Infinite family of solvable and integrable planar quantum systems

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Let us consider the planar Hamiltonian $= \text{the Schroedinger operator}$

$$\mathcal{H} = -\Delta + V(x) \ , \quad x \in \mathbb{R}^2$$

The system is called *integrable*, if there exist one or two algebraically-independent operators $I_i$ commuting with $\mathcal{H}$,

$$[\mathcal{H}, I_i] = 0 \ , \quad i = 1, \ldots \ell$$

If $\ell = 1$ the system is called *completely-integrable*.

If $\ell = 2$, the system is called *superintegrable*

The central problem of quantum mechanics is to solve the Schroedinger equation

$$\mathcal{H}\Psi(x) = E\Psi(x) \ , \quad \Psi(x) \in L^2(\mathbb{R}^2)$$

*Integrability does not help to solve the Schroedinger equation!*
However, among integrable systems there are exactly-solvable systems.

Montreal conjecture (2001):

*All superintegrable systems in $\mathbb{R}^2$ are exactly-solvable*

(P. Tempesta, A.T., P. Winternitz)

A "source" of solvability!
To the best of my knowledge all known explicitly (algebraically) eigenfunctions have a form

$$\Psi(x) = (\text{polynomial in } \phi(x)) \times \text{factor}$$

with a non-singular function in the domain $x$ as factor. No real exceptions are known so far ...
Formalism:

★
Take an infinite set of (constructively-defined) linear spaces \( \mathcal{V}_n, \ n = 0, 1 \ldots \). Assume they can be ordered

\[
\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \ldots \subset \mathcal{V}_n \subset \ldots \mathcal{V}
\]

Such a construction is called the *Infinite flag (filtration)* \( \mathcal{V} \)

★
If an operator \( E \) is such that

\[
E : \mathcal{V}_n \mapsto \mathcal{V}_n, \quad n = 0, 1, 2, \ldots
\]

then it implies that \( E \) preserves the flag \( \mathcal{V} \)

★
An operator \( E \) which preserves an infinite flag of (constructively-defined) finite-dimensional spaces \( \mathcal{V} \) is called the *Exactly-solvable operator with flag* \( \mathcal{V} \)
If operator $Q$ preserves a single space $\mathcal{V}_n$ (for some $n$) it is called the *Quasi-exactly-solvable operator* (we study spaces (flags) of polynomials only)

**Example.** The flag made from space

$$\mathcal{P}_n^{(2)} = \langle x_1^{p_1} x_2^{p_2} | 0 \leq p_1 + p_2 \leq n \rangle$$

$n = 0, 1, \ldots$. This flag is called $\mathcal{P}^{(2)}$

Differential Operator of finite order with polynomial coefficient functions is called the *Algebraic operator*
In the connection to flags of polynomials we introduce a notion ‘characteristic vector’.
Let us consider a flag made out of ”triangular” linear space of polynomials

\[ \mathcal{P}^{(2)}_{n,\alpha} = \langle x_1^{p_1} x_2^{p_2} | 0 \leq \alpha_1 p_1 + \alpha_2 p_2 \leq n \rangle \]

where the “grades” \( \alpha \)'s are positive integer numbers and \( n = 0, 1, 2, \ldots \).

**DEFINITION.** Characteristic vector is a vector with components \( \alpha_i \)

\[ \vec{\alpha} = (\alpha_1, \alpha_2) \]

hence, the characteristic vector for flag \( \mathcal{P}^{(2)} \)

\[ \vec{\alpha}_0 = (1, 1) \]
$gl_3$-algebra

(almost degenerate or totally symmetric)

$(n, 0)$

\[ J_i^- = \frac{\partial}{\partial x_i}, \quad i = 1, 2 \]
\[ J_{ij}^0 = x_i \frac{\partial}{\partial x_j}, \quad i, j = 1, 2 \]
\[ J^0 = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} - n \]
\[ J_i^+ = x_i J^0 = x_i \left( \sum_{j=1}^{2} x_j \frac{\partial}{\partial x_j} - n \right), \quad i = 1, 2 \]

▷ 9 generators
if $n = 0, 1, 2 \ldots$, fin-dim irreps

$$\mathcal{P}_n^{(2)} = \langle x_1^{p_1} x_2^{p_2} \mid 0 \leq p_1 + p_2 \leq n \rangle$$

**Remark.** The flag $\mathcal{P}^{(2)}$ is made out of finite-dimensional irreducible representation spaces of the algebra $gl_3$ taken in realization ($\ast$).

