Infinite family of solvable and integrable planar quantum systems

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Let us consider the planar Hamiltonian = the Schroedinger operator

$$\mathcal{H} \;=\; -\Delta \;+\; V(x)\;, \qquad x \in R^2$$

The system is called *integrable*, if there exist one or two algebraically-independent operators I_i commuting with $\mathcal{H}$,

$$[\mathcal{H}, I_i] \;=\; 0\;, \qquad i = 1, \ldots \ell$$

If $\ell = 1$ the system is called *completely-integrable*.

If $\ell = 2$, the system is called *superintegrable*
The central problem of quantum mechanics is to solve the Schroedinger equation

$$\mathcal{H}\Psi(x) \;=\; E\Psi(x) \quad , \quad \Psi(x) \in L^2(R^2)$$

Integrability does not help to solve the Schroedinger equation!

However, among integrable systems there are **exactly-solvable** systems.

Montreal conjecture (2001):

All superintegrable systems in $\mathbf{R}^2$ *are exactly-solvable*

(P.Tempesta, A.T., P. Winternitz)

A "source" of solvability!

To the best of my knowledge all known explicitly (algebraically) eigenfunctions have a form

$$\Psi(x) \;=\; (\text{polynomial in } \phi(x)) \;\times\; \textbf{factor}$$

with a non-singular function in the domain x as **factor**.
No real exceptions are known so far ...

Formalism:

★

Take an infinite set of (constructively-defined) linear spaces $\mathcal{V}_n, \quad n = 0, 1 \ldots$. Assume they can be ordered

$$\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \ldots \subset \mathcal{V}_n \subset \ldots \mathcal{V}$$

Such a construction is called the *Infinite flag (filtration)* $\mathcal{V}$

★

If an operator E is such that

$$E : \mathcal{V}_n \mapsto \mathcal{V}_n, \quad n = 0, 1, 2, \ldots$$

then it implies that E preserves the flag $\mathcal{V}$

★

An operator E which preserves an infinite flag of (constructively-defined) finite-dimensional spaces $\mathcal{V}$ is called the *Exactly-solvable operator with flag* $\mathcal{V}$

★

If operator Q preserves a single space $\mathcal{V}_n$ (for some n) it is called the *Quasi-exactly-solvable operator*
(**we study spaces (flags) of polynomials only**)

Example. The flag made from space

$$\mathcal{P}_n^{(2)} = \langle\ {x_1}^{p_1} {x_2}^{p_2} |\ 0 \leq\ p_1 + p_2 \leq n\ \rangle$$

$n = 0, 1, \ldots.$ This flag is called $\mathcal{P}^{(2)}$

★

Differential Operator of finite order with polynomial coefficient functions is called the *Algebraic operator*

In the connection to flags of polynomials we introduce a notion *'characteristic vector'*.
Let us consider a flag made out of "triangular" linear space of polynomials

$$\mathcal{P}^{(2)}_{n,\alpha} \;=\; \langle x_1^{p_1} x_2^{p_2} | 0 \leq \alpha_1 p_1 + \alpha_2 p_2 \leq n \rangle$$

where the "grades" α's are positive integer numbers and $n = 0, 1, 2, \ldots$.

DEFINITION. Characteristic vector is a vector with components α_i

$$\vec{\alpha} = (\alpha_1, \alpha_2)$$

hence, the characteristic vector for flag $\mathcal{P}^{(2)}$

$$\vec{\alpha}_0 \;=\; (1, 1)$$

gl_3-algebra

(almost degenerate or totally symmetric)

$(n, 0)$

$$
\begin{aligned}
\mathcal{J}_i^- &= \frac{\partial}{\partial x_i}, \qquad i = 1, 2 \\
\mathcal{J}_{ij}{}^0 &= x_i \frac{\partial}{\partial x_j}, \qquad i, j = 1, 2 \\
\mathcal{J}^0 &= x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} - n \\
\mathcal{J}_i^+ &= x_i \mathcal{J}^0 = x_i \left(\sum_{j=1}^{2} x_j \frac{\partial}{\partial x_j} - n \right), \quad i = 1, 2
\end{aligned}
$$

- 9 generators

- if $n = 0, 1, 2 \ldots,$ *fin-dim* irreps

$$\mathcal{P}_n^{(2)} \;=\; \langle {x_1}^{p_1} {x_2}^{p_2} |\; 0 \leq p_1 + p_2 \leq n \rangle$$

Remark. The flag $\mathcal{P}^{(2)}$ is made out of finite-dimensional irreducible representation spaces of the algebra gl_3 taken in realization $(*)$.

