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Correction and linearization of resonant vector fields and diffeomorphisms

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Abstract. We extend the classical Siegel-Brjuno-Rüssmann linearization theorem to the resonant case by showing that under A. D. Brjuno's diophantine condition, any resonant local analytic vector field (resp. diffeomorphism) possesses a well-defined *correction* which (1) depends on the chart but, in any given chart, is unique (2) consists solely of resonant terms and (3) has the property that, when substracted from the vector field (resp. when factored out of the diffeomorphism), the vector field or diffeomorphism thus "corrected" becomes analytically linearizable (with a privileged or "canonical" linearizing map). Moreover, in spite of the small denominators and contrary to a hitherto prevalent opinion, the correction's analyticity can be established by pure combinatorics, without any analysis.

Contents

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1. Introduction

The present paper is a natural sequel to an earlier investigation [E.S.] in so far as it carries on the study of the main "*associates*" of "*local objects*". By *local objects* we mean local vector fields or diffeomorphisms (usually analytic and resonant) and by "*associates*" we mean other natural objects "*functorially*" attached to, or constructed from, these source objects. The most important *associates* are (1) the nilpotent part (2) the correction and corrected forms (3) the normal and prenormal forms (4) the linearizing or normalizing transformations (5) the analytic invariants.

In its original draft, our paper dealt in parallel with two classes of *associates*: the *correction* and *corrected form* on the one hand, and the so-called *continuous prenormal forms* (in particular the "*distinguished*", "*trimmed*", "*regal*", and "*royal*" forms) on the other hand.

This synoptic study had the advantage of highlighting the far- going differences between these two sets of apparently similar notions. However, this resulted in a paper of excessive length and, much to our regret, we had to halve it for publication. So what we present here is simply a study of the *correction* and *corrected form* of resonant objects. The other notions, such as the *nilpotent parts* and *continuous prenormalizations*, get only a cursory mention, for the sake of perspective and contrast.

The *corrected form* (i.e. the object itself *minus its correction*) is constructed by chopping off a suitable resonant part of the source object, so as to restore *formal linearizability*. Relative to any *given* chart, the correction, as a formal object, is unique, but the real challenge is to establish its convergence. Indeed, despite its superficial similarity with the *continuous prenormal forms*, which are generically divergent, and despite the presence of small denominators, we prove (under the probably optimal diophantine condition introduced by A. D. Brjuno) that the *correction* is always *analytic* and that the *corrected form* that goes with it is *analytically linearizable*. In fact, as we argue towards the end of this section (Remark 1), this statement may be viewed as the natural extension to the resonant case of the classical Siegel-Brjuno-Rüssmann linearization theorem. Furthermore, among all the analytic coordinate changes which linearize the *corrected form*, there exists one which is indisputably "canonical" and which turns out to be closely related to the "royal form", which itself is the simplest continuous prenormal form.

As usual in these questions, the investigation splits into a formal and an analytic part. The formal preparation involves a lot of tedium (with notions like *moulds* and *mould expansions*, which many find unsavoury and would gladly do without) but it should not be despised nor neglected. Indeed, when properly conducted, the formal preliminaries, tedious though they are, can spare us a lot of *spurious analytic complications* which, if anything, are ten times more tedious. In the present instance, a careful formal investigation reveals that there is no need at all for the drudgery of *small denominator compensation*, simply because the “highly multiple” small denominators, which were thought to be a central difficulty inherent in the question, *do not exist at all*: they merely *seem to be there*, because faulty (or rather uneconomical) modes of calculations *seem to introduce them*.

But before starting the investigation, a few reminders about *local objects* (this section) and moulds (next section) are in order, along with some useful terminology.

Local objects and their homogeneous components

We shall be dealing with analytic local objects of two sorts, i. e. on the one hand, with local analytic vector fields (or fields for short) on $\mathbf{C}^\nu$ at 0:

$$X = \sum_{1\leq i\leq \nu} X_i(x)\partial_{x_i} \qquad (X_i(0)=0;\ X_i(x)\in \mathbf{C}\{x\}) \tag{1.1}$$

and, on the other hand, with local analytic self-mappings (or diffeos, short for diffeomorphisms) of $\mathbf{C}^\nu$ with 0 as fixed point:

$$f : x \to f_i(x) \qquad (1\leq i\leq \nu;\ f_i(0)=0; f_i(x)\in \mathbf{C}\{x\}) \tag{1.2}$$

or equivalently, with the related substitution operators (capital-lettered):

$$F : \varphi \to F\cdot\varphi \overset{\text{def}}{=} \varphi\circ f \quad (\varphi(x) \text{ and } \varphi\circ f(x)\in \mathbf{C}\{x\}). \tag{1.3}$$

All along, we shall assume the linear part to be diagonalizable, and work with “prepared forms”, i. e. consider analytic charts where the linear part is diagonal.

Thus, we shall consider fields of the form:

$$X = X^{\text{lin}} + \sum_n B_n \tag{1.4}$$

$$X^{\text{lin}} = \sum_{i=1}^{\nu} \lambda_i x_i \partial_{x_i} \quad (\lambda_i\in \mathbf{C}) \tag{1.4*}$$

$$B_n = B_{n_1,\ldots,n_\nu} = \text{ homogeneous part of degree } n (n_i\geq -1) \tag{1.4**}$$

and diffeos of the form:

$$F = \left\{1+\sum_n B_n\right\} F^{\text{lin}} \tag{1.5}$$

$$(1.5^*) \qquad \begin{cases} F^{\mathrm{lin}} \cdot \varphi(x_1,\ldots,x_\nu) \stackrel{\mathrm{def}}{=} \varphi(l_1 x_1,\ldots,l_\nu x_\nu) \\ \quad (l_i \in \mathbf{C}^*;\ \varphi(x) \in \mathbf{C}\{x\}) \end{cases}$$

$$(1.5^{**}) \qquad B_n = B_{n_1,\ldots,n_\nu} = \text{ homogeneous part of degree } n (n_i \geq -1)$$

Of course, "n-homogeneous" means that for each monomial x^m, we have:

$$(1.6) \qquad \begin{cases} B_n \cdot x^m = \beta_{n,m} x^{n+m} \quad \text{with } \beta_{n,m} \in \mathbf{C}; \\ x^m = \prod_i x_i^{m_i}; x^n = \prod_i x_i^{n_i}\ . \end{cases}$$

Each B_n is a differential operator of order one (for a field) or of some finite order (for a diffeo). Note that, for any given B_n, at most one component n_i may assume the value -1.

Resonance, quasiresonance, nihilence

The eigenvalues λ_i or l_i will be referred to as the object's *multipliers* or, collectively, as its spectrum. We say that the local object (field or diffeo) is *resonant*, if there exist non-trivial relations of the form:

$$(1.7) \qquad \sum_{1 \leq i \leq \nu} m_i \lambda_i = 0 \text{ or } \lambda_j \quad (m_i \in \mathbf{N})$$

$$(1.8) \qquad \prod_{1 \leq i \leq \nu} (l_i)^{m_i} = 1 \text{ or } l_j \quad (m_i \in \mathbf{N})$$

We say that the object is *quasiresonant* if, among all the non-vanishing expressions $\alpha(m) = \langle m, \lambda \rangle$ or $\alpha(m) = l^m - 1$ (with coefficients $m_i \geq 0$ except at most one that may be equal to -1) there is a subinfinity that goes to 0 "abnormally fast", thus violating the two equivalent diophantine conditions:

$$(1.9) \qquad S \stackrel{\mathrm{def}}{=} \sum_k 2^{-k} \log\left(\frac{1}{\varpi(2^k)}\right) < +\infty \quad \text{(A.D.Brjuno)}$$

$$(1.9^*) \qquad S^* \stackrel{\mathrm{def}}{=} \sum_k k^{-2} \log\left(\frac{1}{\varpi(k)}\right) < +\infty \quad \text{(H.Rüssmann)}$$

where

$$\varpi(k) \stackrel{\mathrm{def}}{=} \inf|\alpha(m)|,$$

with an inf over all m such that $m_1 + \ldots + m_k \leq k$ and $\alpha(m) \neq 0$.

Lastly, *nihilence* (which presupposes resonance) amounts to the existence of a “first integral” in the guise of a (formal) power series $H(x) \in \mathbf{C}[[x]]$ with the invariance property:

$$X \cdot H(x) = 0 \quad \text{(for a field)} \tag{1.10}$$

$$H \circ f(x) \stackrel{\text{def}}{=} F \cdot H(x) = H(x) \quad \text{(for a diffeo).} \tag{1.10*}$$

Resonance (resp. quasiresonance or nihilence) complicates the study of local objects by creating “*vanishing denominators*” (resp. “*small denominators*”).

With diffeos an additional, though more harmless complication may enter the picture, namely the *torsion* t, where t is the well-defined natural integer such that:

$$(1/t)(2\pi i\mathbf{Z}) = (2\pi i\mathbf{Q}) \cap \left\{ (2\pi i\mathbf{Z}) \underset{1\leq j\leq \nu}{\oplus} ((\log l_j)\mathbf{Z}) \right\} \tag{1.11}$$

(Note that (1.11) makes sense, since the set on the right-hand side does not depend on the actual determination chosen for $\log l_i$). To simplify, we shall restrict ourselves to *torsion-free* diffeos, i. e. diffeos with $t = 1$. Such diffeos are characterized by the fact that their spectrum $\{l_i\}$ has a (non-unique) *coherent logarithm* $\{\lambda_i = \log l_i\}$, i. e. a set $\{\lambda_i\}$ such that each multiplicative resonance relation $l^m = 1$ translates into an additive resonance relation $\langle m, \lambda\rangle = 0$. Clearly, torsion-affected diffeos have torsion-free iterates of order t, so that, for all intents and purposes, studying the torsion-free case is enough.

The nilpotent part of resonant objects

For resonant vector fields X as in (1.4), there is a classical (see [B]) decomposition:

$$X = X^{\text{dia}} + X^{\text{nal}} \quad (X^{\text{dia}} \text{ and } X^{\text{nal}} \text{ are formal fields}) \tag{1.12}$$

$$[X, X^{\text{dia}}] = [X, X^{\text{nal}}] = [X^{\text{dia}}, X^{\text{nal}}] = 0 \tag{1.12*}$$

into a “*diagonalizable part*” X^{dia} and a “*nilpotent part*” X^{nal}. This decomposition is intrinsic (i.e. chart- independent) and fully characterized by the fact that X^{dia} is (formally) linearizable while X^{nal}, on the contrary, has no linear component.

Similarly, any resonant diffeo F as in (1.5) has an intrinsic decomposition:

$$F = F^{\text{die}} F^{\text{nel}} = F^{\text{nel}} F^{\text{die}} \quad (F^{\text{die}} \text{ and } F^{\text{nel}} \text{ are formal diffeos}) \tag{1.13}$$

into a (formally) linearizable part F^{die} and an identity-tangent part F^{nel}.

Remark. Except for the unproblematic *linear parts* X^{lin} and F^{lin}, the objects attached to vector fields (resp. diffeos) will carry the vowel **a** (resp. the vowel **e**). Thus we will have X^{nal}, X^{pran}, X^{carr} etc. as against F^{nel}, F^{pren}, F^{cerr} etc. The reason for this convention lies in the nature of the moulds involved in both cases (see Sect. 2).

Continuous prenormalization versus discontinuous normalization

Though simplest in terms of outward shape, the normal forms X^{narm} and F^{nerm} have definite drawbacks. One is the unavoidably non-continuous nature of the maps:

$$X \to X^{\text{narm}}; \quad F \to F^{\text{nerm}} \tag{1.16}$$

even when we keep the linear parts X^{lin} and F^{lin} *fixed*. Another is the absence, thus far, of general and truly algorithmic procedures for determining them, especially in the case of highly resonant spectra. (See [Be] and [Bai]). A third drawback (operative even in the case of *simple* resonance) is their unsuitability for mechanical computation: the exact shape of the *normal forms* always depends on one or several *discrete invariants* (such as the "*levels*" p.See [E.1]) whose exact value, in turn, depends on whether *certain real or complex numbers* (which depend polynomially on the Taylor coefficients of the object) *do or do not vanish* – a matter which a computer clearly cannot decide.

So, for many purposes, it is preferable to work with *continuous prenormal* forms or rather, to be quite precise, with *prenormal forms which depend continuously on the homogeneous components B_n of the object, whose linear part is kept fixed* (continuous dependence on the linear part would be an obviously impossible demand).

In concrete terms, a *continuous prenormalization* is a universal correspondence of the form:

$$\text{(1.16}^*\text{)}\quad \begin{cases} X = X^{\text{lin}} + \sum B_n \to X^{\text{pran}} = X^{\text{lin}} + \sum \text{Pran}^\bullet B_\bullet \\ \quad = X^{\text{lin}} + \sum \text{Pran}^{\omega_1,\ldots,\omega_r} B_{n_r} \ldots B_{n_1} \end{cases}$$

$$\text{(1.16}^{**}\text{)}\quad \begin{cases} F = \left\{1 + \sum B_n\right\} F^{\text{lin}} \to F^{\text{pren}} = \left\{\sum \text{Pren}^\bullet B_\bullet\right\} F^{\text{lin}} \\ \quad = \left\{\sum \text{Pren}^{\omega_1,\ldots,\omega_r} B_{n_r} \ldots B_{n_1}\right\} \end{cases}$$

which is entirely determined by a given family, $\text{Pran}^\bullet$ or $\text{Pren}^\bullet$, of scalar coefficients depending on sequences made up of the "frequencies" $\omega_i = \langle n_i, \lambda\rangle$.

Although there exist infinitely many continuous prenormalizations (for case-studies see [E. S.], [E. V. 1], [E. V. 2]), one of them, the so-called "royal" prenormalization, stands out as "canonical". A detailed investigation of its properties has already appeared in [E. V. 0] and another one is forthcoming in [E. V. 2], but its definition will be mentioned in Sect. 6, infra, because of its relevance to the canonical linearization of *corrected objects.*

The correction and corrected form

To any resonant vector field X, we associate a "*correction*" X^{carr} and a "*corrected form*" X^{carrd}:

$$X \to X^{\mathrm{carrd}} = X - X^{\mathrm{carr}}. \tag{1.17}$$

The *correction* is nilpotent (i.e. without linear part) and consists only of resonant terms:

$$[X^{\mathrm{lin}}, X^{\mathrm{carr}}] = 0 \tag{1.17*}$$

while the *corrected form* is required to be linearizable under some change of coordinates Θ_{carr}

$$X^{\mathrm{carrd}} = \Theta_{\mathrm{carr}} X^{\mathrm{lin}} \Theta_{\mathrm{carr}}^{-1} \tag{1.17**}$$

It is an easy matter to check (see Sect. 4) that, for any given chart, X^{carr} and X^{carrd} are uniquely characterized by the combination of (1.17), (1.17*), (1.17**) and that both objects are chart-dependent. On the other hand, the *corrected form* is *not* a prenormal form, and we shall see that its analytic properties are strikingly different from those of prenormal forms.