*There exist other flags associated with irreducible, finite-dimensional representation spaces of the Lie algebras of differential (difference) operators.*

Any operator made out of generators has finite-dimensional invariant subspace which is finite-dimensional irreducible representation space and visa versa.
Smorodinsky-Winternitz Potential ('67)

\[ \mathcal{H}_{SW} = - \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) + \omega^2(x_1^2 + x_2^2) + \frac{\alpha}{x_1^2} + \frac{\beta}{x_2^2} \]

Variables are separated in Cartesian, polar and elliptic coordinates.  
∃ two mutually-non-commuting integrals both of the 2nd order:  
\textit{superintegrability!}

Ground state:

\[ \psi_0 = x_1^a x_2^b e^{-\frac{\omega}{2}(x_1^2 + x_2^2)} , \quad E_0 = 2\omega(a + b + 1) \]

where \( \alpha = a(a - 1) \) and \( \beta = b(b - 1) \)
In polar coordinates \((x_1 = r \cos \varphi, x_2 = r \sin \varphi)\)

\[
\mathcal{H}_{SW}(x, y; \alpha, \beta) = -\partial_r^2 - \frac{1}{r} \partial_r - \frac{1}{r^2} \partial_\varphi^2 + \omega^2 r^2 + \frac{\alpha}{r^2 \cos^2 \varphi} + \frac{\beta}{r^2 \sin^2 \varphi}
\]

Configuration space: \(\varphi \in [0, \frac{\pi}{2}], \ r \in [0, \infty)\)

*Modified isotropic 2D-harmonic oscillator in the 1st quadrant as configuration space*
Variables are separated in polar coordinates.
∃ two mutually-\textbf{non}-commuting integrals of the 2nd and 4th orders:

\textbf{superintegrability!}

Ground state:

\[ \Psi_0 = (x_1^2 - x_2^2)^a (x_1 x_2)^b e^{-\frac{\omega}{2} (x_1^2 + x_2^2)} \quad , \quad E_0 = 2\omega [2(a + b) + 1] \]

where \( \alpha = a(a - 1) \) and \( \beta = b(b - 1) \)
In polar coordinates \((x_1 = r \cos \varphi, x_2 = r \sin \varphi)\)

\[
\mathcal{H}_{BC_2}(x, y; \alpha, \beta) = -\partial_r^2 - \frac{1}{r} \partial_r - \frac{1}{r^2} \partial^2_\varphi + \omega^2 r^2 + \frac{4\alpha}{r^2 \cos^2 2\varphi} + \frac{4\beta}{r^2 \sin^2 2\varphi}
\]

Configuration space: \(\varphi \in [0, \frac{\pi}{4}]\), \(r \in [0, \infty)\)

*Modified isotropic 2D-harmonic oscillator in sector \(\varphi \in [0, \frac{\pi}{4}]\) as configuration space*
3-body Calogero model ($A_2$-rational model, Olshanetsky-Perelomov '77)

\[
\mathcal{H}_{A_2} = -\left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) + \omega^2(x_1^2 + x_2^2 + x_3^2) \\
+ \frac{9\beta}{4} \left( \frac{1}{(x_1 - x_2)^2} + \frac{1}{(x_1 - x_3)^2} + \frac{1}{(x_2 - x_3)^2} \right)
\]

After separation of c.m.s. out, the relative variables are separated in polar coordinates.
\[\exists\text{ two mutually-\textbf{non}-commuting integrals: of the 2nd and either 6th order if } \omega \neq 0 \text{ or 3rd order if } \omega = 0:\]

\[\text{superintegrability!}\]

