Periods of Negative-regular Continued Fractions. Rational numbers.

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Outline

In 1882 Smith included in his 'Algebra' a formula for the convergents of the following continued fraction

$$\underbrace{\frac{1}{1-}\,\frac{1}{4-}\,\frac{1}{1-}\,\frac{1}{4-\cdots}\cdots}_{n}=\frac{2n}{n+1}\longrightarrow 2 \text{ as } n\to\infty, \tag{1}$$

These continued fractions go back to Chebyshev

$$\frac{2}{\pi}\int_{-1}^{1}\frac{\sqrt{1-x^2}}{z-x}dx=2(z-\sqrt{z^2-1})=$$
$$=\frac{2}{2z-}\,\frac{1}{2z-}\,\frac{1}{2z-}\cdots=\frac{1}{z-}\,\frac{1/2}{2z}-\frac{1}{2z-}\cdots\frac{1}{z-}\,\frac{1}{4z-}\,\frac{1}{z-}\,\frac{1}{4z-}\cdots. \tag{2}$$

We study pure periodic negative-regular continued fractions

$$x = \underbrace{\frac{-1}{b_1} + \frac{-1}{b_2} + \cdots + \frac{-1}{b_n}}_{n} + \underbrace{\frac{-1}{b_1} + \frac{-1}{b_2} + \cdots + \frac{-1}{b_n}}_{n} + \cdots, \tag{3}$$

representing rational numbers x where b_1, $b_2, \ldots, b_n$ are positive integers.

Convergence

It is easy to see that for every real x

$$x = \frac{-1}{3} + \frac{-1}{1} + \frac{-1}{2} + \frac{-1}{2} + \frac{-1}{1+x}. \tag{4}$$

The corresponding periodic continued fraction

$$\underbrace{\frac{-1}{3} + \frac{-1}{1} + \frac{-1}{2} + \frac{-1}{2} + \frac{-1}{1}}_{5} + \underbrace{\frac{-1}{3} + \frac{-1}{1} + \frac{-1}{2} + \frac{-1}{2} + \frac{-1}{1}}_{5} + \cdots \tag{5}$$

obviously diverges. Indeed, putting $x = 0$ in (4) we obtain that $P_{5n}/Q_{5n} = 0$ for every n. Putting $x = -1/3$ in (4) we obtain that $P_{5n+1}/Q_{5n+1} = -1/3$ for every n. The same is true for any period representing the identity.

Tietze Theorem

Theorem (Tietze)

Let $\{a_k\}_{k\geq 1}$ be a sequence of nonzero integers and $\{b_k\}_{k\geq 1}$ be a sequence of positive integers such that $b_k \geq |a_k|$ for every k and $b_k \geq |a_k| + 1$ if $a_{k+1} < 0$. Then the continued fraction

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} + \cdots$$

converges to an irrational number except for the case if $a_k < 0$ and $b_k = |a_k| + 1$ starting from some place.

Corollary

Corollary

If $\{b_1, b_2, \ldots, b_n\}$ is a period of a negative-regular continued fraction representing a rational number x then $b_k = 1$ for some k.

Negative Integers

Euler related partial sums of series to convergents of continued fractions:

$$\sum_{k=0}^{n-1} c_k = \frac{c_0}{1} - \frac{c_1/c_0}{1 + c_1/c_0} - \frac{c_2/c_1}{1 + c_2/c_1} - \dots - \frac{c_{n-1}/c_{n-2}}{1 + c_{n-1}/c_{n-2}}.$$

Let now $c_0 = c_1 = \cdots = c_n = -1$. Then

$$-n = \frac{-1}{1} + \underbrace{\frac{-1}{2} + \dots + \frac{-1}{2}}_{n-2} + \frac{-1}{2 + n - n}. \tag{6}$$

Iterating we obtain for negative integers $-n$, $n \geq 2$:

$$-n = \frac{-1}{1} + \underbrace{\frac{-1}{2} + \cdots + \frac{-1}{2}}_{n-2} + \frac{-1}{n+2} + \frac{-1}{1} + \underbrace{\frac{-1}{2} + \cdots + \frac{-1}{2}}_{n-2} + \frac{-1}{n+2} + \cdots.$$

Convergence of Periodic Fractions

Theorem

Every rational number is the value of a convergent pure periodic negative-regular continued fraction.