For resonant diffeos F, we have *one* correction but *two* corrected forms, right and left:

$$F \to F^{\mathrm{cerrd}} = (F^{\mathrm{cerr}})^{-1} \cdot F \tag{1.18}$$

$$F \to F^{\mathrm{cerrd}^*} = F \cdot (F^{\mathrm{cerr}})^{-1} \tag{1.19}$$

but the difference is immaterial, since F^{cerrd} and F^{cerrd^*} are conjugate under F^{cerr}. We shall stick with the definition (1.18), supplemented by the resonance requirement for the *correction*:

$$F^{\mathrm{cerr}} F^{\mathrm{lin}} = F^{\mathrm{lin}} F^{\mathrm{cerr}} \quad (F^{\mathrm{cerr}} = 1 + \ldots) \tag{1.18*}$$

and by the linearizability condition for the *corrected form*:

$$F^{\mathrm{cerrd}} = \Theta_{\mathrm{cerr}} F^{\mathrm{lin}} \Theta_{\mathrm{cerr}}^{-1} \tag{1.18**}$$

Main analytic results F_1, F_2, F_3, F_4

(F_1) Analytic linearization
In the absence of resonance *and* quasiresonance, analytic objects are analytically linearizable. (See [B], [R]).

(F_2) Ramified-seriable linearization
Under so-called *ramified-seriable* changes of coordinates (which involve fractional powers of the variables but share with analytic changes of coordinates the property of unique summability), *even resonant or quasiresonant* objects can be effectively linearized in spiral-like domains of optimal size. Underpinning this possibility is the tantalizing phenomenon of *compensation*. See [E.2] and [E.3].

(F_3) Resonance-induced resurgence
For resonant, but non-quasiresonant objects, all the *relevant associated objects* tend to be divergent *and* resurgent (meaning of course that they involve divergent power series which happen to be *resurgent* relative to certain well-specified variables), but the actual pattern of resurgence depends a good deal on the nature of the associated objects:

(F_3') For the *intrinsic* (i.e. chart-independent) *associated objects* (such as the "*formal integral*" $x(z,u)$; the *nilpotent parts* X^{nal} or F^{nel}; the *normalizing change of coordinates* Θ_{nar} and $\Theta_{\mathrm{nar}}^{-1}$; the *fractional iterates of diffeos*; etc.) the resurgence involved is of "canonical type". It is governed by the so-called Bridge Equation:

$$\dot{\Delta}_\omega\, x(z,u) = A_{\boldsymbol{\omega}} x(z,u) \quad (\dot{\omega} \in \Omega) \tag{1.19}$$

written here in connection with the formal integral:

$$x(z,u) = \{x_1(z,u_1,\ldots,u_{\nu-1}),\ldots,x_\nu(z,u_1,\ldots,u_{\nu-1})\} \tag{1.20}$$

$$\partial_z x_i(z,u) = X_i(x(z,u)) \quad \text{(for fields)}\ (1 \le i \le \nu) \tag{1.20*}$$

$$x_i(z+1,u) = f_i(x(z,u)) \quad \text{(for diffeos)}\ (1 \le i \le \nu). \tag{1.20**}$$

Roughly speaking, the Bridge Equation says that applying an *alien derivation* $\dot{\Delta}_\omega$ to the resurgent object at hand is equivalent to applying an ordinary partial differential operator $A_{\boldsymbol{\omega}}$. Moreover, the operators $A_{\boldsymbol{\omega}}$ in question happen to be *analytic invariants* of the original object X or F. Lastly, the "resurgence lattice" (i.e. the set of all indices $\boldsymbol{\omega}$ such that $\dot{\Delta}_\omega$ may act non-trivially) happens in this case to be the *canonical resurgence lattice* Ω spanned by the multipliers $\lambda_1,\ldots,\lambda_\nu$ (with addition of the universal component $2\pi i\mathbf{Z}$ for diffeos). See [E.1], [E.3] etc.

(F_3'') For the main *non-intrinsic associated objects*, on the other hand, such as the "remarkable" prenormal forms, we still have resurgence, but of a very special sort: it no longer involves the holomorphic invariants A_ω; it is of "rigid type"; and the relevant "resurgence lattices" are often more intricate than Ω (See [E. V. 1], [E. V. 2]).

(F_4) Analytic linearizability of the corrected form
The corrected form X^{carrd} or F^{cerrd} of resonant objects stands apart from most other "associated objects" (both intrinsic or non-intrinsic) since it displays neither divergence nor resurgence. More precisely, under the sole diophantine hypothesis of Brjuno, it can be shown to be both *analytic* and *analytically linearizable* (See Sect. 4, Sect. 5 and Sect. 6 below).

The present paper is devoted to the *correction*, the *corrected form*, and the latter's *canonical* linearization. The follow-up paper [E. V. 2] will deal with the four most remarkable *prenormal forms* ("distinguished"; "trimmed"; "regal"; "royal"). The *nilpotent part* and its divergence-resurgence properties was investigated in a previous paper [E. S.]. Lastly, the *formal integral* and the *analytic invariants* are the subject-matter of an entire book [E. 1].

Remark 1. What survives of the linearization theorem in the resonant case?

Since Brjuno's diophantine condition (1.9) still makes sense for resonant objects, and may or may not be fulfilled, *the question arises as to its implications when it is fulfilled*. Clearly, the answer cannot lie in the convergence of the linearizing maps, since even formal linearization is generally impossible in that case. Nor can it lie in the convergence of the normalizing or prenormalizing maps, since these are known to be generically divergent (see (F_3')). So what? Well, it would seem that the above-mentioned theorem (F_4) about the *analyticity of the correction, and analytical linearizability of the corrected form, provides what is probably the closest and most natural counterpart, in the resonant case, of the classical linearization theorems pioneered by C.L. Siegel and improved upon by A.D. Brjuno (for vector fields) and H. Rüssmann (for diffeos)*.

Remark 2. Earlier work by G. Gallavotti and L. H. Eliasson

So far as we know, the notion of "correction" was first introduced (under the name of *Wick invariant* and for *Hamiltonian vector fields*) in [G.] by the theoretical physicist G. Gallavotti. He was motivated by an analogy with renormalization in quantum field theory, and conjectured the correction's analyticity. Prior to us, L. H. Eliasson established this analyticity in a series of recent articles [E.1], [E.2], [E.3], but only for the correction of vector fields (not diffeomorphisms) and under unnecessarily restrictive diophantine conditions (Siegel's rather than Brjuno's). These pioneering papers by Eliasson found a large echo, especially among theoretical physicists, because at the time they represented the first and only attempt at a purely *analytical* or *di-*

rect treatment of the small denominators of classical mechanics and KAM theory. But the main difference between his and our approach lies in this: L. H. Eliasson was under the impression that small denominators with "high multiplicities" were really present, and he had to resort to hard analysis to show that they somehow "compensated" each other. We, on the other hand, establish by purely algebraic manipulations that these small denominators actually do not exist: they are an optical illusion, as it were, and so the question of compensating them does not arise. Since then, we had several exchanges with L.H. Eliasson, who read our first draft and passed useful comments. We wish to thank him for that.

Remark 3. Why bother about the "correction"?

Let us consider vector fields for definitness. Since the correction X^{carr} of a field X depends on the chart, it is sometimes dismissed as an "artificial" notion, of little or no significance for the dynamics of vector fields. This is a gross misconception, which two simple examples will suffice to dispel.

Fix two local vector fields X and Y with *identical linear parts* and assume that Y is linearizable. X and Y may be anything but, to avoid trivial situations, think of them as being *resonant*. Use "$A \sim B$" as short-hand for "A formally conjugate to B" and consider these two problems:

Problem 1. Find a local field Z_1 such that

$$X - Z_1 \sim Y \tag{1.21}$$

$$[X, Z_1] = 0 \tag{1.21*}$$

Problem 2. Find a local field Z_2 such that

$$X - Z_2 \sim Y \tag{1.22}$$

$$[Y, Z_2] = 0 \tag{1.22*}$$

It so happens that each of these problems always admits a unique formal solution, Z_1 or Z_2, which is obviously chart-invariant. In other words, the fields Z_1 and Z_2 are "intrinsic" or "geometric" functions of the *pair* (X, Y) and, as such, they have a lot to say about the *joint dynamics* of X *and* Y.

If we now place ourselves in a chart where the *linearizable* field Y actually *is* linear, we see at once that in that chart:

$$Y = X^{\mathrm{lin}}; Z_1 = X^{\mathrm{nal}}; Z_2 = X^{\mathrm{carr}} \tag{1.23}$$

Thus, despite the deceptive similarity, Problem 1 is akin to the search for the nilpotent part, and its solution Z_1 is, generally speaking, *divergent-resurgent*; while Problem 2 is akin to the search for the correction, and its solution Z_2 is always (under Brjuno's diophantine condition) *convergent*.

That said, it would be easy to generalize both examples, and to produce two long lists, say List 1 and List 2, of Lie-bracket equations or systems, involving local vector fields and exhibiting *exactly* the same features as, respectively, Problem 1 and Problem 2.

Summing up, we may say that the "correction" derives its importance from three facts:

(i) when considered in its proper setting, it is a perfectly geometric notion;
(ii) it is the simplest representative of a whole class ("List 2") of similar problems;
(iii) it is the most elementary object where we come across the already hinted-at "non-appearance of multiple small denominators", and therefore an ideal test-bench for the study of this fascinating phenomenon.

2. Reminder about moulds, amplification and arborification

The operations ($+$, $\times$, $\circ$) on moulds

A *mould* $M^\bullet$ is a family of elements $M^{\boldsymbol{\omega}}$ of a given commutative ring or algebra, with upper indexation by sequences $\boldsymbol{\omega} = (\omega_1, \dots, \omega_r)$, which include the " empty sequence" $\varnothing$. These sequences have arbitrary lengths $r = r(\boldsymbol{\omega}) \geq 0$ and their components ω_i range over a set Ω that may be any abelian group or semigroup. Throughout, we use the notation:

$$\|\boldsymbol{\omega}\| = \omega_1 + \ldots + \omega_r \text{ if } \boldsymbol{\omega} = (\omega_1, \dots, \omega_r) \tag{2.1}$$

and denote by $\boldsymbol{\omega}^1\boldsymbol{\omega}^2$ the *juxtaposition* of two sequences $\boldsymbol{\omega}^1$ and $\boldsymbol{\omega}^2$. For greater clarity, we use *bold-face* with *upper indexation for sequences* ($\boldsymbol{\omega}$, $\boldsymbol{\omega}^j$ etc.) and *italic characters* with *lower indexation* for their components (ω_i, ω_i^j etc.).

The *addition* $A^\bullet + B^\bullet$ on moulds is trivially defined, while the *multiplication*

$$A^\bullet \times B^\bullet = C^\bullet \quad \text{(non-commutative but associative)}$$

and *composition*

$$A^\bullet \circ B^\bullet = D^\bullet \quad \text{(also non-commutative but associative)}$$

are defined by:

$$C^{\boldsymbol{\omega}} = \sum_{\boldsymbol{\omega}^1\boldsymbol{\omega}^2=\boldsymbol{\omega}} A^{\boldsymbol{\omega}^1} B^{\boldsymbol{\omega}^2} \tag{2.2}$$

$$D^{\boldsymbol{\omega}} = \sum_{\substack{\boldsymbol{\omega}^1\boldsymbol{\omega}^2\ldots\boldsymbol{\omega}^s=\boldsymbol{\omega} \\ 1\leq s;\boldsymbol{\omega}^i\neq\varnothing}} A^{\|\boldsymbol{\omega}^1\|,\|\boldsymbol{\omega}^2\|,\ldots,\|\boldsymbol{\omega}^s\|} \quad B^{\boldsymbol{\omega}^1} B^{\boldsymbol{\omega}^2} \ldots B^{\boldsymbol{\omega}^s} \tag{2.3}$$

Observe that the summation rule in (2.2) allows for only two factor sequences $\boldsymbol{\omega}^1$, $\boldsymbol{\omega}^2$, any of which may turn empty, so that the sum includes the terms $A^{\varnothing}B^{\boldsymbol{\omega}}$ and $A^{\boldsymbol{\omega}}B^{\varnothing}$ at both ends; whereas the summation prescription in (2.3) allows for more than two factor sequences, but rules out empty factors $\boldsymbol{\omega}^i$.

The arch-trivial moulds $1^\bullet$ and $I^\bullet$ defined by:

$$1^{\varnothing} = 1;\ 1^{\omega_1,\dots,\omega_r} = 0 \quad \text{if} \quad r \geq 1 \quad (\forall \omega_i) \tag{2.4}$$

$$I^{\varnothing} = 0;\ I^{\omega_1} = 1 \quad (\forall \omega_1);\ I^{\omega_1,\dots,\omega_r} = 0 (\forall r \geq 2, \forall \omega_i) \tag{2.5}$$

clearly behave as *units* for mould multiplication and mould composition respectively.

Mould-comould contractions

Let the "object" Ob be either a local vector field Y without linear part:

$$\text{Ob} = Y = \sum B_n \quad (B_n \text{ homogeneous}) \tag{2.6}$$

or a local, identity-tangent diffeo G:

$$\text{Ob} = G = 1 + \sum B_n \quad (B_n \text{ homogeneous}). \tag{2.7}$$

To any such Ob we associate an operator-valued "comould" $B_\bullet$ by putting:

$$B_{\varnothing} \overset{\text{def}}{=} 1; B_{n_1,\dots,n_r} \overset{\text{def}}{=} B_{n_r} \dots B_{n_2} \cdot B_{n_1} \tag{2.8}$$

and we let moulds $M^\bullet$ with indices ω_i in $\mathbf{C}$ act on these Ob through *mould-comould contraction*:

$$\text{Act}^{M^\bullet}(\text{Ob}) \overset{\text{def}}{=} \sum M^\bullet B_\bullet = \sum_{r \geq 0} M^{\omega_1,\dots,\omega_r} B_{n_1,\dots,n_r} \tag{2.9}$$

Obviously, the action of $M^\bullet$ is chart-dependent, and relative to a *fixed* scalar product:

$$n_1 \to \langle n_1, \lambda \rangle = \omega_1, \dots, n_r \to \langle n_r, \lambda \rangle = \omega_r. \tag{2.9*}$$

Moreover, we have:

$$(\text{Act}^{M^\bullet} \cdot \text{Act}^{N^\bullet})\text{Ob} = \text{Act}^{M^\bullet \circ N^\bullet} \cdot \text{Ob} \tag{2.10}$$

$$(\text{Act}^{M^\bullet}\text{Ob})(\text{Act}^{N^\bullet}\text{Ob}) = \text{Act}^{N^\bullet \times M^\bullet} \cdot \text{Ob}. \tag{2.11}$$

(Note the order reversal in (2.11), but not in (2.10)).