Ground state:

\[
\Psi_0 = (x_1 - x_2)^b(x_1 - x_3)^b(x_2 - x_3)^b e^{-\frac{\omega}{2}(x_1^2+x_2^2+x_3^2)}, \quad E_0 = 2\omega(3b+1)
\]

where \(\beta = b(b - 1)\)
After separation of cms-coordinate $X = x_1 + x_2 + x_3$ in polar (relative) coordinates
\[
\frac{1}{\sqrt{2}}(x_1 - x_2) = r \cos \varphi , \quad \sqrt{\frac{3}{2}}(x_1 + x_2 - 2x_3) = r \sin \varphi ,
\]

\[
\tilde{H}_{A_2}(x, y; 0, \beta) = -\partial_r^2 - \frac{1}{r} \partial_r - \frac{1}{r^2} \partial_\varphi^2 + \omega^2 r^2 + \frac{9\beta}{r^2 \sin^2 3\varphi}
\]

Configuration space: $\varphi \in [0, \frac{\pi}{6}]$, $r \in [0, \infty)$
Wolfes model ’75 (G\textsubscript{2}-rational model, Olshanetsky-Perelomov ’77)

\[ \mathcal{H}_{G_2} = \sum_{i=1}^{3} \left( -\frac{\partial^2}{\partial x_i^2} + \omega^2 x_i^2 \right) + \alpha \sum_{i<j} \frac{1}{(x_i - x_j)^2} + 3\beta \sum_{k<l, \ k,l\neq m} \frac{1}{(x_k + x_l - 2x_m)^2} \]

After separation of c.m.s. out the relative variables are separated in polar coordinates. \exists two mutually-\textbf{non}-commuting integrals of the 2nd and 6th orders: \textit{superintegrability}!

Ground state:

\[ \Psi_0 = \prod_{i<j}^3 |x_i - x_j|^a \prod_{k<l, \ k,l\neq m}^3 |x_l + x_m - 2x_k|^b e^{-\frac{\omega}{2}(x_1^2 + x_2^2 + x_3^2)} \]

where \( \alpha = a(a - 1) \), \( \beta = b(b - 1) \)
After separation of cms-coordinate \( X = x_1 + x_2 + x_3 \) in polar (relative) coordinates

\[
\frac{1}{\sqrt{2}}(x_1 - x_2) = r \cos \varphi , \quad \sqrt{\frac{3}{2}}(x_1 + x_2 - 2x_3) = r \sin \varphi ,
\]

\[
\tilde{\mathcal{H}}_{G_2}(x, y; \alpha, \beta) = -\partial_r^2 - \frac{1}{r} \partial_r - \frac{1}{r^2} \partial^2_{\varphi} + \omega^2 r^2 + \frac{9\alpha}{r^2 \cos^2 3\varphi} + \frac{9\beta}{r^2 \sin^2 3\varphi}
\]

Configuration space: \( \varphi \in [0, \frac{\pi}{6}] , \ r \in [0, \infty) \)

**Modified isotropic 2D-harmonic oscillator in sector** \( \varphi \in [0, \frac{\pi}{6}] \) **as a configuration space.**
Natural generalization (F.Tremblay, A.T., P.Winternitz ’09, TTW model):

\[ \mathcal{H}_k(x, y; \alpha, \beta) = -\partial_r^2 - \frac{1}{r} \partial_r - \frac{1}{r^2} \partial_\varphi^2 + \omega^2 r^2 + \frac{k^2 \alpha}{r^2 \cos^2 k \varphi} + \frac{k^2 \beta}{r^2 \sin^2 k \varphi} \]

- \( k = 1 \) – Smorodinsky-Winternitz
- \( k = 2 \) – \( BC_2 \)-rational
- \( k = 3 \) – Calogero (\( \alpha = 0 \)) and Wolfes (\( \alpha \neq 0 \))

Ground state:

\[ \Psi_0 = r^{k(a+b)} \cos^a k \varphi \sin^b k \varphi \ e^{-\frac{\omega}{2} r^2} , \quad E_0 = 2 \omega [k(a + b) + 1] , \]

where \( \alpha = a(a - 1) \), \( \beta = b(b - 1) \).
(Super)integrability (for integer $k$)

First integral

$$X_k = -\partial_\phi^2 + \frac{k^2 \alpha}{\cos^2 k\phi} + \frac{k^2 \beta}{\sin^2 k\phi}$$

(separability in polar coordinates)

Second integral

$$Y_{2k} = P_{2k}(\partial_r, \partial_\phi; r, \phi)$$

is explicitly known at $k = 1, 2, 3$ and $4$ (at $\alpha = 0$ for $k = 1 \ldots 6, 8$) but for integer $k > 4$?

