior at ∞? (2): question (1). (3) Make sense of (3), and then answer similar questions.
- Solve in closed form new type of eqns. s.a.

  $$A\eta^2 f^{(4)} + 2A\eta f''' + \frac{1}{2}\eta f' - (1+a)f = 0 \tag{20}$$

  arising as the scaling pinching profile $h \sim (t_c - t) f(x(t_c - t)^{-1/2})$ of the thin film PDE, $h_t + (h_{xxx} h)_x = 0$ where t_c is the singularity time. W e solved it from explicitly summing Taylor series at zero, *and we know of no other way*.
- Find all sings. in $\mathbb{C}$ of classes of PDEs, e.g. (w. H. Park and Y. Takei)

  $$w_{qq} = z h_{zz}; h(z,0) = (z^2+1)^{-1};\ w_q(z,0) = 0 \tag{21}$$

Part II

Purpose: Find integral representations of $\sum_k f(k)$ esp. for obtaining global info for three types of expansions:

$$\sum_{k\geq 0} c_k z^k \ (1), \ \sum_{k\geq 0} c_k \frac{z^k}{k!} \text{ (or combinations of Gamma functions)} \tag{22}$$

$$(3) \sum_{k\geq 0} c_k k! z^k; \quad \text{behavior of } c_k \text{ geometric/algebraic} \tag{23}$$

- Poisson summation:

$$\sum_{k=-\infty}^{\infty} \hat{f}(k) = \underbrace{\sum_{k=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x)\ e^{-i2\pi kx} dx \right)}_{\text{"Poisson transform" of } \hat{f}} = \int_{-\infty}^{\infty} f(x) \underbrace{\left(\sum_{k=-\infty}^{\infty} e^{-i2\pi kx} \right)}_{\sum_{n=-\infty}^{\infty} \delta(x-n)} dx$$

$$= \sum_{n=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x)\ \delta(x-n)\ dx \right) = \sum_{n=-\infty}^{\infty} f(n).$$

- Key: the "Poisson transform" $P := f \mapsto (f(n))_{n\in\mathbb{N}} = (T_f(e^{-ng}))_{n\in\mathbb{N}}$ where T_f is linear and $\operatorname{Re} g \geqslant 0$. Then $\sum_1^\infty f(n) = T_f(\frac{e^{-g}}{1-e^{-g}})$ under appropriate conditions; for our purpose T_f will be a bounded integral operator and $\operatorname{Re} g > 0$.
- Such a T_f exists in a wide generality if T_f is a discrete+continuous Laplace transform, because very general classes of functions admit Borel summable transseries representations.

1. Systems of ODEs. Consider a generic system of ODEs with an irregular singularity at infinity, in *normalized form*:

$$\mathbf{y}' = \hat{\Lambda}\mathbf{y} + \mathbf{g}(x^{-1}, \mathbf{y}); \quad \mathbf{g} = O(1/x^2, \mathbf{y}/x, \mathbf{y}^2) \tag{24}$$

analytic near zero; λ_i l.i. over $\mathbb{Q}$, $\arg(\lambda_i)$ distinct.

Theorem (OC, DMJ 1998)

(i) The general transseries solution (say for $\operatorname{Re} x > 0$) is of the form

$$\sum_{\mathbf{k}\geq 0} \mathbf{C}^{\mathbf{k}} e^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} \tilde{\mathbf{y}}_{\mathbf{k}}(1/x); \quad C_i = 0 \text{ if } \operatorname{Re}\lambda_i \leq 0; \ \tilde{\mathbf{y}}_{\mathbf{k}}(1/x) \text{ formal pow. ser.}$$

(ii) All $\tilde{\mathbf{y}}_{\mathbf{k}}$ are generalized Borel summable, $\mathcal{LB}\tilde{\mathbf{y}}_{\mathbf{k}} = \oint_0^\infty e^{-xp}\mathbf{Y}_{\mathbf{k}}(p)dp$ and

$$\sum_{\mathbf{k}\geq 0} \mathbf{C}^{\mathbf{k}} e^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} \oint_0^\infty e^{-xp}\mathbf{Y}_{\mathbf{k}}(p)dp;$$ general small sol'n of (24), or general if (24) is linear

(iii) [The singularities of $\mathbf{Y}_{\mathbf{k}}$ are all regular of known type and location.]

$\mathbf{C} = 0 \Rightarrow \mathbf{y}(x) = \int_0^\infty e^{-xp}\mathbf{Y}_0(p)dp$

Braaksma's class of normalized difference equations:

$$\mathbf{x}(n+1) = \hat{\Lambda}\mathbf{x}(n) + \mathbf{g}(n, \mathbf{x}(n)) \tag{25}$$

where $\hat{\Lambda} = \mathrm{diag} e^{-\mu_k}$, genericity and conditions on $\mathbf{g}$ essentially as for ODEs.