Useful moulds tend to display special “symmetries”. Four classes in particular stand out: they are the *symmetral*, *alternal*, *symmetrel*, *alternel* moulds.

Their quickest characterization is in terms of their action on local objects:

(2.12) *symmetral* turns *field* into *diffeo*

(2.13) *alternal* turns *field* into *field*

(2.14) *symmetrel* turns *diffeo* into *diffeo*

(2.15) *alternel* turns *diffeo* into *field.*

The analytical translation of these properties is as follows:

(2.16)
$$(M^\bullet \text{ symmetral}) \Rightarrow M^{\boldsymbol{\omega}^1} M^{\boldsymbol{\omega}^2} = \sum M^{\boldsymbol{\omega}} \quad (\boldsymbol{\omega} \in \mathrm{sh}(\boldsymbol{\omega}^1, \boldsymbol{\omega}^2))$$

(2.17)
$$(M^\bullet \text{ alternal}) \Rightarrow 0 = \sum M^{\boldsymbol{\omega}} \quad (\boldsymbol{\omega} \in \mathrm{sh}(\boldsymbol{\omega}^1, \boldsymbol{\omega}^2))$$

(2.18)
$$(M^\bullet \text{ symmetrel}) \Rightarrow M^{\boldsymbol{\omega}^1} M^{\boldsymbol{\omega}^2} = \sum M^{\boldsymbol{\omega}} \quad (\boldsymbol{\omega} \in \mathrm{ctsh}(\boldsymbol{\omega}^1, \boldsymbol{\omega}^2))$$

(2.19)
$$(M^\bullet \text{ alternel}) \Rightarrow 0 = \sum M^{\boldsymbol{\omega}} \quad (\boldsymbol{\omega} \in \mathrm{ctsh}(\boldsymbol{\omega}^1, \boldsymbol{\omega}^2))$$

All sums are finite. The identities hold for any pair $\boldsymbol{\omega}^1, \boldsymbol{\omega}^2$, except that in (2.17) and (2.19) we must assume $\boldsymbol{\omega}^1 \neq \varnothing$, $\boldsymbol{\omega}^2 \neq \varnothing$. The abbreviations *sh* and ctsh stand for “*shuffling*” and “*contracting shuffling*”. More precisely, $\mathrm{sh}(\boldsymbol{\omega}^1, \boldsymbol{\omega}^2)$ is the set of all sequences $\boldsymbol{\omega}$ that can be obtained by intermingling the sequences $\boldsymbol{\omega}^1$ and $\boldsymbol{\omega}^2$ under preservation of their internal order; while $\mathrm{ctsh}(\boldsymbol{\omega}^1, \boldsymbol{\omega}^2)$ contains all these sequences $\boldsymbol{\omega}$, plus those that can be derived by contracting (pairwise) adjoining elements from $\boldsymbol{\omega}^1$ and $\boldsymbol{\omega}^2$.

Thus, if $\boldsymbol{\omega}^1 = (\omega_1)$ and $\boldsymbol{\omega}^2 = (\omega_2, \omega_3)$, the set $\mathrm{sh}(\boldsymbol{\omega}^1, \boldsymbol{\omega}^2)$ contains

$$(\omega_1, \omega_2, \omega_3),\ (\omega_2, \omega_1, \omega_3),\ (\omega_2, \omega_3, \omega_1),$$

while $\mathrm{ctsh}(\boldsymbol{\omega}^1, \boldsymbol{\omega}^2)$ contains two additional sequences, namely

$$(\omega_1 + \omega_2, \omega_3) \text{ and } (\omega_2, \omega_1 + \omega_3).$$

Mould action also gives the key to the numerous *stability properties* of moulds, such as:

$$\text{(Symmetral)} \times \text{(Symmetral)} = \text{(Symmetral)} \tag{2.20}$$

$$\text{(Symmetrel)} \times \text{(Symmetrel)} = \text{(Symmetrel)} \tag{2.21}$$

$$\text{(Alternal)} \circ \text{(Alternal)} = \text{(Alternal)} \tag{2.22}$$

$$\text{(Symmetrel-}1^{\bullet}\text{)} \circ \text{(Symmetrel-}1^{\bullet}\text{)} = \text{(Symmetrel-}1^{\bullet}\text{)} \tag{2.23}$$

As often as not, mould expansions in the raw form (2.9) are ill-suited for studying the convergence or divergence properties of local objects, and it takes *adequate regroupings* to disentangle the situation. Sometimes it is necessary to take hold of a "principal part" Ob° of the object Ob (usually, Ob° consists of resonant components only) and to regroup all terms stemming from Ob° in suitable "clusters". In terms of mould expansions, this translates into an operation known as *mould amplification*.

In other situations, one has to replace the *fully ordered* sequences $\boldsymbol{\omega}$ occuring in mould expansions by *partially ordered* sequences $\boldsymbol{\omega}^{<}$. The effect on moulds of these re-orderings is known as *mould arborification*. In certain cases, one must take recourse to both techniques, amplification *and* arborification, at the same time.

The technique of *mould amplification* is pertinent whenever we deal with divergent-resurgent objects, such as the *nilpotent parts* or the *continuous prenormalizations*. But it has no direct bearing on the present investigation, and so we will be content here to describe *mould arborification* (which, *stricto sensu*, applies only to symmetral or alternal moulds, i.e. those that can be made to act on *vector fields*) and its natural variant, *mould erborification* (which applies to symmetrel or alternel moulds, i.e. those acting on diffeos). Our excuse for burdening the reader with these notions is, first, that we absolutely require them and, second, that they have a very wide range of application.

Arborification and coarborification

Arborification consists in replacing fully ordered sequences $\boldsymbol{\omega}$ by sequences $\boldsymbol{\omega}^{<}$ endowed with an *arborescent partial order* (meaning that each element ω_i in $\boldsymbol{\omega}^{<}$ has at most *one* immediate predecessor ω_{i-}).

For any pair $(\boldsymbol{\omega}, \boldsymbol{\omega}^{<})$, we define $\mathrm{sh}\begin{pmatrix}\boldsymbol{\omega}^{<}\\ \boldsymbol{\omega}\end{pmatrix}$ (resp. $\mathrm{ctsh}\begin{pmatrix}\boldsymbol{\omega}^{<}\\ \boldsymbol{\omega}\end{pmatrix}$) as being the number of order preserving bijections (resp. contracting surjections) of $\boldsymbol{\omega}^{<}$ into $\boldsymbol{\omega}$.

For symmetral or alternal (resp. symmetrel or alternel) moulds, we resort to arborification proper (resp. erborification), which obeys the formula:

$$M^{\boldsymbol{\omega}^{<}} = \sum \mathrm{sh}\begin{pmatrix}\boldsymbol{\omega}^{<}\\ \boldsymbol{\omega}\end{pmatrix} M^{\boldsymbol{\omega}} \quad \left(\text{resp. } \sum \mathrm{ctsh}\begin{pmatrix}\boldsymbol{\omega}^{<}\\ \boldsymbol{\omega}\end{pmatrix} M^{\boldsymbol{\omega}}\right) \tag{2.24}$$

The dual relations for comoulds obviously read:

$$(2.25) \qquad B_n = \sum \mathrm{sh}\binom{\mathbf{n}^<}{\mathbf{n}} B_{n^<} \qquad \left(\text{resp. } \sum \mathrm{ctsh}\binom{\mathbf{n}^<}{\mathbf{n}} B_{n^<}\right).$$

But whereas (2.24) fully defines $M^{\overset{<}{\bullet}}$ in terms of $M^\bullet$, (2.25) does not suffice to determine $B_{\overset{<}{\bullet}}$. So we impose additional natural conditions. For instance, for a "cosymmetral" comould $B_\bullet$ built as in (2.8) from the homogeneous components B_{n_i} of a vector field, we define $B_{n^<}$ as being the differential operator that acts on any given test function $\varphi(x)$ in $\mathbf{C}\{x\}$ according to:

$$(2.26) \quad B_{n^<}\varphi(x) \overset{\text{def}}{=} \{B_{n_r} \dots B_{n_1}\}^< \varphi(x) \quad (\text{here } \mathbf{n}^< = (n_1, \dots, n_r)^<)$$

where each operator B_{n_i} within $\{\dots\}^<$ is made to act on $\varphi(x)$ alone if n_i has no predecessor in $\mathbf{n}^<$, or else on the monomial $x^{n_{i-}}$ implicit in $B_{n_{i-}}$ if n_{i-} is the (necessarily unique) predecessor of n_i in $\mathbf{n}^<$.

Since arborification-coarborification (like erborification-coerborification) are dual operations, we have:

$$(2.27) \qquad \sum M^\bullet B_\bullet = \sum M^{\overset{<}{\bullet}} B_{\overset{<}{\bullet}} \quad (\text{formally})$$

but in numerous instances the seemingly harmless passage from $\bullet$ to $\overset{<}{\bullet}$ restores *normal convergence*, meaning that

$$\sum |M^{\overset{<}{\bullet}}| \, \|B_{\overset{<}{\bullet}}\|$$

may converge even when

$$\sum |M^\bullet| \, \|B_\bullet\|$$

diverges.

Some elementary examples (for illustration and future use)

For any sequence $\boldsymbol{\omega} = (\omega_1, \dots, \omega_r)$ with $\omega_i \in \mathbf{C}$, we put:

$$\|\boldsymbol{\omega}\| = \omega_1 + \dots + \omega_r; \ \check{\omega}_i = \omega_1 + \dots + \omega_i;$$
$$(2.28) \qquad \hat{\omega}_i = \omega_i + \dots + \omega_r \quad (\forall i).$$

Then we introduce six moulds $\mathrm{Sa}^\bullet$, $\mathrm{invSa}^\bullet$, $\mathrm{Ta}^\bullet$, $\mathrm{Se}^\bullet$, $\mathrm{invSe}^\bullet$, $\mathrm{Te}^\bullet$ (defined "almost everywhere"), by putting:

$$(2.29) \qquad \mathrm{Sa}^\varnothing = \mathrm{invSa}^\varnothing = \mathrm{Se}^\varnothing = \mathrm{invSe}^\varnothing = 1; \ \mathrm{Ta}^\varnothing = \mathrm{Te}^\varnothing = 0$$

$$(2.30) \qquad \mathrm{Sa}^{\omega_1,\dots,\omega_r} = (-\check{\omega}_1)^{-1} \dots (-\check{\omega}_r)^{-1}$$

(2.31) $\text{invSa}^{\omega_1,\dots,\omega_r} = (\hat{\omega}_1)^{-1}\dots(\hat{\omega}_r)^{-1}$

(2.32) $\text{Ta}^{\omega_1,\dots,\omega_r} = \text{Sa}^{\omega_1,\dots,\omega_{r-1}} = \text{invSa}^{\omega_2,\dots,\omega_r} \quad \text{if } \|\boldsymbol{\omega}\| = 0$
$\qquad = 0 \quad \text{if } \|\boldsymbol{\omega}\| \neq 0$

(2.33) $\text{Se}^{\omega_1,\dots,\omega_r} = e^{-\|\boldsymbol{\omega}\|}(e^{-\check{\omega}_1}-1)^{-1}\dots(e^{-\check{\omega}_r}-1)^{-1}$

(2.34) $\text{invSe}^{\omega_1,\dots,\omega_r} = (e^{-\hat{\omega}_1}-1)^{-1}\dots(e^{-\hat{\omega}_r}-1)^{-1}$

(2.35) $\text{Te}^{\omega_1,\dots,\omega_r} = \text{Se}^{\omega_1,\dots,\omega_{r-1}} = \text{invSe}^{\omega_2,\dots,\omega_r} \quad \text{if } \|\boldsymbol{\omega}\| = 0$
$\qquad = 0 \quad \text{if } \|\boldsymbol{\omega}\| \neq 0$

Lemma 2.1 $\text{Sa}^\bullet$ *and* $\text{invSa}^\bullet$ *are symmetral and mutually inverse (*$\text{Sa}^\bullet \times \text{invSa}^\bullet = 1^\bullet$*)*
$\text{Se}^\bullet$ *and* $\text{invSe}^\bullet$ *are symmetrel and mutually inverse (*$\text{Se}^\bullet \times \text{invSe}^\bullet = 1^\bullet$*)*
$\text{Ta}^\bullet$ *is alternal and* $\text{Te}^\bullet$ *is alternel.*

Lemma 2.2 *For any non-resonant vector field X as in (1.4) and $B_\bullet$ as in (2.8), we have the formal linearization:*

$$X = \Theta X^{\text{lin}} \Theta^{-1} \tag{2.36}$$

with two reciprocal diffeos:

$$\Theta = \sum \text{Sa}^\bullet B_\bullet\, ;\ \Theta^{-1} = \sum \text{invSa}^\bullet B_\bullet \tag{2.37}$$

Lemma 2.3 *For any non-resonant diffeo F as in (1.5) and $B_\bullet$ as in (2.8), we have the formal linearization:*

$$F = \Theta F^{\text{lin}} \Theta^{-1} \tag{2.38}$$

with two reciprocal diffeos:

$$\Theta = \sum \text{Se}^\bullet \text{B}_\bullet\, ; \quad \Theta^{-1} = \sum \text{invSe}^\bullet B_\bullet \tag{2.39}$$

Lemma 2.4 *Under Brjuno's diophantine condition (1.9), although the mould expansions (2.37) and (2.39) are generically non-convergent (in norm), arborification restores normal convergence in (2.37) and erborification restores it in (2.39).*

Lemma 2.5 *Under arborification (resp. erborification) the moulds* $\text{invSa}^\bullet$ *(resp.* $\text{invSe}^\bullet$*) retain their expression (2.31) (resp. (2.34)) except that the sums*

$$\hat{\omega}_1 = \omega_i + \ldots + \omega_r$$

now extend to all components ω_j which are posterior (or equal) to ω_i relative to the arborescent order of $\boldsymbol{\omega}^<$ ($\text{Sa}^\bullet$ *and* $\text{Se}^\bullet$ *do not arborify so simply).*

3. Mould expansion and variance of the correction

Throughout, we will consider *local, analytic, resonant* vector fields

$$X = X^{\mathrm{lin}} + \sum B_n$$

or diffeos

$$F = (1 + \sum B_n) F^{\mathrm{lin}}$$

in a "prepared chart" (1.4) or (1.5). We are going to expand the principal "associated objects" by contracting the *comould* $B_\bullet$ introduced in (2.8) with suitable *moulds*. Although our immediate concern in this paper is with the *corrected forms*, we must briefly recall, for future use, some basic results about the *nilpotent parts*.