A construction of minimal periods

$$x = \frac{-1}{a_1} + \frac{-1}{a_2} + \cdots + \frac{-1}{a_k}, \quad \frac{1}{x} = \frac{-1}{c_1} + \frac{-1}{c_2} + \cdots + \frac{-1}{c_m}.$$

These expansions are obtained by a generalized Euclidean algorithm. Then

$$x \oplus s(x^{-1}) \stackrel{def}{=} \{a_1, a_2, \ldots, a_{k-1}, a_k + c_m, c_{m-1}, \ldots, c_1\} \mapsto x \quad (8)$$

is the minimal period for x. For example

$$-n = \frac{-1}{1} + \underbrace{\frac{-1}{2} + \cdots + \frac{-1}{2}}_{n-1}, \quad \frac{1}{-n} = \frac{-1}{n} \implies \{1, \underbrace{2, \ldots, 2}_{n-2}, 2+n\} \mapsto -n\,.$$

Consider the following rational function $R_n(x_1, x_2, \ldots, x_n)$:

$$x_1 x_2 \cdots x_{n-1} x_n \left(1 - \frac{1}{x_1 x_2}\right) \cdots \left(1 - \frac{1}{x_{n-1} x_n}\right) \left(1 - \frac{1}{x_n x_1}\right)$$

and denote by $D_n(x_1, \ldots, x_n)$ the polynomial part $[R_n(x_1, \ldots x_n)]$ of R_n. Then $x_1 = b_1$, $x_2 = b_2$,…, $x_n = b_n$ is a solution to the Diophantine equation

$$D_n(x_1, \ldots, x_n) = \pm 2\,. \tag{9}$$

The solutions to (9) not assigned to the periods of rational numbers correspond to the identities of the type (4). For example for $n = 2, 3, 4$ we have:

$$\begin{aligned} D_2 &= b_1 b_2 - 2\,, \\ D_3 &= b_1 b_2 b_3 - b_1 - b_2 - b_3 \\ D_4 &= b_1 b_2 b_3 b_4 - b_1 b_2 - b_2 b_3 - b_3 b_4 - b_4 b_1 + 2\,. \end{aligned}$$

$n=$ **1**	**2**	**3**	**4**	**5**
$\{2\}$	$\{2,2\}$	$\{2,2,2\}$	$\{2,2,2,2\}$	$\{2,2,2,2,2\}$
	$\{1,4\}$	$\{\mathbf{1},\mathbf{2},\mathbf{5}\}$	$\{\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{6}\}$	$\{1,1,4,1,1\}$
		$\{1,3,3\}$	$\{1,2,5,3\}$	$\{1,2,1,3,1\}$
		$\{1,1,b\}$	$\{1,3,1,6\}$	$\{\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{7}\}$
			$\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4}\}$	$\{\mathbf{1},\mathbf{2},\mathbf{6},\mathbf{2},\mathbf{3}\}$
			$\{1,4,1,4\}$	$\{\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{3},\mathbf{6}\}$
			$\{1,2,1,b\}$	$\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{1},\mathbf{7}\}$
			$\{\mathbf{1},\mathbf{2},\mathbf{b},\mathbf{2}\}$	$\{1,2,3,5,3\}$
				$\{1,2,2,5,4\}$
				$\{1,2,6,1,4\}$
				$\{\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{3},\mathbf{5}\}$
				$\{\mathbf{1},\mathbf{2},\mathbf{4},\mathbf{3},\mathbf{3}\}$
				$\{\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{4},\mathbf{4}\}$
				$\{\mathbf{1},\mathbf{2},\mathbf{4},\mathbf{1},\mathbf{5}\}$

Convergence of Periodic Fractions

Theorem

A rational x is the value of a convergent periodic negative-integer continued fraction

$$x = \underbrace{\frac{-1}{b_1} + \frac{-1}{b_2} + \cdots + \frac{-1}{b_n}}_{n} + \underbrace{\frac{-1}{b_1} + \frac{-1}{b_2} + \cdots + \frac{-1}{b_n}}_{n} + \cdots \tag{10}$$

if and only if $Q_{n-1} \neq 0$ and $P_{n-1} + Q_n = \pm 2$. In particular, it converges to $x = 0$ if and only if $Q_{n-1} \neq 0$ and $P_n = 0$; if $Q_{n-1} = 0$ and $P_n \neq 0$, then the periodic continued fraction diverges to ∞; if $Q_{n-1} = 0$ and $P_n = 0$, then the periodic continued fraction diverges.