- Theory of DEs is remarkably similar to that of ODEs. Braaksma: general small solution (nonl. case) or general (linear) is a generalized Borel summed transseries:

$$\mathbf{x}(n) := \sum_{\mathbf{k}\in\mathbb{N}^m} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\mu} n} \int_0^\infty e^{-np} \mathbf{X}_{\mathbf{k}}(p) dp \tag{26}$$

- $\mathbf{k}$ takes only finitely many values in linear case. We see the "Poisson transform" appearing, in this generality.

- Simple example (polylog). Take $\sum_{n=1}^{\infty} z^n n^{-b}$ ($b \notin \mathbb{N}$, $\mathrm{Re}\, b \geq 0$ *for simplicity*). We have $\mathcal{L}^{-1} n^{-b} = p^{b-1}/\Gamma(b)$, thus $n^{-b} = \Gamma(b) \int_0^{\infty} p^{b-1} e^{-np} dp$ and

$$f(z) = \sum_{n=1}^{\infty} z^n n^{-b} = \Gamma(b) \int_0^{\infty} p^{b-1} \sum_{n=1}^{\infty} (ze^{-p})^n dp \tag{27}$$

$$= \frac{z\Gamma(b)}{1 - e^{2\pi i b}} \oint_0^{\infty} \frac{p^{b-1} dp}{e^p - z} = \frac{z\Gamma(b)}{1 - e^{2\pi i b}} \oint_0^{\infty} \frac{\ln(1+s)^{b-1} ds}{(s+1)(s - \underbrace{(z-1)}_{\zeta})}$$

- Note that the integral is manifestly analytic on the first Riemann sheet if $0 \neq (1 - z) \notin \mathbb{R}^-$