There exist other flags associated with irreducible, finite-dimensional representation spaces of the Lie algebras of differential (difference) operators.

Any operator made out of generators has finite-dimensional invariant subspace which is finite-dimensional irreducible representation space and visa versa.

Smorodinsky-Winternitz Potential ('67)

$$\mathcal{H}_{\rm SW} = -\left(\frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2}\right) + \omega^2({x_1}^2 + {x_2}^2) + \frac{\alpha}{x_1^2} + \frac{\beta}{x_2^2}$$

Variables are separated in Cartesian, polar and elliptic coordinates.
$\exists$ two mutually-**non**-commuting integrals both of the 2nd order:
superintegrability!

Ground state:

$$\Psi_0 = x_1^a x_2^b e^{-\frac{\omega}{2}({x_1}^2+{x_2}^2)} \quad , \quad E_0 \;=\; 2\omega(a+b+1)$$

where $\alpha = a(a-1)$ and $\beta = b(b-1)$

In polar coordinates $(x_1 = r\cos\varphi, x_2 = r\sin\varphi)$

$$\mathcal{H}_{SW}(x,y;\alpha,\beta) \;=\; -\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\varphi^2 + \omega^2 r^2 + \frac{\alpha}{r^2\cos^2\varphi} + \frac{\beta}{r^2\sin^2\varphi}$$

Configuration space: $\varphi \in [0, \frac{\pi}{2}]$, $r \in [0, \infty)$

Modified isotropic 2D-harmonic oscillator in the 1st quadrant as configuration space

BC_2-rational model (Olshanetsky-Perelomov, '77)

$$\mathcal{H}_{\mathrm{BC}_2} = -\left(\frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2}\right) + \omega^2({x_1}^2 + {x_2}^2)$$
$$+2\alpha\left(\frac{1}{(x_1 - x_2)^2} + \frac{1}{(x_1 + x_2)^2}\right) + \beta\left(\frac{1}{x_1^2} + \frac{1}{x_2^2}\right)$$

Variables are separated in polar coordinates.
$\exists$ two mutually-**non**-commuting integrals of the 2nd and 4th orders:

superintegrability!

Ground state:

$$\Psi_0 = (x_1^2 - x_2^2)^a (x_1 x_2)^b e^{-\frac{\omega}{2}({x_1}^2 + {x_2}^2)} \quad , \quad E_0 \;=\; 2\omega[2(a+b)+1]$$

where $\alpha = a(a-1)$ and $\beta = b(b-1)$

In polar coordinates $(x_1 = r\cos\varphi, x_2 = r\sin\varphi)$

$$\mathcal{H}_{BC_2}(x,y;\alpha,\beta) \;=\; -\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\varphi^2 + \omega^2 r^2 + \frac{4\alpha}{r^2\cos^2 2\varphi} + \frac{4\beta}{r^2\sin^2 2\varphi}$$

Configuration space: $\varphi \in [0, \frac{\pi}{4}]$, $r \in [0, \infty)$

Modified isotropic 2D-harmonic oscillator in sector $\varphi \in [0, \frac{\pi}{4}]$ *as configuration space*

3-body Calogero model (A_2-rational model, Olshanetsky-Perelomov '77)

$$\mathcal{H}_{\mathrm{A}_2} = -\left(\frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2} + \frac{\partial^2}{\partial {x_3}^2}\right) + \omega^2({x_1}^2 + {x_2}^2 + {x_3}^2)$$
$$+\frac{9\beta}{4}\left(\frac{1}{(x_1-x_2)^2} + \frac{1}{(x_1-x_3)^2} + \frac{1}{(x_2-x_3)^2}\right)$$

After separation of c.m.s. out, the relative variables are separated in polar coordinates.

$\exists$ two mutually-**non**-commuting integrals: of the 2nd and either 6th order if $\omega \neq 0$ or 3rd order if $\omega = 0$:

superintegrability!

