Legendre Polynomials and Legendre-Stirling Numbers

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Prelude

- Let $S_n^{(j)}$ denote the *classical Stirling number of the second kind*. This name was coined by Danish mathematician Niels Nielson (1865-1931) in his book *Die Gammafunktion* (Chelsea, New York, 1965).

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- Let $S_n^{(j)}$ denote the *classical Stirling number of the second kind*. This name was coined by Danish mathematician Niels Nielson (1865-1931) in his book *Die Gammafunktion* (Chelsea, New York, 1965).
- James Stirling (1692-1770) discovered properties of these numbers and how they related to Newton series (series of the form

$$f(z) = a_0 + a_1 z + a_2 z(z-1) + a_3 z(z-1)(z-2) + \dots .)$$

In particular,

$$z^1 = z$$
$$z^2 = z + z(z-1)$$
$$z^3 = z + 3z(z-1) + z(z-1)(z-2)$$
$$z^4 = z + 7z(z-1) + 6z(z-1)(z-2) + z(z-1)(z-2)(z-3)$$

etc.

The coefficients above are precisely the Stirling numbers of the second kind.

Picture of the cover of Stirling's 1730 book:

Methodus Differentialis :

SIVE

TRACTATUS

DE

SUMMATIONE

ET

INTERPOLATIONE

SERIERUM INFINITARUM.

AUCTORE *JACOBO STIRLING*, R. S. S.

First table showing Stirling numbers of the second kind - which appears in Stirling's 1730 book:

Tabulam priorem.

1	1	1	1	1	1	1	1	1	&c.
	1	3	7	15	31	63	127	255	&c.
		1	6	25	90	301	966	3025	&c.
			1	10	65	350	1701	7770	&c.
				1	15	140	1050	6951	&c.
					1	21	266	2646	&c.
						1	28	461	&c.
							1	36	&c.
								1	&c.
									&c.

- The classic Laguerre differential expression in Lagrangian symmetric form is

$$\ell[y](x) = \frac{1}{x^{\alpha}e^{-x}}\left(\left(x^{\alpha+1}e^{-x}y'(x)\right)' + kx^{\alpha}e^{-x}y(x)\right);$$

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- With $k = 1$, the n^{th} composite power of this expression is

$$\frac{1}{x^\alpha e^{-x}} \ell^n[y](x) = \sum_{j=0}^{n} (-1)^j \left(S_{n+1}^{(j+1)} x^{\alpha+j} e^{-x} y^{(j)}(x) \right)^{(j)}.$$

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- Question: Why take the n^{th} power of this expression? This is the key point in this lecture and we'll explain 'why' through a study of the classic second-order Legendre differential equation - since the answer will reveal a new set of combinatorial numbers.

Legendre's Differential Equation

Believed to be a portrait of mathematician Adrien-Marie Legendre, and depicted as such in the classic mathematics history books of Eves and Struik ....

Legendre's Differential Equation

.........it was discovered in 2005, by two students at the University of Strasbourg, that it is actually a portrait of Louis Legendre (1755-1799), a figure who participated in the French Revolution. He was no relation to Adrien-Marie Legendre.

Legendre's Differential Equation

Adrien-Marie Legendre (1752-1833)

This caricature is the only known 'image' of A. M. Legendre; it was discovered in the library of the Institut de France in Paris in 2008.

- $\ell[y](x) = -\left((1-x^2)y'(x)\right)' + ky(x)$

  ($k \geq 0$ fixed; $x \in (-1,1)$; we choose $k = 2$)

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  $(k \geq 0$ fixed; $x \in (-1,1)$; we choose $k=2)$
- The r^{th} degree Legendre polynomial $y = P_r(x)$ satisfies

$$\ell[y] = \lambda_r y$$

where $\lambda_r = r(r+1)+2$. $\{P_r\}_{r=0}^{\infty}$ forms a complete orthogonal set in $L^2(-1,1)$.