Mould expansion of the nilpotent part Lemma 3.1. *The nilpotent part X^{nal} (resp. F^{nel}) of a resonant vector field X (resp. diffeo F), which was defined by equation (1.12–12*) (resp. (1.13)), admits a mould expansion:*

$$X^{\mathrm{nal}} = \sum \mathrm{Nal}^\bullet B_\bullet \ (\mathrm{Nal}^\bullet \text{ altern}\mathbf{a}\text{l}) \tag{3.1}$$

$$F^{\mathrm{nel}} = \sum \mathrm{Nel}^\bullet B_\bullet \ (\mathrm{Nel}^\bullet \text{ altern}\mathbf{e}\text{l}) \tag{3.2}$$

with well defined moulds $\mathrm{Nal}^\bullet$ *and* $\mathrm{Nel}^\bullet$ *explicitly given by:*

$$\begin{cases} \mathrm{Nal}^{\boldsymbol{\omega}} = \displaystyle\sum_{\boldsymbol{\omega}=\mathbf{a}\boldsymbol{\omega}^1\boldsymbol{\omega}^2\ldots\boldsymbol{\omega}^s\mathbf{b}} \left(\frac{(\mathrm{daa})^{s-1}}{(s-1)!}\right) \\ \qquad\qquad \times[(\mathrm{invSaa}^{\mathbf{a}}) \cdot \mathrm{Taa}^{\boldsymbol{\omega}^1} \cdot \mathrm{Taa}^{\boldsymbol{\omega}^2} \ldots \mathrm{Taa}^{\boldsymbol{\omega}^s} (\mathrm{Saa}^{\mathbf{b}})] \end{cases} \tag{3.3}$$

$$\begin{cases} \mathrm{Nel}^{\boldsymbol{\omega}} = \displaystyle\sum_{\boldsymbol{\omega}=\mathbf{a}\boldsymbol{\omega}^1\boldsymbol{\omega}^2\ldots\boldsymbol{\omega}^s\mathbf{b}} \left(\frac{(\mathrm{dee})^{s-1}}{(s-1)!}\right) \\ \qquad\qquad \times[(\mathrm{invSee}^{\mathbf{a}}) \cdot \mathrm{Tee}^{\boldsymbol{\omega}^1} \cdot \mathrm{Tee}^{\boldsymbol{\omega}^2} \ldots \mathrm{Tee}^{\boldsymbol{\omega}^s} (\mathrm{See}^{\mathbf{b}})] \end{cases} \tag{3.4}$$

As this lemma is mentioned simply for information, we will be content with expliciting the ingredients of $\mathrm{Nal}^\bullet$ *and* $\mathrm{Nel}^\bullet$.

Whereas the moulds $\mathrm{Sa}^\bullet$, $\mathrm{invSa}^\bullet$, $\mathrm{Ta}^\bullet$, $\mathrm{Se}^\bullet$, $\mathrm{invSe}^\bullet$, $\mathrm{Te}^\bullet$ introduced in (2.29)–(2.35) were defined almost everywhere, but possessed definite "symmetries", we now require similar moulds, marked with a double vowel (aa or ee), which will be defined everywhere, will coincide with them almost everywhere, but will be neither symmetral/el nor alternal/el.

For any given sequence $\boldsymbol{\omega}$ of length $r = r(\boldsymbol{\omega})$ we set:

$$\mathrm{invSaa}^{\boldsymbol{\omega}} = \mathrm{invSa}^{\boldsymbol{\omega}},\ \mathrm{invSee}^{\boldsymbol{\omega}} = \mathrm{invSe}^{\boldsymbol{\omega}} \text{ if } \hat\omega_1\hat\omega_2\ldots\hat\omega_r \neq 0 \tag{3.5}$$

$$(3.5^*)\qquad \mathrm{invSaa}^{\boldsymbol{\omega}} = 0,\ \mathrm{invSee}^{\boldsymbol{\omega}} = 0 \quad \text{otherwise.}$$

$$(3.6)\qquad \mathrm{Saa}^{\boldsymbol{\omega}} = \mathrm{Sa}^{\boldsymbol{\omega}},\ \mathrm{See}^{\boldsymbol{\omega}} = \mathrm{Se}^{\boldsymbol{\omega}} \text{ if } \check{\omega}_1\check{\omega}_2\ldots\check{\omega}_r \neq 0$$

$$(3.6^*)\qquad \mathrm{Saa}^{\boldsymbol{\omega}} = 0,\ \mathrm{See}^{\boldsymbol{\omega}} = 0 \quad \text{otherwise.}$$

$$(3.7)\qquad \begin{cases} \mathrm{Taa}^{\boldsymbol{\omega}} = \mathrm{Ta}^{\boldsymbol{\omega}}\ ,\ \mathrm{Tee}^{\boldsymbol{\omega}} = \mathrm{Te}^{\boldsymbol{\omega}} \\ \quad \text{if } \|\boldsymbol{\omega}\| = 0 \text{ and } \check{\omega}_1\check{\omega}_2\ldots\check{\omega}_{r-1} \neq 0 \neq \hat{\omega}_2\hat{\omega}_3\ldots\hat{\omega}_r \end{cases}$$

$$(3.7^*)\qquad \mathrm{Taa}^{\boldsymbol{\omega}} = 0\ ,\ \mathrm{Tee}^{\boldsymbol{\omega}} = 0 \quad \text{otherwise.}$$

Of course $\mathrm{Saa}^{\varnothing} = \mathrm{invSaa}^{\varnothing} = \mathrm{See}^{\varnothing} = \mathrm{invSee}^{\varnothing} = 1$, $\mathrm{Taa}^{\varnothing} = \mathrm{Tee}^{\varnothing} = 0$. Then we require derivations daa and dee which act as follows on the above symbols:

$$(3.8)\quad \frac{(\mathrm{daa})^s}{s!}(\mathrm{Saa}^{\boldsymbol{\omega}}) \overset{\mathrm{def}}{=} \mathrm{Saa}^{\boldsymbol{\omega}} \sum_{1\leq i_1\leq\ldots\leq i_s\leq r} (\check{\omega}_{i_1})^{-1}\ldots(\check{\omega}_{i_s})^{-1}$$

$$(3.9)\quad \frac{(\mathrm{daa})^s}{s!}(\mathrm{invSaa}^{\boldsymbol{\omega}}) \overset{\mathrm{def}}{=} \mathrm{invSaa}^{\boldsymbol{\omega}} \sum_{1\leq i_1\leq\ldots\leq i_s\leq r} (-\hat{\omega}_{i_1})^{-1}\ldots(-\hat{\omega}_{i_s})^{-1}$$

$$(3.10)\quad \frac{(\mathrm{daa})^s}{s!}(\mathrm{Taa}^{\boldsymbol{\omega}}) \overset{\mathrm{def}}{=} \mathrm{Taa}^{\boldsymbol{\omega}} \sum^{*}_{1\leq i_1\leq\ldots\leq i_s\leq r-1} (\check{\omega}_{i_1})^{-1}\ldots(\check{\omega}_{i_s})^{-1}$$

$$= \mathrm{Taa}^{\boldsymbol{\omega}} \sum^{*}_{2\leq i_1\leq\ldots\leq i_s\leq r} (-\hat{\omega}_{i_1})^{-1}\ldots(-\hat{\omega}_{i_s})^{-1}$$

$$(3.11)\quad \begin{cases} \dfrac{(\mathrm{dee})^s}{s!}(\mathrm{See}^{\boldsymbol{\omega}}) \overset{\mathrm{def}}{=} \mathrm{See}^{\boldsymbol{\omega}} \\ \qquad \times \displaystyle\sum_{1\leq i_1\leq\ldots\leq i_s\leq r} (e^{\check{\omega}_{i_1}}-1)^{-1}\ldots(e^{\check{\omega}_{i_s}}-1)^{-1} \end{cases}$$

$$(3.12)\quad \begin{cases} \dfrac{(\mathrm{dee})^s}{s!}(\mathrm{invSee}^{\boldsymbol{\omega}}) \overset{\mathrm{def}}{=} \mathrm{invSee}^{\boldsymbol{\omega}} \\ \qquad \times \displaystyle\sum_{1\leq i_1\leq\ldots\leq i_s\leq r} (e^{-\hat{\omega}_{i_1}}-1)^{-1}\ldots(e^{-\hat{\omega}_{i_s}}-1)^{-1} \end{cases}$$

$$(3.13)\quad \begin{cases} \dfrac{(\mathrm{dee})^s}{s!}(\mathrm{Tee}^{\boldsymbol{\omega}}) \overset{\mathrm{def}}{=} \mathrm{Tee}^{\boldsymbol{\omega}} \\ \qquad \times \displaystyle\sum^{*}_{1\leq i_1\leq\ldots\leq i_s\leq r-1} (e^{\check{\omega}_{i_1}}-1)^{-1}\ldots(e^{\check{\omega}_{i_s}}-1)^{-1} \\ \quad = \mathrm{Tee}^{\boldsymbol{\omega}} \displaystyle\sum^{*}_{2\leq i_1\leq\ldots\leq i_s\leq r} (e^{-\hat{\omega}_{i_1}}-1)^{-1}\ldots(e^{-\hat{\omega}_{i_s}}-1)^{-1}\,. \end{cases}$$

The sum (3.3) (3.4) extend to all factorizations of $\boldsymbol{\omega}$ into any number of subsequences $\mathbf{a}, \boldsymbol{\omega}^1, \ldots \boldsymbol{\omega}^s, \mathbf{b}$ ($\mathbf{a}$ and $\mathbf{b}$ may be empty) but only those few factorizations which involve non-degenerate, zero-sum factors $\boldsymbol{\omega}^i$ make an effective contribution. Each operator $\frac{(\mathrm{daa})^{s-1}}{(s-1)!}$ is made to act like a derivation on the symbols standing to its right. Thus:

$$(3.14)\quad \begin{cases} \dfrac{(\mathrm{daa})^s}{s!}(\mathrm{Taa}^{\boldsymbol{\omega}^1}\mathrm{Taa}^{\boldsymbol{\omega}^2}) \\ \quad \overset{\mathrm{def}}{=} \displaystyle\sum_{\substack{s_1+s_2=s \\ s_1,s_2\geq 0}} \left(\left(\frac{(\mathrm{daa})^{s_1}}{s_1!}\right)\mathrm{Taa}^{\boldsymbol{\omega}^1}\right)\left(\left(\frac{(\mathrm{daa})^{s_2}}{s_2!}\right)\mathrm{Taa}^{\boldsymbol{\omega}^2}\right) \end{cases}$$

with $(\mathrm{daa})^0$ acting like the identity.

Mould expansion for the "corrected form" of a vector field

Lemma 3.2. *We have the expansions.*

$$(3.15)\qquad X^{\mathrm{carr}} = \sum \mathrm{Carr}^\bullet B_\bullet$$

$$(3.16)\qquad \begin{cases} X^{\mathrm{carrd}} \overset{\mathrm{def}}{=} X - X^{\mathrm{carr}} \\ \qquad\quad = X^{\mathrm{lin}} + \sum (I^\bullet - \mathrm{Carr}^\bullet) B_\bullet \\ \qquad\quad = X^{\mathrm{lin}} + \sum \mathrm{Carrd}^\bullet B_\bullet \end{cases}$$

with a well-defined alternal mould $\mathrm{Carr}^\bullet$ *calculable by stationary mould composition:*

$$(3.17)\qquad \mathrm{Carrd}^\bullet = I^\bullet - \mathrm{Carr}^\bullet = \mathrm{stat}\ \lim_{n\to+\infty}(I^\bullet - M^\bullet)^{\circ n}$$

where $M^\bullet$ *may denote any of the following moulds:*

(3.18) some fixed prenormalizing mould $\mathrm{Pran}^\bullet$ as in (1.16 *)

(3.18^*) the mould $\mathrm{Nal}^\bullet$ which expands the nilpotent part X^{nal}

(3.18^{**}) the elementary mould $\mathrm{Taa}^\bullet$ introduced in (3.7).

The adequacy of the first two choices is easily proven: indeed, with $M^\bullet$ as in (3.18) or (3.18*), we have:

$$(3.19)\qquad 0 = I^0 - M^0 = I^{\boldsymbol{\omega}} - M^{\boldsymbol{\omega}} \quad if\ r(\boldsymbol{\omega}) \geq 2 \text{ and } \|\boldsymbol{\omega}\| \neq 0\,.$$

Therefore, for any sequence ω of length r we have:

$$(3.20)\qquad \begin{aligned}(I^\bullet - M^\bullet)^{\circ r} &= (I^\bullet - M^\bullet)^{\circ(r+1)} = (I^\bullet - M^\bullet)^{\circ(r+2)}\\ &= \ldots (\operatorname{mod} r(\bullet))\end{aligned}$$

So the stationary limit in (3.17) exists and, due to the meaning of $M^\bullet$ (related to either prenormal forms or the nilpotent part) that limit *cannot but coincide* with $I^\bullet - \mathrm{Carr}^\bullet$.

The real difficulty lies in establishing the legitimacy of the last choice, which paradoxically involves the far simpler mould $\mathrm{Taa}^\bullet$. For that choice, (3.19) still holds and the stationary limit in (3.17) still exists, but there is no obvious reason why it should coincide with $I^\bullet - \mathrm{Carr}^\bullet$. We will postpone the proof to the end of Sect. 4, where the identity (3.17) with $M^\bullet = \mathrm{Taa}^\bullet$ will yield elementary linearizability criteria.

We may observe, meanwhile, that $I^\bullet - M^\bullet$ is alternal with $M^\bullet$ as in (3.18) or (3.18*), but not with the choice (3.18**). That, however, involves no contradiction: the composition of alternal moulds always yields alternal moulds, but so does, on occasion, the composition of non-alternal moulds.

