Lemma \[ B \in \mathbb{R}_+^2, \varphi \in \mathbb{R}_+ \]
\[ \Rightarrow \exists \varphi \in \mathbb{R}_+ \quad \text{and} \quad \varphi(\varphi) \in \mathbb{R}_+ \]
\[ -\frac{d\varphi}{dz} \in \mathbb{R}_+ \quad \frac{d^2\varphi}{dz^2} \in \mathbb{R}_+ \]

Prop. Hence \[ \varphi \in \mathbb{R}_+ \]
\[ H \times \varphi \in \mathbb{R}_+ \]

Cor: \[ \overline{B} \in \mathbb{R}^{+2}, \overline{\varphi} \in \mathbb{R}_+ \Rightarrow \overline{B} \times \overline{\varphi} \in \mathbb{R}_+ \]

Thus \[ \varphi \times \varphi \in \mathbb{R}_+ \]

Ex: \[ \varphi(\varphi) = \frac{1}{z-W_1}, \varphi = \frac{1}{(z-W_2)} \]
\[ \varphi \times \varphi = \int_0^1 \frac{1}{z-W_1} \frac{1}{z-W_2} \, dz \]

\[ \mathbb{R} = \text{regular function in } \mathbb{R}^3 \]
\[ \mathbb{R} = \text{discrete set of ring} \]

Thus \[ \varphi \times \varphi(z) = \frac{1}{z-W_1} \left( \int_0^1 \frac{1}{z-W_1} \, dz + \int_0^1 \frac{1}{z-W_2} \, dz \right) \]

Note also the log singularity by convolution of 2 poles.

From now on we assume \[ \mathbb{R} \times \mathbb{R} \]

Corollary Then \[ \varphi \in \mathbb{R}_+ \text{ regular formal series} \]
then also \[ \varphi \times \varphi \in \mathbb{R}_+ \]
Later we will see that also commutates.

\[ \Phi_0(\hat{\Delta} + \hat{\Phi}) = \hat{\Phi} \] is basically since you expand

**Resurgent function**

def. Given \( \mathbb{Z} \) discrete

analytic at 0

has not along any path

**Single singularity**

Lift to the Riemann surface of log \( \sqrt{\tau} \).

\[ e^{i\theta} \in \hat{\mathbb{C}} \quad \text{if} \quad e^{i\theta} \in \hat{\mathbb{C}} \quad \text{where} \quad \theta \in [0, 2\pi) \]

& & \text{for} \quad r > 0
c & & \pi(c) = \pi(c)

\text{(1) The set ANA (from analytic)}

define ANA = \{ (\hat{\Phi}, h) \mid h: R_{(0, +\infty)}, \hat{\Phi}: \mathcal{V}(h) \to \mathbb{C}^{1/2} \}

\text{germ of holomorphic at the origin in} \hat{\mathbb{C}}.

**fan in D, 0 \in D**

f \in \text{ANA if it has a spiral continuation around 0 on} \mathcal{V}(h)

\[ \mathcal{V}(h) = \{ r e^{i\theta} \mid 0 < r < h(\theta)^{1/2} \} \]
Examples of $f$ in $AN$, $\mathbb{A}^{n}$

1) $f \in \mathcal{O}(D^{n}_{\rho})$, $\rho > 0$  
$D^{n}_{\rho}$ is open disk of radius $\rho$

such an $f$ has

$$\sum_{n \in \mathbb{Z}} a_{n} z^{n} = S \left( \frac{1}{z} \right) + R(z)$$

$$\uparrow \quad \updownarrow$$

we make sense of this

2) $f(z) = \frac{1}{2 \pi} \log z$, $z \in \mathbb{C} \setminus \{0\}$ where $\log$ is principal branch

Log: $\mathbb{C} \setminus \{0\} \to \mathbb{C}$

$\log \circ \exp : \mathbb{C} \to \mathbb{C}$

$AN$ is a linear space, contains "e\textsuperscript{\textit{Sf}}" more precisely.

we can embed $\mathbb{C} \setminus \{0\} \to AN$

\[ \text{find an at } 0 \]

$\mathbb{C} \setminus \{0\} = \text{form. can at } 0$

\[ \text{Sing} := AN / e^{\text{Sf}} \]

\[ \text{Sing} : \mathbb{A}^{n} \to \text{Sing} \]

$\text{Sing} : f \mapsto \overline{f} = \text{Sing}(f)$

\[ \text{Def. Any } \overline{f} \text{ representative of } f \text{ is called a } \text{sing. of } f \]

\[ \text{Ex. Sing}(f) \text{ of example 1 } = f \text{ Sing}(\frac{1}{z}) \]

\[ \text{Note. } \mathcal{E} = \text{ Sing} \left( \frac{1}{2 \pi \xi} \right) \in \text{Sing} \]
\[\sigma(k) = \sin \varphi \frac{(-1)^k k!}{2\pi i^k h}, \quad k \geq 0\]

So then in ex. 1: \(\sin \varphi = \sin \varphi \frac{\varphi(\frac{1}{\xi})}{\varphi(\xi)} = \sin \varphi\)

\[= 2\pi i \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\varphi(\xi)}{\varphi(\xi)} \sigma(k)\]

For example 2):
\[
\hat{f}(\xi) = \hat{\varphi} \log \xi \quad \text{(where } \hat{\varphi} \text{ is analytic)}
\]

with \(\hat{f}(\xi) = \hat{\varphi}(\pi(\xi)) \log \xi\)