- E. C. Titchmarsh (1940) - first to analytically study this expression in $L^2(-1,1)$ [*Eigenfunction expansions associated with second-order differential equations I*, Clarendon Press, Oxford, 1962]

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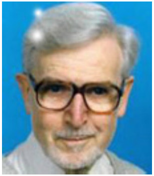
- W. N. Everitt (1980) - discussed the operator theory in $L^2(-1,1)$ and in H_1, the (first) left-definite space [*Legendre polynomials and singular differential operators*, LNM Volume 827, Springer-Verlag, New York, 1980, 83-106]



- $\ell[y] = -\left((1-x^2)y'(x)\right)' + 2y(x)$ is in the limit-circle case at both $x = \pm 1$ in $L^2(-1,1)$ (so two appropriate BC's needed to generate a self-adjoint operator).

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- Define $A : \mathcal{D}(A) \subset L^2(-1,1) \to L^2(-1,1)$ by

$$\begin{aligned}(Af)(x) &= \ell[f](x) \quad (\text{a.e. } x \in (-1,1)) \\ \mathcal{D}(A) &= \{f : (-1,1) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}(-1,1); \\ &\quad f, \ell[f] \in L^2(-1,1); \lim_{x \to \pm 1}(1-x^2)f'(x) = 0\} \\ &= \{f \in \Delta \mid \lim_{x \to \pm 1}(1-x^2)f'(x) = 0\}.\end{aligned}$$

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- Glazman-Krein-Naimark theory $\Rightarrow$ A is self-adjoint in $L^2(-1,1)$, $\{P_m\}_{m=0}^{\infty} \subset \mathcal{D}(A)$, and

$$\sigma(A) = \{m(m+1) + 2 \mid m \in \mathbb{N}_0\}.$$

- For $f, g \in \Delta$, and $[\alpha, \beta] \subset (-1, 1)$, we have **Dirichlet's formula:**

$$\int_\alpha^\beta \ell[f](t)\overline{g}(t)dt$$
$$= -(1-t^2)f'(t)\overline{g}(t) \mid_\alpha^\beta$$
$$+ \int_\alpha^\beta \left((1-t^2)f'(t)\overline{g}'(t) + 2f(t)\overline{g}(t)\right) dt$$

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\end{aligned}$$

- It is tempting (but wrong!) to let $\alpha \to -1$ and $\beta \to +1$; indeed, it is easy to find $f, g \in \Delta$ for which

$$\lim_{x \to -1} (1-t^2)f'(t)\overline{g}(t) \text{ and/or } \lim_{x \to +1} (1-t^2)f'(t)\overline{g}(t)$$

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do not exist.

- However, for $f, g \in \mathcal{D}(A)$, it can be shown that

$$(Af, g) = \int_{-1}^{1} \left((1-t^2)f'(t)\overline{g}'(t) + 2f(t)\overline{g}(t) \right) dt;$$

in particular,

$$(Af, f) \geq 2(f, f) \quad (f \in \mathcal{D}(A))$$

so that A is bounded below by $2I$ in $L^2(-1, 1)$.

Abstract Left-Definite Theory

[L. L. Littlejohn and R. Wellman: *A general left-definite theory for certain self-adjoint operators with applications to differential equations*, J. Differential Equations, 181(2), 2002, 280-339.]

Definition: $H = (V, (\cdot,\cdot))$: Hilbert space; $A : \mathcal{D}(A) \subset H \to H$ self-adjoint and bounded below by kI, $k > 0$; that is, $(Ax, x) \geq k(x, x)$ $(x \in \mathcal{D}(A))$; V_1 linear manifold in V and $(\cdot,\cdot)_1$ is an inner product on $V_1 \times V_1$, and let $H_1 = (V_1, (\cdot,\cdot)_1)$. We say that H_1 is a left-definite space associated with (H, A) if

- (1) H_1 is a Hilbert space

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- (1) H_1 is a Hilbert space
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- (3) $\mathcal{D}(A)$ is dense in H_1
- (4) $(x, x)_1 \geq k(x, x)$ $(x \in V_1)$
- (5) $(x, y)_1 = (Ax, y)$ $(x \in \mathcal{D}(A), y \in V_1)$.