Mould expansion for the "corrected form" of a diffeo

Lemma 3.3. *We have the expansions.*

$$(3.21)\qquad \begin{cases} F^{\mathrm{cerr}} = \sum \mathrm{Cerr}^\bullet B_\bullet\,;\\ (F^{\mathrm{cerr}})^{-1} = \sum \mathrm{invCerr}^\bullet B_\bullet \end{cases}$$

$$(3.22)\qquad \begin{cases} F^{\mathrm{cerrd}} \overset{\mathrm{def}}{=} (F^{\mathrm{cerr}})^{-1} F\\ \qquad = \left(\sum((1^\bullet + I^\bullet) \times (\mathrm{invCerr}^\bullet))B_\bullet\right) F^{\mathrm{lin}}\\ \qquad = \left(\sum \mathrm{Cerrd}^\bullet B_\bullet\right) F^{\mathrm{lin}} \end{cases}$$

with two pairs of well-defined, mutually inverse symmetrel moulds:

$$(3.23)\qquad \mathrm{Cerr}^\bullet \times \mathrm{invCerr}^\bullet = 1^\bullet\,;\ \mathrm{Cerrd}^\bullet \times \mathrm{invCerrd}^\bullet = 1^\bullet$$

which may be calculated by stationary mould composition:

$$(3.24)\qquad \begin{cases} \mathrm{invCerrd}^\bullet = ((1^\bullet + I^\bullet)^{-1} \times (\mathrm{Cerr}^\bullet) - 1^\bullet)\\ \qquad = \mathrm{stat\,lim}_{n\to+\infty}((1^\bullet + I^\bullet)^{-1} \times (M^\bullet) - 1^\bullet)^{\circ n} \end{cases}$$

$$(3.25)\qquad \begin{cases} \mathrm{Cerrd}^\bullet = ((1^\bullet + I^\bullet) \times (\mathrm{invCerr}^\bullet) - 1^\bullet)\\ \qquad = \mathrm{stat\,lim}_{n\to+\infty}((1^\bullet + I^\bullet) \times (M^\bullet)^{-1} - 1^\bullet)^{\circ n} \end{cases}$$

where $M^\bullet$ *may denote any of the following moulds:*

(3.26) some fixed prenormalizing mould $\mathrm{Pren}^\bullet$ as in (1.16**)

(3.26*) the mould $\mathrm{Nel}^\bullet$ which expands the nilpotent part F^{nel}

(3.26**) the elementary mould $\mathrm{Tee}^\bullet$ introduced in (3.7) .

As with vector fields, the adequacy of the first two choices is easily proven; the tricky part is to justify the third choice, which paradoxically involves the far simpler (though admittedly non- symmetrel) mould $\mathrm{Tee}^\bullet$.

Remark. The identities (3.17), (3.24) with $M^\bullet$ as in (3.18) etc. or (3.26) etc., yield useful information about $\mathrm{Carr}^\bullet$ and $\mathrm{Cerr}^\bullet$ (and elementary linearizability criteria will be based on them) but, from a practical viewpoint, the *variance rules*, which we will proceed to derive, lead to even simpler calculations.

Variance of the correction

If to a vector field X there corresponds a vector field Y under some mould action:

$$X = X^{\mathrm{lin}} + \sum B_n \to Y = X^{\mathrm{lin}} + \sum M^\bullet B_\bullet \ (M^\bullet \text{ alternal}) \tag{3.27}$$

then any infinitesimal automorphism acting on X:

$$X \to X + \delta X = (1-\varepsilon C)X(1+\varepsilon C) = X + \varepsilon[X, C] \ (\mathrm{mod}\, \varepsilon^2) \tag{3.27*}$$

$$C \text{ homogeneous vector field of degree } n_0 \text{ with } \langle n_0, \lambda\rangle = \omega_0 \tag{3.27**}$$

induces a concomitant variation of Y:

$$Y \to Y + \delta Y \quad \text{with} \quad \delta Y = \sum_i (\mathrm{var}_i M^\bullet)(C^i B_\bullet) \ (\mathrm{mod}\, \varepsilon^2) \tag{3.28}$$

with:

$$\begin{cases} (C^i B)_{\omega_1,\ldots,\omega_r} \overset{\mathrm{def}}{=} B_{\omega_r} \ldots B_{\omega_{i+1}} C B_{\omega_{i-1}} \ldots B_{\omega_1} \ (\text{resp.} = 0) \\ \qquad \text{if} \quad \omega_i \ = \ \omega_0 (\text{resp.} \neq \omega_0) \end{cases} \tag{3.29}$$

$$\begin{cases} (\mathrm{var}_i M)^{\omega_1,\ldots,\omega_r} \overset{\mathrm{def}}{=} \omega_i M^{\omega_1,\ldots,\omega_r} \\ \qquad\qquad + M^{\omega_1,\ldots,\omega_i+\omega_{i+1},\ldots,\omega_r} \\ \qquad\qquad - M^{\omega_1,\ldots,\omega_{i-1}+\omega_i,\ldots,\omega_r} \end{cases} \tag{3.30}$$

Similarly, if to a diffeo F there corresponds a diffeo G under:

(3.31) $$\begin{cases} F = \left(1+\sum B_n\right) F^{\mathrm{lin}} \to G \\ \quad = \left(\sum M^\bullet B_\bullet\right) F^{\mathrm{lin}} \quad (M^\bullet \text{ symmetrel}) \end{cases}$$

then any infinitesimal automorphism acting on F:

(3.31*) $$F \to F + \delta F = (1-\varepsilon C) F (1+\varepsilon C)$$

(with C as in (3.27**)) induces a concomitant variation of G:
(3.31**)
$$G \to G + \delta G \text{ with } \delta G = \left(\sum_i (ver_i M^\bullet)(C^i B_\bullet)\right) F^{\mathrm{lin}} (\mathrm{mod}\, \varepsilon^2)$$

with $C^i B_\bullet$ as above (see (3.29)) and:

(3.32) $$\begin{cases} (ver_i M)^{\omega_1,\dots,\omega_r} \overset{\mathrm{def}}{=} (e^{\omega_i}-1)\, M^{\omega_1,\dots,\omega_r} \\ \qquad\qquad\qquad\quad +e^{\omega_i} M^{\omega_1,\dots,\omega_i+\omega_{i+1},\dots,\omega_r} \\ \qquad\qquad\qquad\quad -M^{\omega_1,\dots,\omega_{i-1}+\omega_i,\dots,\omega_r} \end{cases}$$

It is often useful to consider the decomposition $ver_i = ver_i^+ - ver_i^-$ with

(3.32*) $$(ver_i^+ M)^{\omega_1,\dots,\omega_r} \overset{\mathrm{def}}{=} e^{\omega_i} (M^{\omega_1,\dots,\omega_r} + M^{\omega_1,\dots,\omega_i+\omega_{i+1},\dots,\omega_r})$$

(3.32**) $$(ver_i^- M)^{\omega_1,\dots,\omega_r} \overset{\mathrm{def}}{=} M^{\omega_1,\dots,\omega_r} + M^{\omega_1,\dots,\omega_{i-1}+\omega_i,\dots,\omega_r}$$

The more direct the geometric meaning of a mould $M^\bullet$, the simpler its "variance" $\mathrm{var}_i M^\bullet$ or $ver_i M^\bullet$ tends to be.

Let us concentrate here on the *correction moulds*.

Proposition 3.1. *Variance of* $\mathrm{Carr}^\bullet$.
The alternal mould $\mathrm{Carr}^\bullet$ *is calculable by the induction:*

(3.33) $$\begin{cases} (\mathrm{var}_i\, \mathrm{Carr})^{\boldsymbol{\omega}} = \sum_{\mathbf{a}\omega_i\mathbf{bc}=\boldsymbol{\omega}} \mathrm{Carr}^{\mathbf{a}\omega_i\mathbf{c}} \mathrm{Carr}^{\mathbf{b}} \\ \qquad\qquad\qquad\quad - \sum_{\mathbf{ab}\omega_i\mathbf{c}=\boldsymbol{\omega}} \mathrm{Carr}^{\mathbf{b}} \mathrm{Carr}^{\mathbf{a}\omega_i\mathbf{c}} \end{cases}$$

with var_i *as in (3.30) and with the initial conditions:*

(3.34) $$\mathrm{Carr}^{\varnothing} = 0;\ \mathrm{Carr}^0 = 1;\ \mathrm{Carr}^{\omega_1} = 0 \text{ if } \omega_1 \neq 0.$$

Proposition 3.2. *Variance of* $\mathrm{Cerr}^\bullet$ *and* $\mathrm{invCerr}^\bullet$.

The symmetrel moulds $\mathrm{Cerr}^\bullet$ *and* $\mathrm{invCerr}^\bullet$ *are calculable by any one of the four inductive systems:*

$$ver_i^+\,\mathrm{Cerr}^{\boldsymbol{\omega}} = \sum_{\mathbf{a}\mathbf{b}\omega_i\mathbf{c}\mathbf{d}=\boldsymbol{\omega}} \mathrm{invCerr}^{\mathbf{b}}(ver_i^-\,\mathrm{Cerr}^{\mathbf{a}\omega_i\mathbf{d}})\mathrm{Cerr}^{\mathbf{c}} \tag{3.35}$$

$$ver_i^-\mathrm{Cerr}^{\boldsymbol{\omega}} = \sum_{\mathbf{a}\mathbf{b}\omega_i\mathbf{c}\mathbf{d}=\boldsymbol{\omega}} \mathrm{Cerr}^{\mathbf{b}}(ver_i^+\mathrm{Cerr}^{\mathbf{a}\omega_i\mathbf{d}})\mathrm{invCerr}^{\mathbf{c}} \tag{3.36}$$

$$ver_i^+\mathrm{invCerr}^{\boldsymbol{\omega}} = \sum_{\mathbf{a}\mathbf{b}\omega_i\mathbf{c}\mathbf{d}=\boldsymbol{\omega}} \mathrm{Cerr}^{\mathbf{b}}(ver_i^-\,\mathrm{Cerr}^{\mathbf{a}\omega_i\mathbf{d}})\mathrm{Cerr}^{\mathbf{c}} \tag{3.37}$$

$$ver_i^-\mathrm{invCerr}^{\boldsymbol{\omega}} = \sum_{\mathbf{a}\mathbf{b}\omega_i\mathbf{c}\mathbf{d}=\boldsymbol{\omega}} \mathrm{Cerr}^{\mathbf{b}}(ver_i^+\mathrm{Cerr}^{\mathbf{a}\omega_i\mathbf{d}})\mathrm{Cerr}^{\mathbf{c}} \tag{3.38}$$

with ver_i^+, ver_i^- *as in (3.32***), (3.32****) and with the following initial conditions:*

$$\mathrm{Cerr}^{\varnothing} = \mathrm{invCerr}^{\varnothing} = 1 \tag{3.39}$$

$$\mathrm{Cerr}^{0} = 1 \quad \mathit{but} \quad \mathrm{invCerr}^{0} = -1 \tag{3.40}$$

$$\mathrm{Cerr}^{\omega_1} = \mathrm{invCerr}^{\omega_1} = 0 \text{ if } \omega_1 \neq 0 \tag{3.41}$$

Remark 1. The above relations *do* provide an inductive scheme, by expressing $\mathrm{Carr}^{\boldsymbol{\omega}}$, $\mathrm{Cerr}^{\boldsymbol{\omega}}$, etc., in terms of a finite number of values $\mathrm{Carr}^{\boldsymbol{\omega}^*}$, $\mathrm{Cerr}^{\boldsymbol{\omega}^*}$, etc., for strictly *shorter* sequences $\boldsymbol{\omega}^*$. Since relations (3.33), (3.34) etc. can be applied for any i less than $r(\boldsymbol{\omega})$, we have in fact a *heavily overdetermined* induction – which is very convenient for cross-checking the calculations and also for eliminating illusory poles (see Sects. 4, 5).

Practically, we may take $i = 1$ or $r(\boldsymbol{\omega})$, so as to cancel one of the two sums on the right-hand side. Thus, for $i = 1$, the relation (3.33) becomes:

$$\omega_1\mathrm{Carr}^{\omega_1,\omega_2,\ldots,\omega_r} + \mathrm{Carr}^{\omega_1+\omega_2,\omega_3,\ldots,\omega_r} = \sum_{\omega_1\mathbf{b}\mathbf{c}=\boldsymbol{\omega}} \mathrm{Carr}^{\omega_1\mathbf{c}}\mathrm{Carr}^{\mathbf{b}}\,. \tag{3.42}$$

Likewise, we get a closed induction for $\mathrm{Cerr}^\bullet$ (resp. $\mathrm{invCerr}^\bullet$), that is to say, an induction involving only $\mathrm{Cerr}^\bullet$ (resp. $\mathrm{invCerr}^\bullet$) by taking either (3.35) with $i = 1$ or (3.36) with $i = r(\boldsymbol{\omega})$ (resp. either (3.37) with $i = r(\boldsymbol{\omega})$ or (3.38) with $i = 1$).

For the purpose of eliminating the “illusory denominators”, however, we shall have to apply the variance rules with other, suitably chosen values of i (see Sects. 4, 5). The fact that the intermediary values of i yield a *mixed* *induction*, involving both $\mathrm{Cerr}^\bullet$ and $\mathrm{invCerr}^\bullet$, is of course inconsequential, since $\mathrm{Cerr}^\bullet$ and $\mathrm{invCerr}^\bullet$, being mutually inverse, symmetrel moulds, interrelate under the standard involution:

$$\mathrm{invCerr}^{\boldsymbol{\omega}} = (-1)^{r(\boldsymbol{\omega})} \sum \mathrm{Cerr}^{\|\boldsymbol{\omega}^s\|,\ldots,\|\boldsymbol{\omega}^2\|,\|\boldsymbol{\omega}^1\|}$$
$$\mathrm{Cerr}^{\boldsymbol{\omega}} = (-1)^{r(\boldsymbol{\omega})} \sum \mathrm{Cerr}^{\|\boldsymbol{\omega}^s\|,\ldots,\|\boldsymbol{\omega}^2\|,\|\boldsymbol{\omega}^1\|}$$

with finite sums $\sum$ extending to all factorizations $\boldsymbol{\omega} = \boldsymbol{\omega}^1\boldsymbol{\omega}^2\ldots\boldsymbol{\omega}^s$ $(s \geq 1; \boldsymbol{\omega}^i \neq 0)$ (mark the order reversal).

Remark 2. *Special values of the correction moulds.*

It is clear, both from the induction (3.33) or the obvious addition rule:

$$(3.43) \qquad (X + X^0)^{\mathrm{carr}} = X^{\mathrm{carr}} + X^0 \quad \text{if} \quad [X^{\mathrm{lin}}, X^0] = 0$$

(which itself follows from the correction's uniqueness) that

$$\mathrm{Carr}^0 = 1 \text{ and } \mathrm{Carr}^{\boldsymbol{\omega}} = 0$$

if $\|\boldsymbol{\omega}\| \neq 0$ or again if $\boldsymbol{\omega}$ has length $r(\boldsymbol{\omega}) \geq 2$ and at least one zero-component $\omega_i = 0$.