It is easy to see that if (10) converges to x then x is a root of *the characteristic quadratic equation*

$$Q_{n-1}X^2 + (Q_n - P_{n-1})X - P_n = 0 \tag{11}$$

of the period $\{b_1, \dots, b_n\}$. As Theorem 4 shows the convergence of (10) under assumption $P_{n-1} + Q_n = \pm 2$ is determined by the entries of the matrix

$$A = \begin{pmatrix} P_{n-1} & P_n \\ Q_{n-1} & Q_n \end{pmatrix} .$$

The equation (11) can be rewritten as

$$x = \frac{-1}{b_1} + \frac{-1}{b_2} + \dots + \frac{-1}{b_n + x} = \frac{P_{n-1}x + P_n}{Q_{n-1}x + Q_n} . \tag{12}$$

Corollary

The continued fraction (10) *with* $P_{n-1} + Q_n = \pm 2$ *diverges if and only if its period represents the identity.*

Proof.

By the determinant identity for continued fractions

$$P_n Q_{n-1} - P_{n-1} Q_n = -1\,.$$

If (10) diverges then $P_{n-1} Q_n = 1$ by Theorem 4. It follows that $P_{n-1} = Q_n = \pm 1$. Hence the period $\{b_1, b_2, \dots, b_n\}$ represents the identity by (12). The converse arguments were illustrated on the example of (5). □

Multiplication of periods

We consider periods $\{b_1, b_2, \dots, b_n\}$ as well-ordered finite sets of not necessarily different positive integers. The set $\mathbb{P}$ of all periods is a semigroup with respect to the following operation:

$$\{b_1, b_2, \dots, b_n\} \cdot \{c_1, c_2, \dots, c_k\} = \{b_1, \dots, b_n, c_1, \dots, c_k\}\,.$$

By (12) the mapping

$$\mathfrak{m}\{b_1, b_2, \dots, b_n\} = \begin{pmatrix} P_{n-1} & P_n \\ Q_{n-1} & Q_n \end{pmatrix} \tag{13}$$

is a homomorphism of $\mathbb{P}$ into the special linear group $SL(2, \mathbb{Z})$ of all 2×2 matrices A with integer entries satisfying $\det(A) = 1$. The kernel $\mathrm{Ker}(\mathfrak{m})$ of $\mathfrak{m}$ consists of all periods representing the identity. Every $A \in SL(2, \mathbb{Z})$ determines the Möbius transform

$$\kappa(A)(z) = \mu(z) = \frac{az + b}{cz + d}\,. \tag{14}$$

The modular group

The image of $SL(2,\mathbb{Z})$ under κ is the modular group Γ. The group operation in Γ is a composition of Möbius transforms. There are exactly two matrices $X = A$ and $X = -A$ satisfying $\kappa(X) = \mu$. The mapping $\kappa : A \longmapsto \mu$ is a homomorphism of groups with the kernel $\{I, -I\}$, I being the identity matrix. The composition of κ and $\mathfrak{m}$ is

$$\mathbb{P} \xrightarrow{\mathfrak{m}} SL(2,\mathbb{Z}) \xrightarrow{\kappa} \Gamma\,. \tag{15}$$

It is well known that Γ is generated by two transformations $S(z) = -1/z$ and $T(z) = z + 1$ satisfying the following relations

$$S^2 = I \quad (ST)^3 = I\,. \tag{16}$$

Theorem

For every $A \in SL(2,\mathbb{Z})$ there is a period $p \in \mathbb{P}$ of only 1's and 2's such that $\mathfrak{m}(p) = A$. Hence $\kappa \circ \mathfrak{m}(\mathbb{P}) = \Gamma$.

Definition

Given $\mu \in \Gamma$ we denote by $\mathbb{P}(\mu)$ the set of all periods $p \in \mathbb{P}$ such that $\kappa \circ \mathfrak{m}(p) = \mu$. To indicate that x is the value of a negative-regular continued fraction (3) *with period $\{b_1, b_2, \dots, b_n\}$ we write for brevity*

$$\{b_1, b_2, \dots, b_n\} \mapsto x\,. \tag{17}$$

We denote by $\mathbb{P}_x$ the set of all periods in $\mathbb{P}$ satisfying (17).