Observation: If A is self-adjoint and bounded below by kI, then A^r is self-adjoint and bounded below by $k^r I$ for each $r > 0$. We can therefore generalize our Definition. We note, however, that the literature contained no examples of "higher" left-definite spaces.

Definition: Let $r > 0$. V_r linear manifold in V and $(\cdot,\cdot)_r$ is an inner product on $V_r \times V_r$. Let $H_r = (V_r, (\cdot,\cdot)_r)$. H_r is a r^{th} *left-definite space* associated with (H, A) if:

(1) H_r is a Hilbert space

(2) $\mathcal{D}(A^r)$ is a subspace of V_r

(3) $\mathcal{D}(A^r)$ is dense in H_r

(4) $(x,x)_r \geq k^r(x,x)$ $(x \in V_r)$

(5) $(x,y)_r = (A^r x, y)$ $(x \in \mathcal{D}(A^r),\, y \in V_r)$.

Of course, existence of H_r is certainly in question at this point. In a sense, the most important property is (5).

Theorem *Suppose* A is a self-adjoint operator in $H = (V, (\cdot,\cdot))$ that is bounded below by kI. Let $r > 0$ and

$$V_r := \mathcal{D}(A^{r/2})$$
$$(x, y)_r := (A^{r/2}x, A^{r/2}y) \quad (x, y \in V_r)$$
$$H_r := (V_r, (\cdot,\cdot)_r).$$

Then H_r is the unique r^{th} left-definite space associated with (H, A). Moreover,

- if A is bounded, then $V = V_r$ and $(\cdot,\cdot)$ and $(\cdot,\cdot)_r$ are equivalent for all $r > 0$.

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- if A is bounded, then $V = V_r$ and $(\cdot, \cdot)$ and $(\cdot, \cdot)_r$ are equivalent for all $r > 0$.
- if A is unbounded, then V_r is a proper subspace of V and, for $0 < r < s$, V_s is a proper subspace of V_r; moreover, none of the inner products $(\cdot, \cdot)$, $(\cdot, \cdot)_r$, or $(\cdot, \cdot)_s$ are equivalent.

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- Moreover, if $\{\phi_n\}$ is a (complete) set of orthogonal eigenfunctions of A in H then they are also a (complete) orthogonal set in each H_r.

Left-definite operators

Definition: Suppose $H = (V, (\cdot, \cdot))$ is a Hilbert space and A is a self-adjoint operator in H that is bounded below by kI. Let $r > 0$ and $H_r = (V_r, (\cdot, \cdot)_r)$ be the r^{th} left-definite space associated with (H, A). If there exists a self-adjoint operator A_r in H_r that is a restriction of A; i.e.

$$A_r x = Ax$$
$$x \in \mathcal{D}(A_r) \subset \mathcal{D}(A),$$

we call A_r an r^{th} left-definite operator associated with (H, A).

Existence of A_r is also at question at this point.

Theorem Suppose A is a self-adjoint operator in $H = (V, (\cdot,\cdot))$ that is bounded below by kI. Let $r > 0$ and let $H_r = (V_r, (\cdot,\cdot)_r)$ be the r^{th} left-definite space associated with (H, A). Define A_r in H_r by

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Then A_r is the unique left-definite operator associated with (H, A). Moreover, $\sigma(A) = \sigma(A_r)$. Furthermore,

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Legendre Left-definite Analysis

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- Pleijel [1975, 1976] was the first to study the Legendre expression $\ell[\cdot]$ in the first left-definite space H_1 generated by the inner product

$$(f,g)_1 = \int_{-1}^{1} \Big((1-t^2) f'(t)\overline{g}'(t) + 2f(t)\overline{g}(t) \Big)\, dt.$$

He first observed that $\ell[\cdot]$ is limit-point at both $x = \pm 1$ in H_1.