Things change slightly for $\mathrm{Cerr}^{\boldsymbol{\omega}}$ and $\mathrm{invCerr}^{\boldsymbol{\omega}}$. These coefficients still vanish if $\|\boldsymbol{\omega}\| \neq 0$, but we have:

$$(3.44) \qquad \mathrm{Cerr}^0 = 1; \quad \mathrm{Cerr}^{0,0} = \mathrm{Cerr}^{0,0,0} = \ldots = 0$$

$$(3.45)\begin{cases} \mathrm{invCerr}^0 & = -1;\ \mathrm{invCerr}^{0,0} = 1; \mathrm{invCerr}^{0,0,0} = -1;\ \ldots; \\ \mathrm{invCerr}^{0,\ldots,0} & = (-1)^s\ (s \text{ zeros}). \end{cases}$$

More generally, $\mathrm{Cerr}^{\boldsymbol{\omega}} = 0$ as soon as $\boldsymbol{\omega}$ has length $r \geq 2$ and begins *or* ends with a zero-component (i.e. $\omega_1\omega_r = 0$), and

$$\mathrm{Cerr}^{\boldsymbol{\omega}} = (-1)^s \mathrm{Cerr}^{\boldsymbol{\omega}^*}$$

if $\boldsymbol{\omega}$ carries exactly s zero-components $\omega_i = 0$ at inside positions only (so that $\omega_1\omega_r \neq 0$) and if $\boldsymbol{\omega}^*$ denotes the sequence $\boldsymbol{\omega}$ deprived of its zero-components. For $\mathrm{invCerr}^{\boldsymbol{\omega}}$ the rules are somewhat simpler: for any sequence $\boldsymbol{\omega}$ with exactly s zero-components, at whatever location (interior or lateral), we have:

$$\mathrm{invCerr}^{\boldsymbol{\omega}} = (-1)^s \mathrm{invCerr}^{\boldsymbol{\omega}^*}$$

where again $\boldsymbol{\omega}^*$ is $\boldsymbol{\omega}$ minus its zero-components.

Short proof of Proposition 3.1 and 3.2. The proof splits into three auxiliary lemmas. But first, we must introduce suitably compact notations. For any Lie element B_{ω_i}, let $\bar{B}_{\omega_i}$ denote the adjoint operator $\mathrm{adj}(B_{\omega_i})$ (so that $\bar{B}_{\omega_i} B_{\omega_j} \overset{\text{def}}{\equiv} [B_{\omega_i}, B_{\omega_j}]$) and for any sequence $\boldsymbol{\omega}$, let us put:

$$(\bar{B})_{\boldsymbol{\omega}} = (\bar{B})_{\omega_1,\ldots,\omega_r} \overset{\text{def}}{=} \bar{B}_{\omega_r} \ldots \bar{B}_{\omega_1} \tag{3.46}$$

$$B_{[\boldsymbol{\omega}]} = B_{[\omega_1,\ldots,\omega_r]} \overset{\text{def}}{=} \bar{B}_{\omega_r} \ldots \bar{B}_{\omega_2} B_{\omega_1} = [B_{\omega_r} \ldots [B_{\omega_2}, B_{\omega_1}] \ldots]. \tag{3.47}$$

Lemma 3.3. *Let $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r)$ be a sequence of length r and assume that the number ω_0 occurs exactly r_0 times in it. Then, for any alternal mould $M^\bullet$, the identities hold:*

$$\sum M^{\boldsymbol{\omega}'} B_{\boldsymbol{\omega}'} = (1/r) \sum M^{\boldsymbol{\omega}'} B_{[\boldsymbol{\omega}']} = (1/r_0) \sum M^{\boldsymbol{\omega}''} B_{[\boldsymbol{\omega}'']} \tag{3.48}$$

with the first two sums extending to all sequences $\boldsymbol{\omega}'$ equivalent to $\boldsymbol{\omega}$ upto order, and with a third sum extending to all sequences $\boldsymbol{\omega}''$ beginning with ω_0 and equivalent to $\boldsymbol{\omega}$ upto order.

Proof of Lemma 3.3. $M^\bullet$ being alternal, the first sum is in fact a Lie element and so, by the classical projection theorem, it is equal to the sum of the corresponding brackets divided by the length r. Hence the equality with the last sum easily follows, by using the identity:

$$M^{\mathbf{a}\omega_0\mathbf{b}} = (-1)^{r(\mathbf{a})} \sum M^{\omega_0\mathbf{c}} \quad (\mathbf{c} \in \mathrm{sh}(\tilde{\mathbf{a}}, \mathbf{b})) \tag{3.49}$$

which holds for any alternal mould $M^\bullet$ and allows us to move any intermediary element ω_0 to the front position. Note that the sum (3.49) extends to all sequences $\mathbf{c}$ obtainable by shuffling $\mathbf{b}$ with $\tilde{\mathbf{a}}$, the latter being the reverse sequence of $\mathbf{a}$, i.e:

$$\tilde{\mathbf{a}} = (a_{r(\mathbf{a})}, \ldots, a_1) \text{ if } \mathbf{a} = (a_1, \ldots, a_{r(\mathbf{a})}) .$$

Lemma 3.4. *If Proposition 3.1 holds, then any infinitesimal automorphism (3.27*) taking X into $X + \delta X$, automatically takes X^{carrd} into $X^{\mathrm{carrd}} + \delta X^{\mathrm{carrd}}$ with:*

$$\delta X^{\mathrm{carrd}} = \varepsilon[X, C] - \varepsilon \sum_{\boldsymbol{\omega}} \mathrm{Carr}^{\omega_0\boldsymbol{\omega}} (\bar{B})_{\boldsymbol{\omega}} [X^{\mathrm{carr}}, C] \tag{3.50}$$

Conversely, if (3.50) holds for any infinitesimal automorphism induced by an homogeneous field C of arbitrary degree n_0 (with $\omega_0 = \langle n_0, \lambda \rangle$ as usal), then Proposition 3.1 is true.

Proof of Lemma 3.4. According to (3.28) we may write:

$$\begin{aligned} \delta X^{\mathrm{carrd}} &= \delta X - \delta X^{\mathrm{carr}} \\ &= \varepsilon[X, C] - \varepsilon \sum_i \sum_\bullet (\mathrm{var}_i \mathrm{Carr}^\bullet)\ (C^i B_\bullet) \end{aligned} \tag{3.51}$$

But if the formula (3.33) holds, then it is an easy matter to check that, for any fixed integer i, the mould $\mathrm{var}_i\ \mathrm{Carr}^\bullet$ inherits the alternality of $\mathrm{Carr}^\bullet$ itself. We may therefore take advantage of the identities (3.48) to move the index ω_0, implicit in the last sum (3.51), into front position, and thus to transform (3.51) successively into:

$$\delta X^{\mathrm{carrd}} = \varepsilon[X, C] - \varepsilon \sum_{\boldsymbol{\omega}} (\mathrm{var}_1 \mathrm{Carr}^{\omega_0 \boldsymbol{\omega}})(\bar{B})_{\boldsymbol{\omega}} C \tag{3.52}$$

$$\delta X^{\mathrm{carrd}} = \varepsilon[X, C] - \varepsilon \sum_{\boldsymbol{\omega}^1, \boldsymbol{\omega}^2} \mathrm{Carr}^{\omega_0 \boldsymbol{\omega}^2} \mathrm{Carr}^{\boldsymbol{\omega}^1} (\bar{B})_{\boldsymbol{\omega}^2} (\bar{B})_{\boldsymbol{\omega}^1} C \tag{3.53}$$

$$\delta X^{\mathrm{carrd}} = \varepsilon[X, C] - \varepsilon \sum_{\boldsymbol{\omega}^2} \mathrm{Carr}^{\omega_0 \boldsymbol{\omega}^2} (\bar{B})_{\boldsymbol{\omega}^2} [X^{\mathrm{carr}}, C] \tag{3.54}$$

which is the same as (3.50). The only point calling for comment is the absence from (3.54) of the multiplicity factor r_0 occuring in the last sum of (3.48): this is because $(C^i B)_{\boldsymbol{\omega}}$ is non-zero for exactly r_0 distinct values of the index i, so that, by linearity, the multiplicity r_0 cancels out. Thus the truth of Proposition 3.1 implies that of (3.50) for any C. Conversely, by going through the above identities (3.52), (3.53), (3.54) in reverse order, we easily see that the validity of (3.50) for any C implies the splitting rule (3.48).

Lemma 3.5. *For any homogeneous field C of degree n_0 (with $\omega_0 = \langle n_0, \lambda \rangle$ as usual), the following identity holds:*

$$[X, C] - \sum_{\boldsymbol{\omega}} \mathrm{Carr}^{\omega_0 \boldsymbol{\omega}} (\bar{B})_{\boldsymbol{\omega}} [X^{\mathrm{carr}}, C] \equiv [X^{\mathrm{carrd}}, C - C^{**}] \tag{3.55}$$

with a vector field C^{**} given by:

$$C^{**} = \sum_{\boldsymbol{\omega}} \mathrm{Rect}^{\omega_0 \boldsymbol{\omega}} (\bar{B})_{\boldsymbol{\omega}} [X^{\mathrm{carr}}, C] \tag{3.56}$$

relative to a mould $\mathrm{Rect}^\bullet$ defined by:

$$\mathrm{Rect}^{\omega_0 \bullet} \stackrel{\mathrm{def}}{=} (\mathrm{Saa}^{\omega_0 \bullet}) \circ (I^\bullet - \mathrm{Carr}^\bullet) \quad (\text{with } \mathrm{Saa}^\bullet \text{ as in (3.5)}) \tag{3.57}$$

or more explicitly:

$$\begin{cases} \mathrm{Rect}^{\omega_0 \boldsymbol{\omega}} = \displaystyle\sum_{\boldsymbol{\omega}^1 \ldots \boldsymbol{\omega}^s = \boldsymbol{\omega}} \mathrm{Saa}^{\omega_0, \|\boldsymbol{\omega}^1\|, \ldots, \|\boldsymbol{\omega}^s\|} \\ \qquad\qquad \times (I^{\boldsymbol{\omega}^1} - \mathrm{Carr}^{\boldsymbol{\omega}^1}) \ldots (I^{\boldsymbol{\omega}^s} - \mathrm{Carr}^{\boldsymbol{\omega}^s}). \end{cases} \tag{3.57*}$$

Proof of Lemma 3.5. The mould translation of (3.55) reads:

$$C^{**} = \sum_{\boldsymbol{\omega}} \mathrm{Reg}^{\omega_0 \boldsymbol{\omega}} (\bar{B})_{\boldsymbol{\omega}} C \tag{3.58}$$

with a mould $\mathrm{Reg}^{\bullet}$ such that:

$$(3.59) \begin{cases} (\omega_0 + \|\boldsymbol{\omega}\|)\, \mathrm{Reg}^{\omega_0 \boldsymbol{\omega}} + \sum_{\boldsymbol{\omega}^1 \boldsymbol{\omega}^2 = \boldsymbol{\omega}} \mathrm{Reg}^{\omega_0 \boldsymbol{\omega}^1} (I^{\boldsymbol{\omega}^2} - \mathrm{Carr}^{\boldsymbol{\omega}^2}) \\ \qquad \equiv -\mathrm{Carr}^{\boldsymbol{\omega}} + \sum_{\boldsymbol{\omega}^1 \boldsymbol{\omega}^2 = \boldsymbol{\omega}} \mathrm{Carr}^{\omega_0 \boldsymbol{\omega}^2} \mathrm{Carr}^{\boldsymbol{\omega}^1} \end{cases}$$

If we look for a solution $\mathrm{Reg}^{\bullet}$ of the form:

$$\mathrm{Reg}^{\omega_0 \boldsymbol{\omega}} = \sum_{\boldsymbol{\omega}^1 \boldsymbol{\omega}^2 = \boldsymbol{\omega}} \mathrm{Rect}^{\omega_0 \boldsymbol{\omega}^2} \mathrm{Carr}^{\boldsymbol{\omega}^1} \tag{3.60}$$

we see that (3.59) transforms into:

$$(\nabla \mathrm{Rect}^{\bullet}) + (1^{\bullet} + \mathrm{Rect}^{\bullet}) \times (I^{\bullet} - \mathrm{Carr}^{\bullet}) \equiv 0 \tag{3.61}$$

with ∇ denoting multiplication by $\|\boldsymbol{\omega}\|$ and with the initial conditions:

$$\mathrm{Rect}^{\varnothing} = 0;\ \mathrm{Rect}^{\omega_1} = 0 \quad (\forall \omega_1). \tag{3.61*}$$

Now, the mould $\mathrm{Reg}^{\bullet}$ defined by (3.60) verifies, like the auxiliary mould $\mathrm{Saa}^{\bullet}$ (see (3.6))

$$\mathrm{Reg}^{\boldsymbol{\omega}} = \mathrm{Saa}^{\boldsymbol{\omega}} = 0 \quad \text{if} \quad \|\boldsymbol{\omega}\| = 0 \tag{3.61**}$$

and, just like $\mathrm{Saa}^{\bullet}$ and $\mathrm{Carr}^{\bullet}$, it is defined for any sequence $\boldsymbol{\omega}$. Using this remark, it is immediate to check that the mould $\mathrm{Rect}^{\bullet}$, as defined by (3.57), satisfies the mould equation (3.61), so that the corresponding mould $\mathrm{Reg}^{\bullet}$ satisfies (5.59) and, by way of consequence, the identity (5.55) holds.

We are now in a position to complete the proof of Proposition 3.1. Indeed, for any infinitesimal automorphism (3.27*) acting on X, X^{carr} and δX^{carr} are characterized respectively by:

$$[X^{\mathrm{lin}}, X^{\mathrm{carr}}] = 0 \quad \text{and} \quad \{X^{\mathrm{carrd}} \text{ conjugate to } X^{\mathrm{lin}}\} \tag{3.63}$$

$$(3.64) \qquad \begin{cases} [X^{\mathrm{lin}}, X^{\mathrm{carr}} + \delta X^{\mathrm{carr}}] = 0 \\ \text{and } \{X^{\mathrm{carrd}} + \delta X^{\mathrm{carrd}} \text{ conjugate to } X^{\mathrm{lin}}\} \end{cases}$$

Therefore the characterization (3.64) may be replaced by:

$$(3.65) \qquad \begin{cases} [X^{\mathrm{lin}}, \delta X^{\mathrm{carr}}] = 0 \\ \text{and } \{X^{\mathrm{carrd}} + \delta X^{\mathrm{carrd}} \text{ conjugate to } X^{\mathrm{carrd}}\}. \end{cases}$$

But the identity (3.51) shows precisely that, when defining $\delta X^{\mathrm{carrd}}$ as in (3.50), the field $X^{\mathrm{carrd}} + \delta X^{\mathrm{carrd}}$ is conjugate to X^{carrd} under the infinitesimal automorphism:

$$(3.66)\qquad \begin{cases} X^{\mathrm{carrd}} \to X^{\mathrm{carrd}} + \delta X^{\mathrm{carrd}} \\ \qquad = (1 - \varepsilon(C - C^{**}))X^{\mathrm{carrd}}(1 + \varepsilon(C - C^{**})) \end{cases}$$

with C^{**} as in (3.56), (3.57).