The signature of a period

By (13) the sum $P_{n-1} + Q_n$ is the trace of the matrix $\mathfrak{m}\{b_1, b_2, \dots, b_n\}$. In what follows we denote by

$$\epsilon_n = \text{sign}\{b_1, \dots, b_n\} = \pm 1$$

the sign of $P_{n-1} + Q_n$. It is called the signature of $\{b_1, \dots, b_n\}$. By Theorem 4 if (3) converges to x then $P_{n-1} + Q_n = 2\epsilon_n$. A rational x can be recovered from its period $\{b_1, \dots, b_n\}$ by the following formula:

$$\boxed{x = -\frac{Q_n - \epsilon_n}{Q_{n-1}}}. \tag{18}$$

Using operations of insertion of 1's inside the period $\{b_1, \dots, b_n\}$ one can obtain infinitely many other periods in $\mathbb{P}(\mu)$. These operations are based on simple identities:

$$\begin{aligned} &\frac{-1}{1}\,{}_{+}\,\frac{-1}{1}\,{}_{+}\,\frac{-1}{1+w} = w\,; \\ &a + \frac{-1}{1}\,{}_{+}\,\frac{-1}{1}\,{}_{+}\,\frac{-1}{b+w} = a+b-1+w\,; \\ &a + \frac{-1}{1}\,{}_{+}\,\frac{-1}{b+w} = a-1+\frac{-1}{b-1+w}\,, \end{aligned} \tag{19}$$

which being applied in proper places do not change the matrix $\mathfrak{m}\{b_1, \dots, b_n\}$ of the period $\{b_1, \dots, b_n\}$.

Definition

A period $\{b_1, \dots, b_n\}$ *in* $\mathbb{P}(\mu)$ *is called primitive if it cannot be shorten with the operations described in* (19) *so that a shorter period remains in* $\mathbb{P}(\mu)$. *A primitive period in* $\mathbb{P}_x$ *is called minimal if it has the shortest length from all primitive periods for* x.

For example

$$\{1,2,5\} \mapsto -3\,;\{1,2,3,4\} \mapsto -3\,;\{1,2,2,b,3\} \mapsto -3\,, b \neq 1,$$

are different primitive periods for -3. The first period in the list above is minimal. The signature of the first three periods is $+1$. The signature of the last period in the list is -1 for every $b \neq 1$. The list of primitive periods for 0 is given by

$$\{b,1,1\}, b > 1\,;\{1,1,\underbrace{2,\dots 2}_{k},1\},\ k = 1,2,\dots\,. \qquad (20)$$

Every $\mu \in \Gamma$ can be uniquely factored as:

$$\mu = S^{\epsilon}(ST)^{\alpha_1} S (ST)^{\alpha_2} S \cdots S (ST)^{\alpha_r} S^{\delta}, \tag{21}$$

where ϵ and δ take values 0 or 1 and α_k equal 1 or 2.
Let $\mathbb{P}'$ be the set of all primitive periods. We prove that for every $\mu \in \Gamma$ there is a *unique* primitive period $p \in \mathbb{P}'$ such that $\mu = \kappa \circ \mathfrak{m}(p)$. In other words the set of all products of the form

$$ST^{b_1} ST^{b_2} \cdots ST^{b_n} \tag{22}$$

such that $\{b_1, \ldots, b_n\}$ is a primitive period coincides with Γ. We give a simple algorithm for the construction of the primitive period of μ starting from the unique factorization (21) of μ and vice versa. The primitive factorization (22) has an advantage compared with the standard factorization (21). It is not only much shorter but also has a nice interpretation in terms of periodic negative-regular continued fractions.

Theorem

Let $\{b_1, \ldots, b_n\}$ be the minimal period of a rational number x. Then the primitive period of x of length $n + k$, $k \geq 1$ is obtained from

$$\{\underbrace{\underbrace{b_1, \ldots, b_n}, \underbrace{b_1, \ldots, b_n}, \ldots, \underbrace{b_1, \ldots, b_n}}_{k+1}\}$$

by eliminating 'interior' 1's using the last formula of (19).

Definition

All primitive periods obtained form the minimal period of a rational number x by the algorithm of Theorem (9) are called direct periods of x.

Theorem

A primitive period of a rational $x \neq 0$ is not direct if and only if its signature equals the sign of x.

Theorem

If n is the length of the minimal period for a rational number $x \neq 0$ then all not direct periods for x have the length $n+2$ and differ from each other only at one position of their period.

Theorem

If $x < 0$ and $-x = [a_0; a_1, \dots, a_m]$ is the regular continued fraction for $-x$ then the length of the minimal period for x is $a_0 + \dots + a_m$.