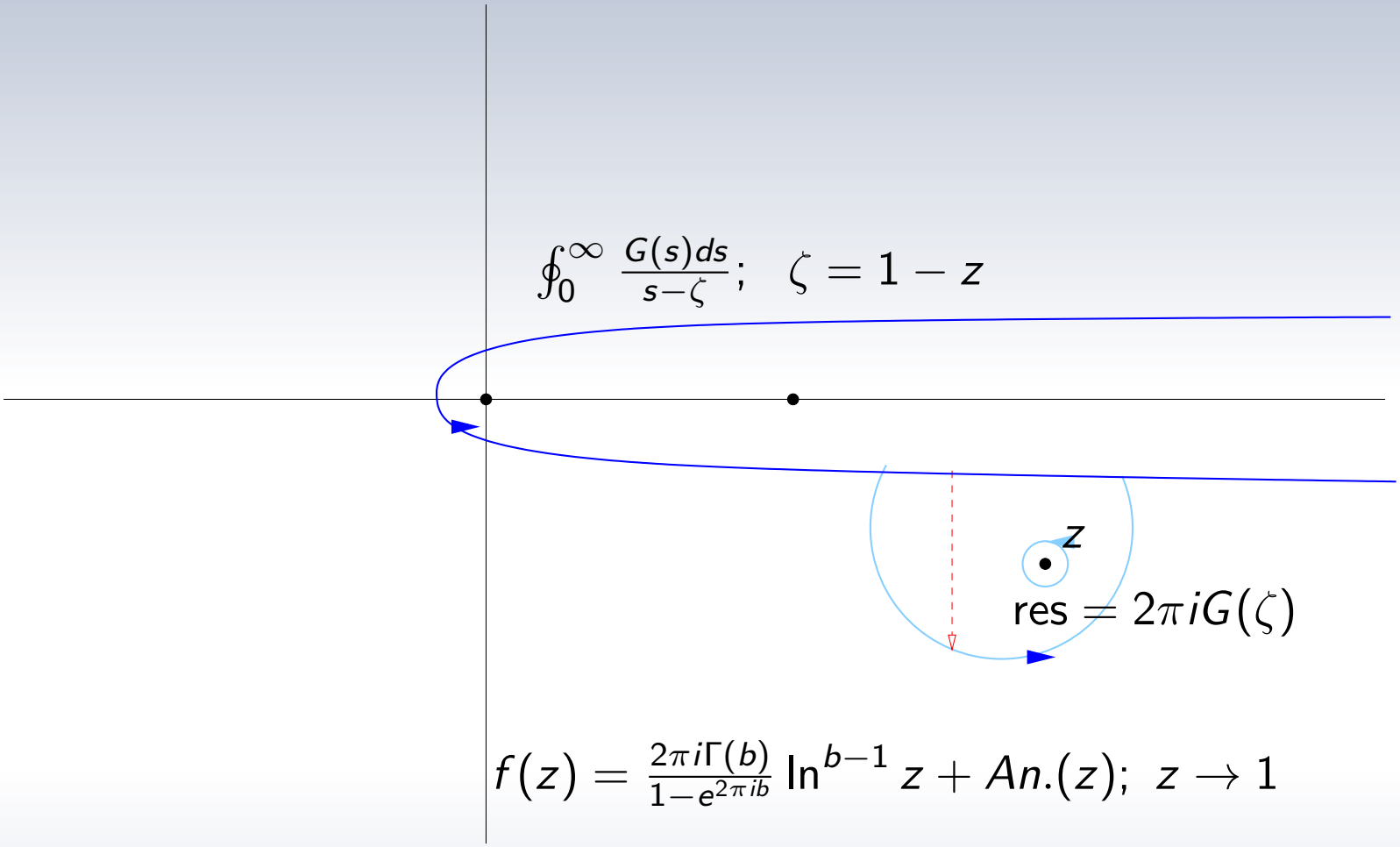

Figure: Deformation of contour $\oint_0^{\infty e^{i\phi}}$ to bring z in the interior of the (new) curve. The integral on the deformed contour is manifestly analytic when z is in its interior **even at points where the integrand is singular in the interior of the curve.** In the example $f(z)$, f is analytic on univ. covering of $\mathbb{C} \setminus \{0, 1\}$.

Definition

Let $\{0 \neq a_j \in \mathbb{C} : 1 \leq j \leq N\}$ have distinct arguments. Let $\mathcal{M}$ consist of the functions algebraically bounded at ∞ and analytic in $\mathbb{C} \setminus \bigcup_{j=1}^{N} \{a_j t : t \geq 1\}$. (This is one of the simplest settings often occurring in applications; the results are more general).

Theorem

Let $f = \sum_{k \geq 1} c_k z^k$. *Then* $f \in \mathcal{M}$ *iff* $\exists k_0$, *for large enough* $\forall k > k_0$, $c_k = \sum_{j=1}^{N} a_j^{-k} \oint_0^\infty e^{-kp} F_j(p) dp$ *with* $F_j \in \mathcal{M}$ *and* a_j *as above. More precisely,*

$$f(z) = \sum_{j=0}^{k_0} c_j z^j + z \oint_0^\infty \sum_{j=1}^{N} \frac{F_j(\ln(1+s))ds}{(1+s)((1+s)a_j - z)} \tag{28}$$

Conversely, $c_k = \dfrac{1}{2\pi i} \displaystyle\sum_{j=1}^{N} (a_j)^{-k} \oint_0^\infty e^{-ks} f(a_j e^s) ds, \; k > k_0, \sup_{z \in \mathbb{C}} |z|^{-k_0} f(z) < \infty$

Example 1: $\sum_{n\geq 1} \frac{z^k}{k^b+\ln k}$

- $(k^b + \ln k)^{-1} = \int_0^\infty e^{-kp} G(p)dp$, where

$$G(p) = \frac{1}{2\pi i} \oint_0^\infty \frac{e^{-up}}{(-u)^b + \ln(-u)} du = \frac{1}{2\pi i} \oint_0^\infty \frac{e^{-s} ds}{(-s^b)p^{-b} + \ln(-s) - \ln p}$$

Note that G is singular (ramified) only at zero. Define

$$\tilde{G}(s) = G(\ln(1+s))/(1+s)$$

$$\sum_{k=1}^\infty \frac{z^k}{k^b + \ln k} = z \oint_0^\infty \frac{\tilde{G}(s)}{s-(z-1)} ds$$

essentially a Hilbert transform of $1_{\mathbb{R}^+}\tilde{G}$. This is analytic except at $z = 1$, goes to zero at ∞ and has a Frobenius expansion at $z = 1$. Singularities in z and p (Borel variable associated to k) are dual to each-other, as we have seen.

Entire functions

Theorem

Assume that the entire function f is given by

$$f(z) = \sum_{k=1}^{\infty} \frac{c_k z^k}{k!} \tag{29}$$

with c_k as before. Then,

$$f(z) = \oint_0^{\infty} \sum_{j=1}^{N} \left[\left(e^{\frac{z}{s+a_j}} - 1 \right) \frac{F_j(\ln(1+s))}{(1+s)} \right] ds \tag{30}$$

- As in the simple example, the behavior at infinity follows from the integral representation by classical means.