Ground state:

$$\Psi_0 = (x_1-x_2)^b(x_1-x_3)^b(x_2-x_3)^b e^{-\frac{\omega}{2}({x_1}^2+{x_2}^2+{x_3}^2)}\,,\ E_0 = 2\omega(3b+1)$$

where $\beta = b(b-1)$

After separation of cms-coordinate $X = x_1 + x_2 + x_3$
in polar (relative) coordinates
$\frac{1}{\sqrt{2}}(x_1 - x_2) = r\cos\varphi$, $\sqrt{\frac{3}{2}}(x_1 + x_2 - 2x_3) = r\sin\varphi$,

$$\tilde{\mathcal{H}}_{A_2}(x, y; 0, \beta) \;=\; -\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\varphi^2 + \omega^2 r^2 + \frac{9\beta}{r^2 \sin^2 3\varphi}$$

Configuration space: $\varphi \in [0, \frac{\pi}{6}]$, $r \in [0, \infty)$

Wolfes model '75 (G_2-rational model, Olshanetsky-Perelomov '77)

$$\mathcal{H}_{G_2} = \sum_{i=1}^{3}\left(-\frac{\partial^2}{\partial {x_i}^2} + \omega^2 x_i^2\right) + \alpha \sum_{i<j}^{3} \frac{1}{(x_i - x_j)^2}$$
$$+ 3\beta \sum_{k<l,\ \ k,l\neq m}^{3} \frac{1}{(x_k + x_l - 2x_m)^2}$$

After separation of c.m.s. out the relative variables are separated in polar coordinates. $\exists$ two mutually-**non**-commuting integrals of the 2nd and 6th orders: *superintegrability*!

Ground state:

$$\Psi_0 = \prod_{i<j}^{3} |x_i - x_j|^a \prod_{k<l,\ k,l\neq m}^{3} |x_l + x_m - 2x_k|^b e^{-\frac{\omega}{2}({x_1}^2 + {x_2}^2 + {x_3}^2)}$$

where $\alpha = a(a-1)$, $\beta = b(b-1)$

After separation of cms-coordinate $X = x_1 + x_2 + x_3$
in polar (relative) coordinates
$\frac{1}{\sqrt{2}}(x_1 - x_2) = r\cos\varphi$, $\sqrt{\frac{3}{2}}(x_1 + x_2 - 2x_3) = r\sin\varphi$,

$$\tilde{\mathcal{H}}_{G_2}(x, y; \alpha, \beta) \;=\; -\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\varphi^2 + \omega^2 r^2 + \frac{9\alpha}{r^2\cos^2 3\varphi} + \frac{9\beta}{r^2\sin^2 3\varphi}$$

Configuration space: $\varphi \in [0, \frac{\pi}{6}]$, $r \in [0, \infty)$

Modified isotropic 2D-harmonic oscillator in sector $\varphi \in [0, \frac{\pi}{6}]$ *as a configuration space.*

Natural generalization (F.Tremblay, A.T., P.Winternitz '09, **TTW** model):

$$\mathcal{H}_k(x,y;\alpha,\beta) \;=\; -\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\varphi^2 + \omega^2 r^2 + \frac{k^2\alpha}{r^2\cos^2 k\varphi} + \frac{k^2\beta}{r^2\sin^2 k\varphi}$$

- $k=1$ – Smorodinsky-Winternitz
- $k=2$ – BC_2-rational
- $k=3$ – Calogero ($\alpha=0$) and Wolfes ($\alpha\neq 0$)

Ground state:

$$\Psi_0 = r^{k(a+b)}\cos^a k\varphi\,\sin^b k\varphi\; e^{-\frac{\omega}{2}r^2}\;,\;\; E_0 \;=\; 2\omega[k(a+b)+1]\;,$$

where $\alpha = a(a-1)\;,\;\beta = b(b-1)\;.$

(Super)integrability (for integer k)

First integral

$$X_k \;=\; -\partial_\varphi^2 + \frac{k^2\alpha}{\cos^2 k\varphi} + \frac{k^2\beta}{\sin^2 k\varphi}$$

(separability in polar coordinates)

Second integral

$$Y_{2k} \;=\; P_{2k}(\partial_r, \partial_\varphi; r, \varphi)$$

is explicitly known at $k = 1, 2, 3$ and 4 (at $\alpha = 0$ for $k = 1 \ldots 6, 8$)
but for integer $k > 4$?