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  He first observed that $\ell[\cdot]$ is limit-point at both $x = \pm 1$ in H_1.
- Everitt [1980] continued the study of $\ell[\cdot]$ in H_1 and obtained a self-adjoint operator A_1 in

$$H_1 = \{f : (-1,1) \to \mathbb{C} \mid f \in AC_{\text{loc}}(-1,1); \quad f,\ (1-x^2)^{1/2} f' \in L^2(-1,1)\}$$

  having $\{P_m\}_{m=0}^{\infty}$ as eigenfunctions.

- Everitt, Marić, Littlejohn [2002]: the first left-definite operator A_1 is explicitly given by

$$
\begin{aligned}
(A_1 f)(x) &= \ell[f](x) \quad (\text{a.e. } x \in (-1,1)) \\
\mathcal{D}(A_1) &= \{f : (-1,1) \to \mathbb{C} \mid f, f', f'' \in AC_{\text{loc}}(-1,1); \\
&\qquad (1-x^2)^{3/2} f''' \in L^2(-1,1)\}.
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- What are the left-definite spaces $\{H_r\}$ and left-definite operators $\{A_r\}$ associated with A? Since $\{H_r\}_{r>0}$ and the inner products $(\cdot,\cdot)_r$ are determined from the powers A^r of the A, we can only determine these spaces and operators for $r \in \mathbb{N}$.

  [Everitt, Littlejohn, Wellman: *Legendre polynomials, Legendre-Stirling numbers, and the left-definite spectral analysis of the Legendre differential expression*, J. Comput. Appl. Math.,148, 213-238, 2002.]

Powers of the Legendre Expression & Legendre-Stirling Numbers

- The key: with $\ell[\cdot]$ denoting the Legendre differential expression, we have, for each $n \in \mathbb{N}$,

$$\ell^n[y](x) = \sum_{j=0}^{n} (-1)^j \left(c_j(n)(1-x^2)^j y^{(j)}(x) \right)^{(j)},$$

where, for $j \in \{1, 2, \ldots, n\}$,

$$c_j(n) := PS_{n+1}^{(j+1)}$$

where $PS_n^{(j)}$ is, what we call, a Legendre-Stirling number.

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where $PS_n^{(j)}$ is, what we call, a Legendre-Stirling number.

- These Legendre-Stirling numbers are given explicitly by:

$$PS_n^{(j)} := \sum_{r=1}^{j} (-1)^{r+j} \frac{(2r+1)(r^2+r)^n}{(r+j+1)!(j-r)!} > 0.$$

- For each $n \in \mathbb{N}$, we can compute the n^{th} left-definite space

$$H_n = (V_n, (\cdot,\cdot)_n)$$

associated with the pair $(L^2(-1,1), A)$. Indeed,

$$\begin{aligned} V_n &= \{f \mid f \in AC_{loc}^{(n-1)}(-1,1); (1-t^2)^{n/2} f^{(n)} \in L^2(-1,1)\} \\ &= \mathcal{D}(A^{n/2}) \end{aligned}$$

and

$$(f,g)_n = \sum_{j=0}^{n} c_j(n) \int_{-1}^{1} f^{(j)}(t)\overline{g}^{(j)}(t)(1-t^2)^j dt.$$

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- In particular, we obtain yet another characterization of the domain of A :

$$\mathcal{D}(A) = \{f : (-1,1) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}(-1,1); \\ (1-t^2) f'' \in L^2(-1,1)\}.$$