Let \(\hat{\varphi}(\xi) = \hat{\varphi}(\pi(\xi)) (\log \xi + 2\pi i)\)

then \(\hat{\varphi} = \hat{\varphi}\text{ an. }\sin \varphi \hat{\varphi} = \sin \varphi \hat{\varphi} \equiv \frac{b_{\lambda} \hat{\varphi} = \sin \varphi \hat{\varphi}}{2\pi i}.\)

Structure of \(\sin \varphi\):

In \(\sin \varphi\) there is \(\varphi\) and we have

\[
\left(\hat{\varphi}(\xi) \frac{\log \xi}{2\pi i}\right) \hat{\varphi} = \left(\hat{\varphi}(\xi) \frac{\log \xi}{2\pi i}\right) \hat{\varphi} = \left(\hat{\varphi}(\pi(\xi)) \log \xi + 2\pi i\right) \hat{\varphi} = \varphi \hat{\varphi} \sigma = \hat{f}
\]

Ex. 3: \(I_\sigma(\xi) = \frac{\varphi^{\sigma-1}}{(1 - e^{-2\pi i \sigma}) \Gamma(\sigma)}\)

in \(\sigma \in \mathbb{C} \setminus \mathbb{Z}\)

and in \(\sigma \in \mathbb{Z}\):

\[
I_\sigma(\xi) = \frac{(-1)^k k!}{2\pi i^k h}, \quad \text{so } \sigma = \sigma(k)
\]

For \(k > 0\):

\[
I_k(\xi) = \frac{(-1)^k k!}{2\pi i^k h}
\]

In \(\mathbb{N}\): \(I_n(\xi)\) must be changed by adding an an. part

\[
\lim_{\sigma \to n} \frac{\varphi^{\sigma-1} - \xi^{n}}{(1 - e^{-2\pi i \sigma}) \Gamma(\sigma)} = \frac{\xi^n \log \xi}{(n-1) 2\pi i}
\]

\(= I_n(\xi)\)
\[ I_n := \sin \theta \frac{n \log \xi}{(n\theta)! 2n\theta} \]

\[ \left( \frac{\eta}{\theta} \right)_{\theta \in \mathbb{C}} \] is a family of sing. either poles or logs

We can define Borel transform not only for functions but also for a larger class with \( \mathcal{P} \to \left. I_0 = B(\mathcal{P}) \right. \)

He then restricts to poles and logs

---

**Def. (minors)**

\[ \text{Sing} \to \text{ANA} \]

\[ \tilde{f} \mapsto \tilde{f}(\tilde{\theta}) - \tilde{f}(\tilde{\theta} e^{-2\pi i}) : = \tilde{f}^\vartheta \]

\( \tilde{f} = \text{Van} (\tilde{f}) \) is called the minor of \( \tilde{f} \)

**Ex.** \( \text{Van} (s(\eta)) = 0 \)

**Ex.** \( \text{Van} \left( \frac{s \log \xi}{2n\xi} \right) = \varphi \) if \( \varphi \) is analytic

**Ex.** \( \text{Van} (I_0) = ? = (1 - e^{-2\pi i \theta}) I_0 \)
Motivation: one result/application of alien derivings.

Vector field $\mathbf{X}_B_i = \frac{\partial}{\partial x} + B_1(x, y) \frac{\partial}{\partial y}, \quad i=1, 2$ (don't bne)

where $B_1(x, y) = \sum_{i \geq 0} a_i(x) \delta^i$, $a_i(x) \in \mathbb{C}[z^{-1}]$ $\quad a_0 = \delta(x)$ $\quad a_1 = 1 + \delta(x)$

Has formal sol. $\tilde{\phi}_0$. They show it is meromorphic $\mathbf{X}_B_1$ is an envj to $B_2 \iff$ both give the same $c_1, c_2, c_2, \ldots$

\[ \Delta_{-1} \tilde{\phi}_0 = c_1 \tilde{\phi}_0 \]
\[ \Delta_{-1} \tilde{\phi}_1 = c_2 \tilde{\phi}_2 \]

\[ \frac{\partial}{\partial x} = B_1(x, \phi(z)) \]

\[ * \quad * \quad * \]

\[ \text{SING}^{\text{simp}} = \left\{ f \in \text{SING} \mid f = \lim_{z \to \infty} z^a f(z) \right\}, \quad f(z) = \frac{a}{2\pi i} + \frac{\phi(z)}{2\pi i} \frac{\log z}{2\pi i} + R(z) \]

Where $R$ is analytic $\text{Re}(\phi(z))$

\[ \log z = \text{any branch of log} \]

\[ \mathbb{C}[z^{-1}] \xrightarrow{B} \mathbb{C}[z] \xrightarrow{\phi} \text{SING}^{\text{simp}} \]

\[ a \delta + \phi(z) \mapsto f(z), \quad f(z) = a \cdot \frac{\delta + \phi(z)}{2\pi i} \frac{\log z}{2\pi i} \]
\text{Observation} \\
L (a \delta + \varphi(z)) = a + \int_0^\infty e^{-i \delta \varphi(z)} d\xi \\
= \int_0^\infty e^{-i \varphi(z)} d\xi \\
= \int_0^\infty e^{-i \varphi(z)} f(z) d\xi \\
This is laplace of majors \\
\text{Sin}_{\infty} \subset \text{Sin} \\
\text{Def} \varphi \in \text{Sin}_{\infty} \quad \varphi \text{ even } \varphi \in \text{ANA} \\
\varphi \text{ continuous}