(Dunkl operators formalism in polar coordinates:
C Quesne, '96 – for $k = 3$ and odd $k > 3$, '10)

For even k – W Miller et al, '10

For rational $k = \frac{p}{q}$ – W Miller et al, '10 - '11

$$Y_{2k} \;=\; P_{2(p+q-1)}(\partial_r, \partial_\varphi; r, \varphi)$$

For all known cases

$$[X_k, Y_{2k}] \neq 0$$

(Super)integrability (for any real $k > 0$)

First integral

$$X_k \;=\; -\partial_\varphi^2 + \frac{k^2\alpha}{\cos^2 k\varphi} + \frac{k^2\beta}{\sin^2 k\varphi}$$

(separability in polar coordinates)

Does a second integral exist?

interesting open question

Exact-solvability (for real $k > 0$)

Make a gauge rotation (similarity transformation)

$$h \;=\; \Psi_0^{-1}(\mathcal{H}_k - E_0)\Psi_0$$

and get the operator

$$h \;=\; -\partial_r^2 + (2\omega r - \frac{2k(a+b)+1}{r})\partial_r - \frac{1}{r^2}\partial_\varphi^2 - \frac{2k}{r^2}(-a\tan k\varphi + b\cot k\varphi)\partial_\varphi$$

for which the lowest eigenfunction is a constant with zero eigenvalue.
Separating variables

$$\Psi_{N,n} \;=\; r^{2nk} L_N^{(k(2n+a+b))}(\omega r^2) P_n^{(a-1/2,b-1/2)}(\cos 2k\varphi)\;\Psi_0$$

$$E_{N,n} \;=\; 2\omega[2(N+nk)+k(a+b)+1]$$

where $N, n = 0, 1, 2, \ldots$.

Degeneracy:

$$N + kn = \text{integer}$$

At $a, b = 0$ it is the spectra of anisotropic harmonic oscillator in $\mathbf{R}^2$

$$V(x,y) \;=\; \omega^2 x^2 + k^2\omega^2 y^2$$

Isotropic oscillator in the sector becomes anisotropic!

Exact-solvability (for integer k)

Make change of variables in h_k

$$t = r^2 \ , \ u = r^{2k} \sin^2 k\varphi$$

- invariants of dihedral group $I_2(k)$.
It takes amazingly simple algebraic form:

$$h_k = -4t\partial_t^2 - 8ku\partial_{tu}^2 - 4k^2 t^{k-1} u\partial_u^2$$
$$+4[\omega t - (a+b)k - 1]\partial_t + [4\omega ku - 2k^2(2b+1)t^{k-1}]\partial_u$$

h_k has infinitely-many finite-dimensional invariant subspaces in polynomials

$$\mathcal{P}_n^{(k)} = (t^p u^q | 0 \leq (p+kq) \leq n) \ , \ n = 0, 1, 2, \ldots$$

Flag:

$$\vec{\alpha} = (1, k)$$

The Lie algebra:

$$J^1 = \partial_t$$

$$J^2_n = t\partial_t - \frac{n}{3}\ ,\ J^3_n = su\partial_u - \frac{n}{3}$$

$$J^4_n = t^2\partial_t + stu\partial_u - nt$$

$$R_i = t^i\partial_u\ ,\ i = 0, 1, \ldots, s\ , \quad \mathcal{R}^{(s)} \equiv (R_0, \ldots, R_s)$$

they span non-semi-simple algebra $gl(2, \mathbf{R}) \ltimes \mathcal{R}^{(s)}$ at $s = 2, 3, \ldots$

S. Lie, $\sim$1890 at $n = 0$, W. Miller '68 and then
A. González-Lopéz, P Olver, N Kamran, '91 at $n \neq 0$
(Case 24, complete classification)

$$\mathcal{P}^{(s)}_n = (t^p u^q | 0 \leq (p + sq) \leq n)$$

common invariant subspace (reducible)

By adding

$$T_0^{(s)} \;=\; u\partial_t^s$$

to $gl(2,\mathbf{R}) \ltimes \mathcal{R}^{(s)}$, the action on $\mathcal{P}_n^{(s)}$ gets irreducible.