(Dunkl operators formalism in polar coordinates: C Quesne, '96 – for $k = 3$ and odd $k > 3$, '10)

For even $k$ – W Miller et al, '10
For rational $k = \frac{p}{q}$ – W Miller et al, ’10 - ’11

$$Y_{2k} = P_{2(p+q-1)}(\partial_r, \partial_\varphi; r, \varphi)$$

For all known cases

$$[X_k, Y_{2k}] \neq 0$$
(Super)integrability (for any real \( k > 0 \))

First integral

\[
X_k = -\partial^2 \varphi + \frac{k^2 \alpha}{\cos^2 k\varphi} + \frac{k^2 \beta}{\sin^2 k\varphi}
\]

(separability in polar coordinates)

Does a second integral exist?

*interesting open question*
Exact-solvability (for real $k > 0$)

Make a gauge rotation (similarity transformation)

$$ h = \psi_0^{-1}(\mathcal{H}_k - E_0)\psi_0 $$

and get the operator

$$ h = -\partial_r^2 + (2\omega r - \frac{2k(a + b) + 1}{r})\partial_r - \frac{1}{r^2}\partial_\varphi^2 - \frac{2k}{r^2}(-a \tan k\varphi + b \cot k\varphi)\partial_\varphi $$

for which the lowest eigenfunction is a constant with zero eigenvalue.

Separating variables

$$ \Psi_{N,n} = r^{2nk}L_N^{(k(2n+a+b))}(\omega r^2)P_n^{(a-1/2,b-1/2)}(\cos 2k\varphi)\psi_0 $$
\[ E_{N,n} = 2\omega [2(N + nk) + k(a + b) + 1] \]

where \( N, n = 0, 1, 2, \ldots \). 

Degeneracy: 
\[ N + kn = \text{integer} \]

At \( a, b = 0 \) it is the spectra of anisotropic harmonic oscillator in \( \mathbb{R}^2 \).

\[ V(x, y) = \omega^2 x^2 + k^2 \omega^2 y^2 \]

Isotropic oscillator in the sector becomes anisotropic!
Exact-solvability (for integer $k$)

Make change of variables in $h_k$

$$t = r^2, \ u = r^{2k} \sin^2 k\varphi$$

- invariants of dihedral group $I_2(k)$.
It takes amazingly simple algebraic form:

$$h_k = -4t\partial_t^2 - 8kuv\partial_{tu} - 4k^2 t^{k-1} u\partial_u$$
$$+ 4[\omega t - (a + b)k - 1]\partial_t + [4\omega ku - 2k^2 (2b + 1)t^{k-1}]\partial_u$$

$h_k$ has infinitely-many finite-dimensional invariant subspaces in polynomials

$$\mathcal{P}_n^{(k)} = (t^p u^q | 0 \leq (p + kq) \leq n), \ n = 0, 1, 2, \ldots$$

Flag:

$$\vec{\alpha} = (1, k)$$
The Lie algebra:

\[ J^1 = \partial_t \]

\[ J^2_n = t\partial_t - \frac{n}{3}, \quad J^3_n = su\partial_u - \frac{n}{3} \]

\[ J^4_n = t^2\partial_t + stu\partial_u - nt \]

\[ R_i = t^i\partial_u, \quad i = 0, 1, \ldots, s, \quad \mathcal{R}^{(s)} \equiv (R_0, \ldots, R_s) \]

they span non-semi-simple algebra \( gl(2, \mathbb{R}) \rtimes \mathcal{R}^{(s)} \) at \( s = 2, 3, \ldots \)

S. Lie, \( \sim1890 \) at \( n = 0 \), W. Miller ’68 and then
A. González-Lopéz, P Olver, N Kamran, ’91 at \( n \neq 0 \)
(Case 24, complete classification)

\[ \mathcal{P}^{(s)}_n = (t^p u^q | 0 \leq (p + sq) \leq n) \]

common invariant subspace (reducible)
By adding

\[ T_0^{(s)} = u \partial^s_t \]

to \( gl(2, \mathbb{R}) \times \mathcal{R}^{(s)} \), the action on \( \mathcal{P}_n^{(s)} \) gets irreducible.