Example 2

$$\sum_{k=1}^{\infty} \frac{x^k}{k^{k+1}} = \int_0^{\infty} (1+u)^{-1} G(\ln(1+u)) \left[\exp\left(\frac{x}{1+u}\right) - 1\right] du =: f_2(x)$$

where now $G(p) = s_2'(1+p) - s_1'(1+p)$ and $s_{1,2}$ are two branches of the functional inverse of $s - \ln s$. It is easy to see that the behavior of the integral is $e^{x+o(x)}$ as $x \to \infty$ (the complete asymptotics is also easy to obtain). This was clear from the sum too.
But, as $x \to -\infty$, $f_2 \sim C + x^{-1/2} e^{-x} \tilde{y}(1/x)$ where $\tilde{y}$ is a factorially divergent series (whose terms, and C, can be calculated).

Borel summation: $\tilde{f}(z) := \sum_{k=1}^{\infty} c_k k! z^{k+1}$

Theorem

The series is (generalized) Borel summable with Borel transform in $\mathcal{M}$ iff this property is shared by $k \mapsto c_k$. In particular,

$$\mathcal{LB}\tilde{f} = \int_0^{\infty} dp e^{-p/z} p \sum_{j=1}^{N} \oint_0^{\infty} \frac{F_j(\ln(1+s))}{(1+s)(a_j s + a_j - p)} ds$$

$$= -\sum_{j=1}^{N} \oint_0^{\infty} \frac{F_j(\ln(1+s))}{1+s} \left(z - a_j(s+1) e^{-\frac{a_j(s+1)}{z}} \mathrm{Ei}\left(\frac{a_j(s+1)}{z} \right) \right) ds \tag{31}$$

Reconstruction of solutions from Taylor coeffs.

$$A\eta^2 f^{(4)} + 2A\eta f''' + \frac{1}{2}\eta f' - (1+a)f = 0 \tag{32}$$

has the general solution

$$\eta^{2a+2} \oint_0^\infty p^{7/4-a} B(\omega_j p^{1/4}) e^{-p/\eta^2} dp$$

where where ω_j are the quartic roots of -1, B is the Bessel function J or the modified Bessel function K –depending on the parameters a, A.

PDEs

- $w_{qq} = zw_{zz}$ has singular manifolds

$$Si = \{(q,z) : q = 2c_1\sqrt{z} + c_2\sqrt{2}(1+c_3 i); c_1, c_2, c_3 \in \{-1,1\}\} \tag{33}$$

where

$$w(z,q) \sim \frac{f_{c_1,c_2,c_3}(\sqrt{z},q)}{(q - 2c_1\sqrt{z} - c_2\sqrt{2}(1+c_3 i))}$$

for $c_1, c_2, c_3 \in \{-1,1\}$, where f_{c_1,c_2,c_3} are analytic at each point on the singular manifold.

- The point $z = 0$ is special: near $z = 0$ the singularity is to leading order of the form

$$\frac{(q - c_2\sqrt{2}(1+c_3 i))\mathrm{E}\left(\frac{q+2c_1\sqrt{z}-c_2\sqrt{2}(1+c_3 i)}{q-2c_1\sqrt{z}-c_2\sqrt{2}(1+c_3 i)}\right) + 2c_1\sqrt{z}\mathrm{K}\left(\frac{q+2c_1\sqrt{z}-c_2\sqrt{2}(1+c_3 i)}{q-2c_1\sqrt{z}-c_2\sqrt{2}(1+c_3 i)}\right)}{3(q + 2c_1\sqrt{z} - c_2\sqrt{2}(1+c_3 i))\sqrt{q - 2c_1\sqrt{z} - c_2\sqrt{2}(1+c_3 i)}} \tag{34}$$

Here E is a complete elliptic integral of second kind, K is a complete elliptic integral of first kind.

Euler-Maclaurin

- Recall that $\sum_n \mathbf{y}(n)$ can be expressed as an integral for any transseriable $\mathbf{y}$, in particular for $\mathbf{y}$ satisfying a generic system of (O)(P)DEs. Modified Bessel:

$$x^2y'' + xy' - (\nu^2 + x^2)y = 0 \quad (\text{solns: } I_\nu, K_\nu) \tag{35}$$

- By Borel summation we get

$$K_\nu(x) = e^{-x}\int_0^\infty \frac{1}{2}\left[\frac{[p+1+\sqrt{p(p+2)}]^\nu + [p+1+\sqrt{p(p+2)}]^{-\nu}}{\sqrt{p(p+2)}}\right] e^{-px}dp =: e^{-x}\int_0^\infty e^{-px}G(p)dp$$

- implying in particular,

$$\sum_{n=1}^\infty K_\nu(n) = \int_0^\infty \frac{G(p)dp}{e^{p+1}-1}; \quad \sum_{n=M+1}^\infty K_\nu(n) = \int_0^\infty \frac{G(p)dp}{e^{p+1}-1}e^{-M(p+1)}dp \tag{36}$$

(similarly for $\sum_{n=M+1}^\infty K_\nu(an+b)$ etc), from which one can easily obtain asymptotics in M or ν or both.

Thank you.