Property:

$$T_i^{(s)} = \underbrace{[J^4,[J^4,[\ldots J^4,\, T_0^{(s)}]\ldots]}_{i} = u\partial_t^{s-i} J_0(J_0+1)\ldots(J_0+i-1)\ ,$$

$i = 1,\ldots s$, all $T_i^{(s)}$ of the fixed degree s, $\quad J_0 = t\partial_t \;+\; su\partial_u \;-\; n$

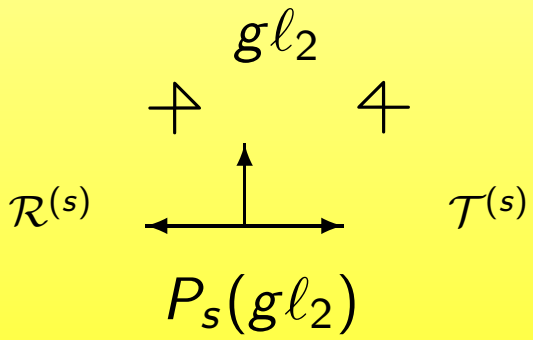
Nilpotency:

$$T_i^{(s)} = 0\ ,\ \ i > s\ .$$

Commutativity:

$$[T_i^{(s)}, T_j^{(s)}] = 0 \ , \quad i,j = 0, \ldots s \ , \quad \mathcal{T}^{(s)} \equiv (T_0^{(s)}, \ldots, T_s^{(s)})$$

Decomposition: $\mathcal{T}^{(s)} \rtimes gl_2 \ltimes \mathcal{R}^{(s)}$



Infinite-dimensional, $(2s+6)$ **generated algebra with** $\mathcal{P}_n^{(s)}$ **irreps space** (for $s = 1 \to gl(3)$)

At $s = k$,

$$-\frac{h_k}{4} = (J_n^2 + 2J_n^3)J^1 + kJ^3 R_{k-1}$$

$$[(a+b)k + 1 + n]J^1 - \omega(J_n^2 + J_n^3) + \frac{k}{6}[2n + 3k(2b+1)]R_{k-1} - \frac{2\omega}{3}$$

Hence, $gl(2, \mathbf{R}) \ltimes \mathcal{R}^{(k)}$ is the hidden algebra,

Generators J^4 as well as $T^{(k)}$'s are absent in representation of h_k, hence, finite-dimensional invariant subspaces of h are finite-dimensional representation spaces of $gl(2, \mathbf{R}) \ltimes \mathcal{R}^{(k)}$.

Such a sequence of linear spaces $\mathcal{P}_n^{(k)}$, $n = 0, 1, \ldots$ forms the *flag* and h_k preserves this flag.

The flag is invariant with respect to weighted-projective transformations:

$$t \to t + A_0$$

$$u \to u + B_k t^k + B_{k-1} t^{k-1} \ldots + B_0$$

algebraic Hamiltonian remains algebraic!

A meaning of this invariance for the Hamiltonian is unclear so far ...

Integrability for integer k (algebraic forms)

There must exist the algebraic form of the integral X_k as well as $\mathcal{Y}_{2k}$ (all common eigenfunctions are polynomials)

Gauge rotation: $\quad x_k \;=\; \Psi_0^{-1}(X_k - c_k)\Psi_0$

in (t, u) coordinates:

$$x_k = -4k^2 u(t^k - u)\partial_u^2 - 4k^2[(b + \frac{1}{2})t^k - (a + b + 1)u]\partial_u$$

where $c_k = k^2(a + b)^2$ is the lowest eigenvalue.

In the generators

$$x_k = -4kJ^3 R_k + 4J^3 J^3 - 4k^2(b + \frac{1}{2})R_k + 4k(a + b)J^3$$

x_k preserves the **same** *flag $(1, k)$ as the Hamiltonian h_k*

♣ Algebraic forms y_{2k} in $x-$space for Y_{2k} known explicitly for $k = 1, 2, 3, 4$ only - they contain $T_0^{(1,2,3,4)}$ generators, respectively.

♣ Evidently, algebraic forms y_{2k} exist for any integer (rational) k - joint eigenfunctions with h_k are polynomials.