Property:

\[
T_i^{(s)} = [J^4, [J^4, \ldots, [J^4, T_0^{(s)}], \ldots]] = u \partial_t^{s-i} J_0(J_0+1) \cdots (J_0+i-1), \]

\[ i = 1, \ldots, s, \text{ all } T_i^{(s)} \text{ of the fixed degree } s, \quad J_0 = t \partial_t + su \partial_u - n \]

Nilpotency:

\[ T_i^{(s)} = 0, \quad i > s. \]
Commutativity:

\[ [T_i^{(s)}, T_j^{(s)}] = 0, \quad i, j = 0, \ldots, s, \quad \mathcal{T}^{(s)} \equiv (T_0^{(s)}, \ldots, T_s^{(s)}) \]

Decomposition:

\[ \mathcal{T}^{(s)} \rtimes gl_2 \rtimes \mathcal{R}^{(s)} \]

Infinite-dimensional, \((2s + 6)\) generated algebra with \(\mathcal{P}_n^{(s)}\) irreps space \((\text{for } s = 1 \rightarrow gl(3))\)
At $s = k$,

$$\frac{-h_k}{4} = (J^2_n + 2J^3_n)J^1 + kJ^3 R_{k-1}$$

$$[(a + b)k + 1 + n]J^1 - \omega(J^2_n + J^3_n) + \frac{k}{6}[2n + 3k(2b + 1)]R_{k-1} - \frac{2\omega}{3}$$

Hence, $gl(2, \mathbb{R}) \ltimes \mathcal{R}^{(k)}$ is the hidden algebra,

Generators $J^4$ as well as $T^{(k)}$'s are absent in representation of $h_k$, hence, finite-dimensional invariant subspaces of $h$ are finite-dimensional representation spaces of $gl(2, \mathbb{R}) \ltimes \mathcal{R}^{(k)}$.

Such a sequence of linear spaces $\mathcal{P}_n^{(k)}$, $n = 0, 1, \ldots$ forms the flag and $h_k$ preserves this flag.
The flag is invariant with respect to weighted-projective transformations:

\[ t \rightarrow t + A_0 \]

\[ u \rightarrow u + B_k t^k + B_{k-1} t^{k-1} \ldots + B_0 \]

**algebraic Hamiltonian remains algebraic!**

A meaning of this invariance for the Hamiltonian is unclear so far ...
Integrability for integer $k$ (algebraic forms)

There must exist the algebraic form of the integral $X_k$ as well as $\mathcal{V}_{2k}$ (all common eigenfunctions are polynomials)

Gauge rotation: $x_k = \psi_0^{-1}(X_k - c_k)\psi_0$

in $(t, u)$ coordinates:

$$x_k = -4k^2u(t^k - u)\partial_u^2 - 4k^2[(b + \frac{1}{2})t^k - (a + b + 1)u]\partial_u$$

where $c_k = k^2(a + b)^2$ is the lowest eigenvalue.

In the generators

$$x_k = -4kJ^3R_k + 4J^3J^3 - 4k^2(b + \frac{1}{2})R_k + 4k(a + b)J^3$$

$x_k$ preserves the same flag $(1, k)$ as the Hamiltonian $h_k$
Algebraic forms $y_{2k}$ in $x$–space for $Y_{2k}$ known explicitly for $k = 1, 2, 3, 4$ only - they contain $T_{0}^{(1,2,3,4)}$ generators, respectively.

Evidently, algebraic forms $y_{2k}$ exist for any integer (rational) $k$ - joint eigenfunctions with $h_{k}$ are polynomials.