Combinatorics

[G. E. Andrews, W. Gawronski, L. L. Littlejohn, *The Legendre-Stirling Numbers,* Discrete Math., 311 (2011), 1255-1272]

j/n	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$j=1$	1	1	1	1	1	1	1
$j=2$	–	1	3	7	15	31	63
$j=3$	–	–	1	6	25	90	301
$j=4$	–	–	–	1	10	65	350
$j=5$	–	–	–	–	1	15	140
$j=6$	–	–	–	–	–	1	21
$j=7$	–	–	–	–	–	–	1

Stirling numbers of the second kind (e.g. $S_6^{(4)} = 65$)

j/n	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$j=1$	1	2	4	8	16	32	64
$j=2$	–	1	8	52	320	1936	11648
$j=3$	–	–	1	20	292	3824	47824
$j=4$	–	–	–	1	40	1092	25664
$j=5$	–	–	–	–	1	70	3192
$j=6$	–	–	–	–	–	1	112
$j=7$	–	–	–	–	–	–	1

Legendre-Stirling numbers (e.g. $PS_6^{(4)} = 1092$)

Property	Stirling 2nd Kind	Legendre-Stirling
VRR	$S_n^{(j)} = \sum_{r=j}^{n} S_{r-1}^{(j-1)} j^{n-r}$	$PS_n^{(j)} = \sum_{r=j}^{n} PS_{r-1}^{(j-1)} (j^2+j)^{n-r}$
RGF	$\prod_{r=1}^{j} \frac{1}{1-rx} = \sum_{n=0}^{\infty} S_n^{(j)} x^{n-j}$	$\prod_{r=1}^{j} \frac{1}{1-r(r+1)x} = \sum_{n=0}^{\infty} PS_n^{(j)} x^{n-j}$
TRR	$S_n^{(j)} = S_{n-1}^{(j-1)} + jS_{n-1}^{(j)}$	$PS_n^{(j)} = PS_{n-1}^{(j-1)} + j(j+1)PS_{n-1}^{(j)}$
	$S_n^{(0)} = S_0^{(j)} = 0; S_0^{(0)} = 1$	$PS_n^{(0)} = PS_0^{(j)} = 0; PS_0^{(0)} = 1$
HGF	$x^n = \sum_{j=0}^{n} S_n^{(j)} (x)_j$, where	$x^n = \sum_{j=0}^{n} PS_n^{(j)} \langle x \rangle_j$, where
	$(x)_j = x(x-1)\ldots(x-j+1)$	$\langle x \rangle_j = x(x-2)\ldots(x-(j-1)j))$
1st Kind	$(x)_n = \sum_{j=0}^{n} s_n^{(j)} x^j$	$\langle x \rangle_n = \sum_{j=0}^{n} ps_n^{(j)} x^j$

Zoom in:

Property	Stirling 2nd Kind	Legendre-Stirling
RGF	$\prod_{r=1}^{j} \frac{1}{1-rx} = \sum_{n=0}^{\infty} S_n^{(j)} x^{n-j}$	$\prod_{r=1}^{j} \frac{1}{1-r(r+1)x} = \sum_{n=0}^{\infty} PS_n^{(j)} x^{n-j}$

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- $S_n^{(j)}$ is the number of ways of putting n objects into j non-empty, indistinguishable boxes.

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- For positive integers $p, q \leq n$ and $i, j \in \{1, 2\}$, we say that $p_i > q_j$ if $p > q$.

- To describe what the Legendre-Stirling number $PS_n^{(j)}$ counts, we describe two rules on how to fill $j+1$ 'boxes' with the numbers
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- **Theorem**: For $n, j \in \mathbb{N}_0$ and $j \leq n$, the Legendre-Stirling number $PS_n^{(j)}$ is the number of different distributions according to the above two rules.

  [G. E. Andrews and L. L. Littlejohn, *A Combinatorial Interpretation of the Legendre-Stirling Numbers*, Proc. Amer. Math. Soc., 137(8), 2009, 2581-2590.]