This ends the proof of Proposition 3.1. We omit that of Proposition 3.2, since it is exactly on the same lines. But we shall mention an alternative proof of Proposition 3.1, which is both simpler and more informative. It involves proving the following:

Lemma 3.6. *When X is subjected to an infinitesimal automorphism:*

$$(3.67)\qquad X \to X + \delta X = X + \varepsilon[X, C] \quad (\mathrm{hom} C = \omega_0 \neq 0)$$

as in (3.27*), the correction and corrected form change according to the rules:

$$(3.68)\qquad \delta X^{\mathrm{carr}} = \varepsilon[X^{\mathrm{carr}}, C^*]$$

$$(3.69)\qquad \delta X^{\mathrm{carrd}} = \varepsilon[X^{\mathrm{carrd}}, C - C^{**}]$$

with

$$(3.68^*)\qquad C^* = \sum_{\boldsymbol{\omega}} \mathrm{Carr}^{\omega_0 \boldsymbol{\omega}}(\bar{B})_{\boldsymbol{\omega}}[X^{\mathrm{carr}}, C]$$

$$(3.69^*)\qquad C^{**} = \sum_{\boldsymbol{\omega}} \mathrm{Carr}^{\omega_0 \boldsymbol{\omega} \omega_{00}}(\bar{B})_{\boldsymbol{\omega}}[X^{\mathrm{carr}}, C]$$

with sums extending to all sequences $\boldsymbol{\omega}$ (including $\boldsymbol{\omega} = \varnothing$, in which case $(\bar{B})_{\varnothing} = 1$) and with a last componant ω_{00} defined by

$$(3.69^{**})\qquad \omega_0 + \|\boldsymbol{\omega}\| + \omega_{00} = 0$$

Remark. We observe the usual "complementarity" between the field C^*, which carries only resonant terms, and the field C^{**}, which carries only non-resonant terms. Indeed, the resonant terms of C^{**}, if there were any, would correspond to $\omega_{00} = 0$ in (3.69**), but when ω_{00} vanishes, so does $\mathrm{Carr}^{\omega_0 \boldsymbol{\omega} \omega_{00}}$. Conversely, one may show that C^{**} is the only vector field without resonant terms which, when bracketed with X^{carrd} as in (3.69), yields the variation $\delta X^{\mathrm{carrd}}$.

Simultaneous proof of the variance rule (3.33) and of Lemma 3.6.

Due to the variational characterization (3.65) of the correction, all that is required is *establishing the compatibility of the variance rules (3.33) with the explicit variations of Lemma 3.6.* Now, (3.33) trivially implies (3.68), (3.68*). But it also implies (3.69), (3.69*).

Indeed, if we consider the expression:

$$(3.70)\quad \begin{cases} -\delta X^{\mathrm{carrd}} + \varepsilon[X^{\mathrm{carrd}}, C - C^{**}] \\ \quad \equiv -\varepsilon[X, C] + \varepsilon[X^{\mathrm{carr}}, C^{*}] + \varepsilon[X - X^{\mathrm{carr}}, C - C^{**}] \end{cases}$$

with C^* and C^{**} as in Lemma 3.6, we see that, for sequences $\boldsymbol{\omega}$ of length $r(\boldsymbol{\omega}) \geq 2$, the term:

$$(3.71)\qquad (\bar{B})_{\boldsymbol{\omega}} C \overset{\mathrm{def}}{=} [B_{\omega_r} \ldots [B_{\omega_2}, [B_{\omega_1}, C]] \ldots]$$

occurs in (3.70) with a scalar factor $\Gamma_{\boldsymbol{\omega}}$ of the form:

$$(3.72)\qquad \Gamma_{\boldsymbol{\omega}} \equiv -\mathrm{Carr}^{\boldsymbol{\omega}} + \Gamma^1_{\boldsymbol{\omega}} + \Gamma^2_{\boldsymbol{\omega}} + \Gamma^3_{\boldsymbol{\omega}} + \Gamma^4_{\boldsymbol{\omega}}$$

with:

$$(3.73)\qquad \Gamma^1_{\boldsymbol{\omega}} = + \sum_{\mathbf{ab}=\boldsymbol{\omega}} \mathrm{Carr}^{\omega_0 \mathbf{b}} \mathrm{Carr}^{\mathbf{a}} = + \sum \Gamma^1_{\mathbf{a};\mathbf{b}}$$

$$(3.74)\qquad \Gamma^2_{\boldsymbol{\omega}} = + \sum_{\mathbf{ab}=\boldsymbol{\omega}} \omega_* \mathrm{Carr}^{\omega_0 \mathbf{b} \omega_*} \mathrm{Carr}^{\mathbf{a}} = + \sum \Gamma^2_{\mathbf{a};\mathbf{b}}$$

$$(3.75)\qquad \Gamma^3_{\boldsymbol{\omega}} = - \sum_{\substack{\mathbf{ab}=\boldsymbol{\omega} \\ \mathbf{b} \neq \varnothing}} \mathrm{Carr}^{\omega_0 \mathbf{b}' \omega_{**}} \mathrm{Carr}^{\mathbf{a}} = + \sum \Gamma^3_{\mathbf{a};\mathbf{b}}$$

$$(3.76)\qquad \Gamma^4_{\boldsymbol{\omega}} = + \sum_{\mathbf{acd}=\boldsymbol{\omega}} \mathrm{Carr}^{\mathbf{d}} \mathrm{Carr}^{\omega_0 \mathbf{c} \omega_{***}} \mathrm{Carr}^{\mathbf{a}} = + \sum \Gamma^4_{\mathbf{a};\mathbf{c};\mathbf{d}}$$

Here, $\boldsymbol{\omega}, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ denote *sequences* and $\mathbf{b}'$ in (3.75) denotes the sequence $\mathbf{b}$ deprived of its last component (which makes sense, since $\mathbf{b} \neq \varnothing$). As for the components $\omega_*, \omega_{**}, \omega_{***}$, they are so defined as to secure a vanishing sum for the sequences which they conclude:

$$(3.77)\quad \omega_0 + \|\mathbf{b}\| + \omega_* = 0;\ \omega_0 + \|\mathbf{b}'\| + \omega_{**} = 0;\ \omega_0 + \|\mathbf{c}\| + \omega_{***} = 0$$

The sum $\Gamma^1_{\boldsymbol{\omega}}$ stems from $[X^{\mathrm{carr}}, C^*]$, and all other terms stem from the bracket $[X - X^{\mathrm{carr}}, C - C^{**}]$. It is then an easy matter to check that $\Gamma_{\boldsymbol{\omega}} = 0$. Indeed, when $\omega_0 + \|\boldsymbol{\omega}\| \neq 0$, we see that:

$$(3.78)\qquad \Gamma^1_{\boldsymbol{\omega}} = 0;\ -\mathrm{Carr}^{\boldsymbol{\omega}} + \Gamma^2_{\boldsymbol{\omega};\varnothing} = 0$$

$$\Gamma^2_{\mathbf{a};\mathbf{b}} + \Gamma^3_{\mathbf{a};\mathbf{b}} + \sum_{\mathbf{cd}=\mathbf{b}} \Gamma^4_{\mathbf{a};\mathbf{c};\mathbf{d}} = 0 \tag{3.79}$$

The latter identity results from applying (3.33) to $\mathrm{Carr}^{\omega_0 \mathbf{b} \omega_*}$ with $i = r(\omega_0 \mathbf{b} \omega_*)$.

Similarly, when $\omega_0 + \|\boldsymbol{\omega}\| = 0$

$$\mathrm{Carr}^{\boldsymbol{\omega}} = 0 \tag{3.80}$$

$$\Gamma^2_{\boldsymbol{\omega}} = 0 \quad (\text{since } \omega_* = 0 \text{ if } \|\mathbf{a}\| = 0) \tag{3.81}$$

$$\Gamma^4_{\boldsymbol{\omega}} = 0 \quad (\text{since } \omega_{***} = 0 \text{ if } \|\mathbf{a}\| = \|\mathbf{d}\| = 0) \tag{3.82}$$

$$\Gamma^1_{\boldsymbol{\omega}} + \Gamma^3_{\boldsymbol{\omega}} = 0 \quad (\text{since } (\mathbf{b}'\omega_{**}) = \mathbf{b} \text{ if } \|\mathbf{a}\| = 0) \tag{3.83}$$

One deals in the same way with the far simpler case when the sequence $\boldsymbol{\omega}$ has length $r(\boldsymbol{\omega}) = 1$. Thus, $\Gamma_{\boldsymbol{\omega}} \equiv 0$. Therefore, the variance rule (3.33) is compatible with the variation formulae of Lemma 3.6 and, *as a consequence* (due once again to (3.65)), *both are valid!*

Remark. We may check that our earlier formula (3.56) + (3.57) for $\delta X^{\mathrm{carrd}}$ agrees with our new and simpler formula (3.69). Indeed, (3.57) may be rephrased as:

$$\mathrm{Rect}^{\omega_0 \bullet} \equiv (\mathrm{Taa}^{\omega_0 \bullet \omega_{00}}) \circ (I^\bullet - \mathrm{Carr}^\bullet) \tag{3.84}$$

with $\omega_0 + \|\bullet\| + w_{00} = 0$. But since, as we shall establish towards the end of Sect. 4:

$$\mathrm{Taa}^\bullet \circ (I^\bullet - \mathrm{Carr}^\bullet) \equiv 0 \tag{3.85}$$

and since $\mathrm{Taa}^0 = 0$, it is plain that the right-hand side of (3.84) reduces to:

$$\mathrm{Rect}^{\omega_0 \bullet} \equiv 0 - (\mathrm{Taa}^{\omega_0 + \|\bullet\| + \omega_{00}})(I - \mathrm{Carr})^{\omega_0 \bullet \omega_{00}} \equiv +\mathrm{Carr}^{\omega_0 \bullet \omega_{00}} \tag{3.86}$$

Uniqueness of the correction

Lemma 3.7. *For any vector field* $X = X^{\mathrm{lin}} + \sum B_n$ $(n \in N)$ *with diagonal linear part* X^{lin} *and homogeneous components* B_n*, there exists exactly one "correction"* Y *characterized by:*

$$(X - Y) \text{ formally conjugate to } X^{\mathrm{lin}} \tag{3.87}$$

$$[Y, X^{\mathrm{lin}}] = 0 \tag{3.88}$$

Proof. In search of a contradiction, let us assume that there exists a correction Y distinct from X^{carr}. It is necessarily of the form:

$$Y = X^{\mathrm{carr}} + Z = \sum_{n \in \mathbf{N}_0} (X^{\mathrm{carr}})_n + \sum_{n \in \mathbf{N}_0^*} Z_n \tag{3.89}$$

Here, $\mathbf{N}_0$ denotes the set of all multi-integers n orthogonal to λ (i.e. $\langle n, \lambda \rangle = 0$) and $\mathbf{N}_0^*$ denotes the *non-empty* subset of $\mathbf{N}_0$ which corresponds to non-vanishing components Z_n of Z:

$$\{n \in \mathbf{N}_0^*\} \Leftrightarrow \{Z_n \neq 0\} \tag{3.90}$$

Further, let $\min(\mathbf{N}_0^*)$ denote the set of all *minimal elements* of $\mathbf{N}_0^*$ (relative to the natural order on N) and let $\mathrm{int}(\mathbf{N}_0^*)$ be the set of all n in $\mathbf{N}_0$ such that n be strictly superior to some n^* in $\min(\mathbf{N}_0^*)$, again relatively to the natural order.

We may write $X - Y$ as a sum of homogeneous components:

$$\left\{ \begin{aligned} X - Y &\equiv X - X^{\mathrm{carr}} - Z \equiv X^{\mathrm{carrd}} - Z \\ &\equiv X^{\mathrm{lin}} + \textstyle\sum_{n \in N} (X^{\mathrm{carrd}})_n - \sum_{n \in \mathbf{N}_0^*} Z_n \end{aligned} \right\} \tag{3.91}$$

But since we assumed Y to be a "correction", we must have:

$$(X - Y)^{\mathrm{nal}} = 0 \quad \text{i.e.} \quad (X^{\mathrm{carrd}} - Z)^{\mathrm{nal}} = 0 \tag{3.92}$$

where $(\ldots)^{\mathrm{nal}}$ denotes the *nilpotent part* of $(\ldots)$, expanded as in (3.1). Within the resulting expansion, let us sort out the parts A, B, C which are respectively of degree 0; 1; or $\geq$2 in Z. We clearly have:

$$(X^{\mathrm{carrd}} - Z)^{\mathrm{nal}} = A + B + C = 0 \tag{3.93}$$

$$A \equiv 0 \quad (\text{degree } 0 \text{ in } Z) \tag{3.94}$$

$$B \equiv \sum_{n \in \min(\mathbf{N}_0^*)} Z_n + \sum_{n \in int(\mathbf{N}_0^*)} (\ldots) \quad (\text{degree } 1 \text{ in } Z) \tag{3.95}$$

$$C \equiv \sum_{n \in int(\mathbf{N}_0^*)} (\ldots) \quad (\text{degree } \geq 2 \text{ in } Z) \tag{3.96}$$

Therefore:

$$(X^{\mathrm{carrd}} - Z)^{\mathrm{nal}} = \sum_{n \in \min(\mathbf{N}_0^*)} Z_n + \sum_{n \in int(\mathbf{N}_0^*)} (\ldots) = 0 \tag{3.97}$$

But the first bracket on the right-hand side of (3.97) represents, by construction, a non-zero vector field. So $(X^{\mathrm{carrd}} - Z)^{\mathrm{nal}}$ itself ought to be $\neq 0$. This contradiction establishes the lemma.