$y_{2k}$ at $k = 1, 2, 3, 4$ preserves the same flag $(1, k)$ as for the Hamiltonian $h_{k}$

Is it true for any integer $k$? What about rational $k$?
Classical Mechanics

Hamiltonian

\[ \mathcal{H}_k(x, y; \alpha, \beta) = p_1^2 + p_2^2 + \omega^2 r^2 + \frac{k^2 \alpha}{r^2 \cos^2 k \varphi} + \frac{k^2 \beta}{r^2 \sin^2 k \varphi} \]

likely superintegrable (yes, for rational \( k \))

\[ X_k = L_3^2 + \frac{k^2 \alpha}{\cos^2 k \varphi} + \frac{k^2 \beta}{\sin^2 k \varphi} \]

\[ Y_{2k} =? \]

For any real \( k > 0 \) all bounded trajectories are closed!

(periodic motion with period \( \frac{\pi}{2\omega} \) → isochronicity)
Quasi-Exact-solvability (for integer $k$)

$$H_{k}^{(qes)}(x, y; \alpha, \beta) = -\partial_{r}^{2} - \frac{1}{r} \partial_{r} - \frac{1}{r^2} \partial_{\varphi}^{2} + g^2 r^6 + 2g\omega r^4$$

$$+ [\omega^2 - 2g(2N + 2 + k(a + b))] r^2 + \frac{\alpha k^2}{r^2 \cos^2 k\varphi} + \frac{\beta k^2}{r^2 \sin^2 k\varphi}$$

where about $\left(\frac{N^2}{2k} + 1\right)$ eigenstates are known explicitly (algebraically). Their eigenfunctions have a form of a polynomial $p(t, u) \in \mathcal{P}_{N}^{(k)}$ multiplied by

$$\Psi_{0}^{(qes)} = r^{(a+b)k} \cos^{a} k\varphi \sin^{b} k\varphi e^{-\frac{\omega r^2}{2} - \frac{gr^4}{4}} \quad \text{(ground state at } N = 0)$$

$gl(2, \mathbb{R}) \ltimes \mathbb{R}^{k+1}$ is hidden algebra. Integrability? - Yes, $X_k$ remains. Superintegrability? - "Y_{2k}"
Observation:

the operator

\[ i_{\text{par}}^{(n)}(t, u) = \prod_{j=0}^{n} (\mathcal{J}^0(n) + j) \]

with

\[ \mathcal{J}^0(n) = t \frac{\partial}{\partial t} + ku \frac{\partial}{\partial u} - n \]

commutes with \( h^{(qes)} = (\Psi_0^{(qes)})(-1) H_k^{(qes)}(x, y; \alpha, \beta) \Psi_0^{(qes)} \),

\[ [h^{(qes)}(t, u), i^{(n)}_{\text{par}}(t, u)] : \mathcal{P}_N^{(k)} \mapsto 0 \]

hence, \( i^{(n)}_{\text{par}} \) is particular integral
From Harmonic Oscillator TTW to planar "Coulomb" problem
(S.Post, P.Winternitz '10)
(coupling constant metamorphosis)

\[ \mathcal{H}_k^c(x, y; \alpha, \beta) = -\partial_\rho^2 - \frac{1}{\rho} \partial_\rho - \frac{1}{\rho^2} \partial_\theta^2 - \frac{Q}{\rho} + \frac{k^2 \alpha}{4 \rho^2 \cos^2 k\frac{\theta}{2}} + \frac{k^2 \beta}{4 \rho^2 \sin^2 k\frac{\theta}{2}} \]

Configuration space: \( \theta \in [0, \frac{\pi}{k}] \), \( \rho \in [0, \infty) \)

\[ E_{N,n} = -\frac{Q^2}{[2(N + kn) + 1 + k(a + b)]^2}, \quad N, n = 0, 1, 2, \ldots \]

Ground state:

\[ \psi_{0,0}^c = \rho^{(a+b)\frac{k}{2}} \cos^a\left(k\frac{\theta}{2}\right) \sin^b\left(k\frac{\theta}{2}\right) e^{-2\sqrt{-E_{0,0}}} \rho \]

where \( \alpha = a(a - 1) \), \( \beta = b(b - 1) \).
Eigenfunctions:

\[ \Psi_{N,n}^c = \rho^{(2n+a+b)\frac{k}{2}} L_N^{((2n+a+b)k)} \left( 2\sqrt{-E_{N,n}} \rho \right) P_n^{(a-1/2, b-1/2)} \left( -\cos k \theta \right) \]

\[ \cos^a \left( k \frac{\theta}{2} \right) \sin^b \left( k \frac{\theta}{2} \right) e^{-2\sqrt{-E_{N,n}} \rho} \]

where \( N, n = 0, 1, 2, \ldots \) 
\((N + kn)\) plays a role of principal quantum number

*System remains superintegrable* like *TTW* ...