For example for $x = -5/3$ with the minimal period $\{1, 3, 5, 2\}$ we obtain

$$\frac{5}{3} = 1 + \frac{1}{1 + \frac{1}{2}}; \quad 1 + 1 + 2 = \boxed{4}.$$

For the Metius' approximation to π

$$\frac{355}{113} = 3 + \frac{1}{7} + \frac{1}{16}$$

implying that the length of the period of $-355/113$ is $3 + 7 + 16 = 26$. The length of the period of $-103993/33102$ is 318 since $103993/33102$ is the convergent which follows next to Metius' convergent: $318 = 3 + 7 + 15 + 293$.

Alternatively, all minimal periods for negative rational numbers can be obtained from the period $\{2\}$ by induction. If a minimal period $\{b_1, \ldots, b_n\}$ of length n is already obtained then the minimal periods of length $n+1$ are

$$\{1, b_1+1, b_2, \ldots, b_{n-1}, b_n+1\}\,, \quad \{b_1+1, b_2, \ldots, b_{n-1}, b_n+1, 1\}\,. \tag{23}$$

The number of regular continued fractions $[a_0; a_1, \dots, a_m]$ satisfying

$$a_0 + a_1 + \cdots + a_m = n$$

equals 2^{n-1}. This gives 2^{n-1} different negative rational numbers with the length of the minimal period n. These rational numbers are closely related to Farey's series and the Stern-Brocot tree. The formulas (23) are in fact equivalent to the formulas used for the construction of the Calkin-Wilf tree. From the point of view of the present paper the Stern-Brocot tree describes and classifies rational regular continued fraction whereas the Calkin-Wilf tree does the same for negative-regular continued fractions.

Identities

Theorem

A periodic negative-regular continued fraction (3) *diverges if and only if at least one of the three following conditions holds:*

(1) $\{b_1, \dots, b_n\}$ *represents the identity;*

(2) $\{b_1, \dots, b_n, b_1, \dots, b_n\}$ *represents the identity;*

(3) $\{b_1, \dots, b_n, b_1, \dots, b_n, b_1, \dots, b_n\}$ *represents the identity.*

Elliptic elements

Theorem

Let $P = \{b_1, \ldots, b_n\}$ be the minimal period of a rational number $x \neq -1$ with marked element b_k. Then the period P^*, in which b_k is replaced with $b_k - 2$, corresponds to a Möbius transform of order 2. Any Möbius transform of order 2 in the modular group is obtained this way.

The correspondence in the Theorem is one-to-one.

Elliptic elements

Theorem

Let $\{b_1, \dots, b_n\}$ be a primitive period. The following conditions are equivalent:

(1) $\{b_1, \dots, b_n\}$ *is a period of an elliptic transform of order* 2*;*

(2) *there is a minimal period* $\{b_1, \dots, b_k + 2, \dots, b_n\}$ *of a rational number such that* $b_k + 2$ *is its marked element;*

(3) *the numerators and denominators of the last two convergents for* $\{b_1, \dots, b_n\}$ *satisfy*

$$P_n Q_{n-1} = -(Q_n^2 + 1)\,;$$

Elliptic elements

Theorem

Let $\{b_1, \dots, b_n\}$ be a primitive period. The following conditions are equivalent:

(1) *$\{b_1, \dots, b_n\}$ is a period of an elliptic transform of order 2;*

(2) *The rational number $|P_n/Q_n|$ can be represented as the value of a symmetric regular continued fraction*

$$\left|\frac{P_n}{Q_n}\right| = a_0 + \frac{1}{a_1} {}_{+\cdots+} \frac{1}{a_{2d}}\,, a_k = a_{2d-k}\,, k = 0, \dots 2d\,.$$

Elliptic elements

Theorem

Let $P = \{b_1, \dots, b_n\}$ be the minimal period of a rational number $x \neq -1$ with marked element b_k. Then the period P^*, in which b_k is replaced with $b_k - 1$, corresponds to a Möbius transform of order 3. Any Möbius transform of order 3 in the modular group is obtained this way.

Again the correspondence can be arranged in a one-to-one way.

Elliptic elements

Corollary

Let $P = \{b_1, \ldots, b_n\}$ be the minimal period of a rational number. Then there is an element $b_k > 3$ in the period P such that the double of the new period with $b_k := b_k - 2$ represents the identity.

This corollary shows that a small modification in one term of the period (namely, subtracting 2) may easily ruin the convergence.