♣ y_{2k} *at* $k = 1, 2, 3, 4$ *preserves the* **same** *flag* $(1, k)$ *as for the Hamiltonian* h_k

Is it true for any integer k **?** *What about rational* k *?*

Classical Mechanics

Hamiltonian

$$\mathcal{H}_k(x,y;\alpha,\beta) \;=\; p_1^2 + p_2^2 + \omega^2 r^2 + \frac{k^2\alpha}{r^2\cos^2 k\varphi} + \frac{k^2\beta}{r^2\sin^2 k\varphi}$$

likely superintegrable (yes, for rational k)

$$X_k \;=\; L_3^2 + \frac{k^2\alpha}{\cos^2 k\varphi} + \frac{k^2\beta}{\sin^2 k\varphi}$$

$$Y_{2k} = ?$$

For any real $k > 0$ all bounded trajectories are closed!
(periodic motion with period $\frac{\pi}{2\omega} \to$ isochronicity)

Quasi-Exact-solvability (for integer k)

$$H_k^{(qes)}(x,y;\alpha,\beta) \;=\; -\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\varphi^2 + g^2 r^6 + 2g\omega r^4$$

$$+[\omega^2 - 2g(2\mathcal{N} + 2 + k(a+b))]r^2 + \frac{\alpha k^2}{r^2\cos^2 k\varphi} + \frac{\beta k^2}{r^2 \sin^2 k\varphi}$$

where about $\left(\frac{\mathcal{N}^2}{2k} + 1\right)$ eigenstates are known explicitly (algebraically). Their eigenfunctions have a form of a polynomial $p(t,u) \in \mathcal{P}_{\mathcal{N}}^{(k)}$ multiplied by

$$\Psi_0^{(qes)} \;=\; r^{(a+b)k}\,\cos^a k\varphi\,\sin^b k\varphi\, e^{-\frac{\omega r^2}{2} - \frac{gr^4}{4}} \text{ (ground state at } \mathcal{N} = 0)$$

$gl(2,\mathbf{R}) \ltimes \mathbf{R}^{k+1}$ is hidden algebra.
Integrability? - Yes, X_k remains. Superintegrability? - "Y_{2k}"

Observation:

the operator

$$i_{par}^{(n)}(t,u) \;=\; \prod_{j=0}^{n}(\mathcal{J}^0(n)+j)$$

with

$$\mathcal{J}^0(n) \;=\; t\frac{\partial}{\partial t} + ku\frac{\partial}{\partial u} - n$$

commutes with $h^{(qes)} \;=\; (\Psi_0^{(qes)})^{(-1)} H_k^{(qes)}(x,y;\alpha,\beta)\Psi_0^{(qes)}$,

$$[h^{(qes)}(t,u)\;,\; i_{par}^{(n)}(t,u)] : \; \mathcal{P}_{\mathcal{N}}^{(k)} \;\mapsto\; 0$$

hence, $i_{par}^{(n)}$ is **particular** integral

From Harmonic Oscillator TTW to planar "Coulomb" problem

(S.Post, P.Winternitz '10)

(coupling constant metamorphosis)

$$\mathcal{H}_k^c(x,y;\alpha,\beta) \;=\; -\partial_\rho^2 - \frac{1}{\rho}\partial_\rho - \frac{1}{\rho^2}\partial_\theta^2 - \frac{Q}{\rho} + \frac{k^2\alpha}{4\rho^2\cos^2 k\frac{\theta}{2}} + \frac{k^2\beta}{4\rho^2\sin^2 k\frac{\theta}{2}}$$

Configuration space: $\theta \in [0, \frac{\pi}{k}]$, $\rho \in [0,\infty)$

$$E_{N,n} \;=\; -\frac{Q^2}{[2(N+kn)+1+k(a+b)]^2}\;, \quad N,n = 0,1,2,\ldots$$

Ground state:

$$\Psi_{0,0}^c \;=\; \rho^{(a+b)\frac{k}{2}}\,\cos^a\!\left(k\frac{\theta}{2}\right)\,\sin^b\!(k\frac{\theta}{2})\;e^{-2\sqrt{-E_{0,0}}\;\rho}$$

where $\alpha = a(a-1)$, $\beta = b(b-1)$.