\( k = 1 \) - Kalnins E G et al, ’96
\( k = 2 \) - Rodriguez M A et al, ’09
Quasi-Exact-solvability (for integer $k$) - planar "Coulomb" case

$$H_{k}^{c,(qes)}(x, y; \alpha, \beta) = -\partial_{\rho}^{2} - \frac{1}{\rho} \partial_{\rho} - \frac{1}{\rho^{2}} \partial_{\theta}^{2} + g^{2} \rho^{2} + 2g\omega\rho$$

$$+ \left[ \omega^{2} - 2g(2N + 2 + k(a + b)) - Q \right] \rho + \frac{\alpha k^{2}}{\rho^{2} \cos^{2} k\theta} + \frac{\beta k^{2}}{\rho^{2} \sin^{2} k\theta}$$

where about $\left( \frac{N^{2}}{2k} + 1 \right)$ eigenstates are known explicitly (algebraically). Their eigenfunctions have a form of a polynomial $p(t, u) \in \mathcal{P}_{N}^{(k)}$ multiplied by

$$\psi_{0}^{c,(qes)} = \rho^{(a+b)k} \cos^{a} \left( \frac{k\theta}{2} \right) \sin^{b} \left( \frac{k\theta}{2} \right) e^{-\frac{\omega\rho}{2} - \frac{g\rho^{2}}{4}} \text{ (ground state at } N = 0)$$

$gl(2, \mathbb{R}) \ltimes \mathbb{R}^{k+1}$ is hidden algebra.
From continuous to discrete:

- Algebraic Differential operators $h$ can be replaced by Difference Operators on uniform lattice with preservation of the property of existence of polynomial eigenfunctions

$$\frac{d}{dx} \rightarrow e^{\frac{d}{dx}}, \ x \rightarrow xe^{-\frac{d}{dx}}$$

(it is a Canonical Transformation, Y.F. Smirnov, A.T. ’95)

- Integrability is preserved also

- Hidden algebra continues to exist, it remains the same Lie algebra but realized by finite difference operators.

Similar holds for exponential lattice (C Chryssomalakos, A.T. ’01).

Lattices are taken in $(t, u)$ coordinates not in Cartesian ones.
CONCLUSION

All known planar integrable and exactly-solvable Hamiltonians with rational potentials are members of a single continuous family!

• What about trigonometric ones, $BC_2$ and $G_2$, can they also be members of some continuous family? Elliptic $BC_2$?

• Classical case - Action-Angle representation (Gonera, 2010-11, Lechtenfeld et al, 2011-12)

• What is a meaning of the parameter $k$??

• Similar inf-dim fin-gen algebras with generalized Gauss decomposition diagram exist at $\mathbb{R}^d$, $d = 4, 6, 7, 8$
Journal of Physics A: Mathematical and Theoretical

Best Paper Prize 2011

is awarded to

Frédérick Tremblay, Alexander V Turbiner and Pavel Winternitz

An infinite family of solvable and integrable quantum systems on a plane

M T Batchelor
Editor-in-Chief

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Publisher

IOP Publishing
Out \(\sim 1000\) papers published at *Journal of Physics A* in 2010 in all branches of physics from fluid mechanics to field/string theory **57 papers** were selected.

Out of **14 papers** selected in *Mathematical Physics* branch, **three papers** were closely related/inspired by TTW paper:

- **Families of classical subgroup separable superintegrable systems,** by E G Kalnins, J M Kress and W Miller Jr  

- **Superintegrability of the Tremblay - Turbiner - Winternitz quantum Hamiltonians on a plane for odd \(k\),**  

- **Periodic orbits for an infinite family of classical superintegrable systems,**  
  by F Tremblay, A V Turbiner and P Winternitz  