Eigenfunctions:

$$\Psi^c_{N,n} \;=\; \rho^{(2n+a+b)\frac{k}{2}} L_N^{((2n+a+b)k)}(2\sqrt{-E_{N,n}}\,\rho) P_n^{(a-1/2,b-1/2)}(-\cos k\theta)$$

$$\cos^a\left(k\frac{\theta}{2}\right)\,\sin^b(k\frac{\theta}{2})\;e^{-2\sqrt{-E_{N,n}}\;\rho}$$

where $N, n = 0, 1, 2, \ldots$
$(N + kn)$ plays a role of *principal* quantum number

System remains **superintegrable** *like TTW ...*

$k = 1$ - Kalnins E G et al, '96
$k = 2$ - Rodrıguez M A et al, '09

Quasi-Exact-solvability (for integer k) - planar "Coulomb" case

$$H_k^{c,(qes)}(x,y;\alpha,\beta) \;=\; -\partial_\rho^2 - \frac{1}{\rho}\partial_\rho - \frac{1}{\rho^2}\partial_\theta^2 + g^2\rho^2 + 2g\omega\rho$$

$$+\frac{[\omega^2 - 2g(2\mathcal{N} + 2 + k(a+b)) - Q]}{\rho} + \frac{\alpha k^2}{\rho^2\cos^2 k\theta} + \frac{\beta k^2}{\rho^2\sin^2 k\theta}$$

where about $\left(\frac{\mathcal{N}^2}{2k} + 1\right)$ eigenstates are known explicitly (algebraically). Their eigenfunctions have a form of a polynomial $p(t,u) \in \mathcal{P}_{\mathcal{N}}^{(k)}$ multiplied by

$$\Psi_0^{c,(qes)} \;=\; \rho^{(a+b)k}\cos^a\left(k\frac{\theta}{2}\right)\sin^b(k\frac{\theta}{2})\, e^{-\frac{\omega\rho}{2}-\frac{g\rho^2}{4}} \text{ (ground state at } \mathcal{N}=0)$$

$gl(2,\mathbf{R}) \ltimes \mathbf{R}^{k+1}$ is hidden algebra.

From continuous to discrete:

- Algebraic Differential operators h can be **replaced** by Difference Operators on uniform lattice with preservation of the property of existence of polynomial eigenfunctions

$$\frac{d}{dx} \to e^{\frac{d}{dx}} \ , \ x \to xe^{-\frac{d}{dx}}$$

  (it is a Canonical Transformation, Y.F.Smirnov, A.T. '95)
- Integrability is preserved also
- Hidden algebra continues to exist, it remains the **same** Lie algebra but realized by **finite difference** operators.

Similar holds for exponential lattice (C Chryssomalakos, A.T. '01).

Lattices are taken in (t, u) coordinates **not** in Cartesian ones.

CONCLUSION

All known planar integrable and exactly-solvable Hamiltonians with rational potentials are members of a single continuous family!

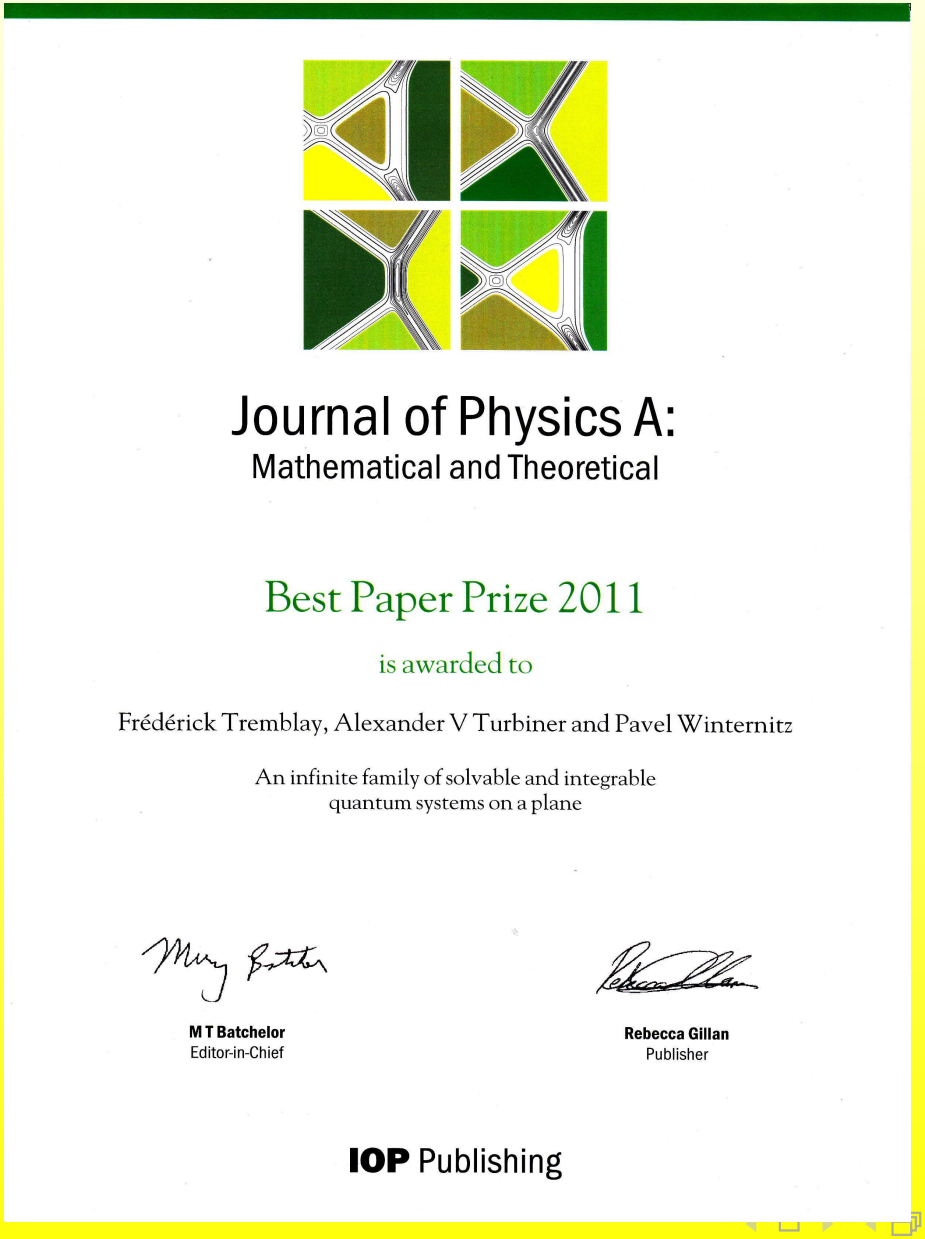
• What about trigonometric ones, BC_2 and G_2, can they also be members of some continuous family? Elliptic BC_2?

• Classical case - Action-Angle representation (Gonera, 2010-11, Lechtenfeld et al, 2011-12)

• What is a meaning of the parameter k??

• Similar inf-dim fin-gen algebras with generalized Gauss decomposition diagram exist at $\mathbf{R}^d, d = 4, 6, 7, 8$

Prize



Journal of Physics A:
Mathematical and Theoretical

Best Paper Prize 2011

is awarded to

Frédérick Tremblay, Alexander V Turbiner and Pavel Winternitz

An infinite family of solvable and integrable
quantum systems on a plane

M T Batchelor
Editor-in-Chief

Rebecca Gillan
Publisher

IOP Publishing

Highlights of 2010

Out ∼1000 papers published at *Journal of Physics A* in 2010 in all branches of physics from fluid mechanics to field/string theory
57 papers were selected.
Out of **14 papers** selected in *Mathematical Physics* branch,
three papers were closely related/inspired by TTW paper:

- *Families of classical subgroup separable superintegrable systems,*
  by E G Kalnins, J M Kress and W Miller Jr
  J. Phys. A: Math. Theor. 43 092001 (2010)
- *Superintegrability of the Tremblay - Turbiner - Winternitz quantum Hamiltonians on a plane for odd k,*
  by C Quesne, J. Phys. A: Math. Theor. 43 082001 (2010)
- *Periodic orbits for an infinite family of classical superintegrable systems,*
  by F Tremblay, A V Turbiner and P Winternitz
  J. Phys. A: Math. Theor. 43 015202 (2010)