4. Analyticity of the corrected form (for vector fields)

Proposition 4.1. *Let X be a local, analytic, resonant vector field whose spectrum $\lambda = (\lambda_1, \dots, \lambda_\nu)$ meets Brjuno's diophantine condition (1.9). Then the correction X^{carr} and corrected form $X^{\mathrm{carrd}} \stackrel{\mathrm{def}}{=} X - X^{\mathrm{carr}}$ are analytic, and their arborified mould expansions:*

$$X^{\mathrm{carr}} = \sum \mathrm{Carr}^{\stackrel{<}{\bullet}} B_{\stackrel{<}{\bullet}} \; ; \; X^{\mathrm{carrd}} = \sum \mathrm{Carrd}^{\stackrel{<}{\bullet}} B_{\stackrel{<}{\bullet}} \tag{4.1}$$

are normally convergent.

Elucidations. "Arborification" of course means that we subject $\mathrm{Carr}^\bullet$ (resp. $\mathrm{Carrd}^\bullet$) and $B_\bullet$ to the dual changes introduced in (2.25), (2.26), (2.27). "Normal convergence" means that:

$$\sum |\mathrm{Carr}^{\stackrel{<}{\bullet}}| \; \|B_{\stackrel{<}{\bullet}}\|_{U,V} < +\infty \; ; \; \sum |\mathrm{Carrd}^{\stackrel{<}{\bullet}}| \; \|B_{\stackrel{<}{\bullet}}\|_{U,V} < +\infty \tag{4.2}$$

for a suitable norm $\| \bullet \|_{U,V}$ defined on the endomorphisms of $\mathbf{C}\{x\}$ and indexed by a pair $U \subset V$ of small enough neighbourhoods of $0 \in \mathbf{C}^\nu$. The definition is as follows:

$$\begin{cases} \|B\|_{U,V} \stackrel{\mathrm{def}}{=} Sup_{\|\varphi\|_V \leq 1} \|B \cdot \varphi\|_U \\ \{B \in \mathrm{Endo}(\mathbf{C}\{x\}); \; \varphi \in \mathbf{C}\{x\}\} \end{cases} \tag{4.3}$$

with the usual sup norm on φ:

$$\|\varphi\|_V \stackrel{\mathrm{def}}{=} Sup_{x \in V} |\varphi(x)| \quad \{\varphi \in \mathbf{C}\{x\}; \; x \in V\} \tag{4.4}$$

Proof of Proposition 4.1. The proof rests entirely on the *variance rules* which govern the inductive calculation of $\mathrm{Carr}^\bullet$ and $\mathrm{Carr}^{\stackrel{<}{\bullet}}$. At each inductive step, these rules allow great flexibility in the choice of the index i in var_i; which flexibility can be taken advantage of to avoid the occurence of *illusory multiple poles* or, to put it another way, to establish the *non-repetition of the "small denominators"*.

Lemma 4.1. *Arborification of the variance rule.*
The variance rule (3.33) arborifies as follows:

$$\begin{cases} \mathrm{var}_i \mathrm{Carr}^{\stackrel{<}{\boldsymbol{\omega}}} = \displaystyle\sum_{\left\{\substack{\omega_i \text{ in } \stackrel{<}{\mathbf{a}} \\ \stackrel{<}{\mathbf{b}} \text{ follows } \omega_i}\right\}} \mathrm{Carr}^{\stackrel{<}{\mathbf{a}}} \, \mathrm{Carr}^{\stackrel{<}{\mathbf{b}}} \\ \qquad\qquad - \displaystyle\sum_{\left\{\substack{\omega_i \text{ in } \stackrel{<}{\mathbf{c}} \\ \stackrel{<}{\mathbf{b}} \text{ precedes } \omega_i}\right\}} \mathrm{Carr}^{\stackrel{<}{\mathbf{b}}} \mathrm{Carr}^{\stackrel{<}{\mathbf{c}}} \end{cases} \tag{4.5}$$

Here, the sums extend to all connected subsequences $\overset{<}{\mathbf{b}}$ of $\overset{<}{\boldsymbol{\omega}}$ either directly following or preceding ω_i, and $\overset{<}{\mathbf{a}}$ (resp. $\overset{<}{\mathbf{c}}$) denotes the remaining part of $\overset{<}{\boldsymbol{\omega}}$, with the arborescent order inherited from the parent sequence $\overset{<}{\boldsymbol{\omega}}$. As for the arborified variance operator var_i, it acts as follows:

$$\mathrm{var}_i \, \mathrm{Carr}^{\overset{<}{\boldsymbol{\omega}}} = \omega_i \, \mathrm{Carr}^{\overset{<}{\boldsymbol{\omega}}} - \mathrm{Carr}^{\overset{<}{\boldsymbol{\eta}}} + \sum \mathrm{Carr}^{\overset{<}{\boldsymbol{\sigma}}} \tag{4.6}$$

where $\overset{<}{\boldsymbol{\eta}}$ denotes the unique arborified sequence obtained by contracting ω_i with its immediate predecessor $\omega_{i\_}$; and the $\overset{<}{\boldsymbol{\sigma}}$ denote the sequences $\overset{<}{\boldsymbol{\sigma}}$ obtained by contracting ω_i with any one of its immediate successors ω_j (i.e. $\omega_{j\_} = \omega_i$).

The verification is straightforward. One first checks that the definition (3.30) of the variance *arborifies* according to (4.6). Then one checks that the right-hand side of (3.33), under arborification, yields the right-hand side of (4.6). Three points, however, deserve mention. *First*, due to the alternality of $\mathrm{Carr}^{\bullet}$, it is enough to consider *connected sequences* $\overset{<}{\mathbf{a}}$, $\overset{<}{\mathbf{b}}$, $\overset{<}{\mathbf{c}}$ i.e. sequences with one root only, since for disconnected sequences $\overset{<}{\mathbf{a}}$, $\overset{<}{\mathbf{b}}$, $\overset{<}{\mathbf{c}}$, the arborified expressions $\mathrm{Carr}^{\overset{<}{\mathbf{a}}}$, $\mathrm{Carr}^{\overset{<}{\mathbf{b}}}$, $\mathrm{Carr}^{\overset{<}{\mathbf{c}}}$ vanish. *Second*, saying that $\overset{<}{\mathbf{b}}$ immediately follows (resp. precedes) ω_i means that the root ω_j of $\overset{<}{\mathbf{b}}$ has ω_i as its direct antecedent (resp. that the direct antecedent $\omega_{i\_}$ of ω_i lies in $\overset{<}{\mathbf{b}}$). *Third*, the sequences $\overset{<}{\mathbf{a}}$, $\overset{<}{\mathbf{b}}$, $\overset{<}{\mathbf{c}}$ inherit the arborescent order of $\overset{<}{\boldsymbol{\omega}}$. For $\overset{<}{\mathbf{b}}$ this presents no difficulty. For $\overset{<}{\mathbf{a}}$ (resp. $\overset{<}{\mathbf{c}}$), however, it means that the components of $\overset{<}{\boldsymbol{\omega}}$ which (possibly) immediately follow $\overset{<}{\mathbf{b}}$ (resp. *the* component which precedes $\overset{<}{\mathbf{b}}$) become direct successors of ω_i in $\overset{<}{\mathbf{a}}$ (resp. becomes the direct predecessor of ω_i in $\overset{<}{\mathbf{c}}$) although, within the original sequences $\overset{<}{\boldsymbol{\omega}}$, these components were "separated" from ω_i by $\overset{<}{\mathbf{b}}$. Thus, $\overset{<}{\mathbf{a}}$ and $\overset{<}{\mathbf{b}}$ are connected "in themselves", but not necessarily as subsequences of $\overset{<}{\boldsymbol{\omega}}$.

Lemma 4.2 *Non-repetition of the denominators in* $\mathrm{Carr}^{\boldsymbol{\omega}}$.

For sequences $\boldsymbol{\omega}$, of a given length r and of a fixed degeneracy pattern of order d (see below) the correction mould $\mathrm{Carr}^{\boldsymbol{\omega}}$, as a rational function of its $(r-d)$ independent variables ω_i, has only poles of the form $(\eta)^{-\mu}$, with linear combinations η of the ω_i obtained by splitting unbreakable zero-sum subsequences $\boldsymbol{\omega}^*$ of $\boldsymbol{\omega}$:

$$\pm\eta = \|\boldsymbol{\omega}'\| = -\|\boldsymbol{\omega}''\| \quad (\boldsymbol{\omega}'\boldsymbol{\omega}'' = \boldsymbol{\omega}^* = (\omega_i, \ldots, \omega_j)) \tag{4.7}$$

and with a multiplicity μ no larger than the number of unbreakable, zero-sum sequences $\boldsymbol{\omega}^*$ which, when split, can produce η. Moreover, although *there is in general no canonical "best way" of decomposing* $\mathrm{Carr}^{\boldsymbol{\omega}}$*, there always exist decompositions of the form:*

$$\text{(4.8)}\qquad \begin{cases} \mathrm{Carr}^{\boldsymbol{\omega}} = \sum_p n_p (\eta_{p,1})^{-\mu_{p,1}} (\eta_{p,2})^{-\mu_{p,2}} \dots (\eta_{p,s_p})^{-\mu_{p,s_p}} \\ \qquad \left(\sum_i \mu_{p,i} \equiv r-1 \right) \end{cases}$$

which involve only effective poles η*, with a multiplicity never exceeding their effective multiplicity* μ*; and with integral coefficients* n_p *bounded by:*

$$\text{(4.9)}\qquad \sum_p |n_p| \leq \frac{(2r-2)!}{(r-1)!r!} \leq 4^r \quad (r = r(\boldsymbol{\omega});\ n_p \in \mathbf{Z} - \{0\})$$

Comments and proof A fixed *degeneracy pattern* of order d is a set of d pairs (i,j) verifying $1 \leq i < j \leq r$ and such that the sequence $\boldsymbol{\omega}^* = (\omega_i, \dots, \omega_j)$ be unbreakable with zero-sum. "*Unbreakability*", once again, *rules out* non-trivial factorizations $\boldsymbol{\omega}^* = \boldsymbol{\omega}'\boldsymbol{\omega}''$ with either $\|\boldsymbol{\omega}'\| = 0$ or $\|\boldsymbol{\omega}''\| = 0$, but it *does not rule out* non-trivial factorizations of the form $\boldsymbol{\omega}^* = \boldsymbol{\omega}'\boldsymbol{\omega}''\boldsymbol{\omega}'''$ with a zero-sum middle factor $\|\boldsymbol{\omega}''\| = 0$.

What the above lemma tells us about the multiplicities μ of the poles $(\eta)^{-\mu}$ amounts to saying, in effect, that the denominators η of $\mathrm{Carr}^{\boldsymbol{\omega}}$ *undergo no repetitions unless they are already repeated within the sequence* $\boldsymbol{\omega}$*, these repetitions being induced by the degeneracy pattern itself.* This is in complete contrast with the behaviour of most other moulds, such as $\mathrm{Sa}^{\boldsymbol{\omega}}$, $\mathrm{invSa}^{\boldsymbol{\omega}}$, $\mathrm{Nal}^{\boldsymbol{\omega}}$, and *all* prenormalizing moulds $Pran^{\boldsymbol{\omega}}$.

Thus, if $\boldsymbol{\omega}$ has an even length $r = 2r'$ and the following degeneracy pattern:

$$\text{(4.10)}\qquad 0 = \omega_1 + \omega_r = \omega_2 + \omega_3 = \omega_4 + \omega_5 = \omega_6 + \omega_7 = \dots = \omega_{r-2} + \omega_{r-1} = 0$$

(which leaves as independent variables ω_1 and $\omega_2, \omega_4, \omega_6, \dots, \omega_{r-2}$) we have on the one hand, for all $Pran^{\boldsymbol{\omega}}$ that are free of parasitical singularities:

$$\text{(4.11)}\qquad Pran^{\boldsymbol{\omega}} = -(\omega_1)^{-r'} \prod_{i=1}^{r'-1} (\omega_1 + \omega_{2i})^{-1}$$

with a "huge" multiplicity (either r' or $1+r'$) for ω_1; and on the other hand, calculating $\mathrm{Carr}^{\boldsymbol{\omega}}$ by the procedure which we shall spell out in a moment, we find:

$$\text{(4.12)}\qquad \mathrm{Carr}^{\boldsymbol{\omega}} \equiv (-1)^{r'-1} (\omega_1)^{-1} \prod_{1 \leq i \leq r'} ((\omega_{2i})(\omega_1 + \omega_{2i}))^{-1}$$

in full agreement with the above lemma, and with *no unwarranted repetitions of poles*.

As for the proof of Lemma 4.2, it is amazingly simple and rests entirely on the repeated application of the *variance rule* (3.33) relative to an index i subject to a simple selection rule $[C1]+[C2]$ (infra), which automatically takes care of the non-repetition of poles.

Let us fix a sequence $\boldsymbol{\omega}$ with a given length $r = r(\boldsymbol{\omega})$ and a given degeneracy pattern of order d, which leaves exactly $r-d$ independent variables ω_i. We make no assumption whatsoever on the nature of the ZUS in $\boldsymbol{\omega}$ (from now on, ZUS will stand for "*zero-sum, unbreakable sequence*") except that all should have length ≥ 2, for otherwise $\mathrm{Carr}^{\boldsymbol{\omega}}$ would be $\equiv 0$. That aside, $\boldsymbol{\omega}$ may have as many as $r-1$ different ZUS and these may overlap in any conceivable way.

We say that a ZUS $\boldsymbol{\omega}^*$ is adjacent to a component ω_i if ω_i either *initiates or terminates* $\boldsymbol{\omega}^*$ (i.e. if it is its first or last element) or if it immediately *precedes or follows* $\boldsymbol{\omega}^*$. Thus, any ω_i has at most four distinct adjacent ZUS. If ω_i *initiates* a (necessarily unique) ZUS $\boldsymbol{\omega}^+$, we set $r_i^+ = r(\boldsymbol{\omega}^+) =$ length of $\boldsymbol{\omega}^+$ and, if not, we set $r_i^+ = 0$. Similarly, if ω_i terminates a (necessarily unique) ZUS $\boldsymbol{\omega}^-$, we set $r_i^- = r(\boldsymbol{\omega}^-)$ and, if not, we set $r_i^- = 0$.

The selection rules for the index i of var_i read:

[C1] $$\{0 < r_i^+ + r_i^-\}$$

[C2] $$\begin{cases} \quad\ \{0 < r_i^+ \leq r_{i+1}^+ \ or\ 0 = r_{i+1}^+\} \\ and\ \ \{0 < r_i^- \leq r_{i-1}^- \ or\ r_{i-1}^- = 0\} \end{cases}$$

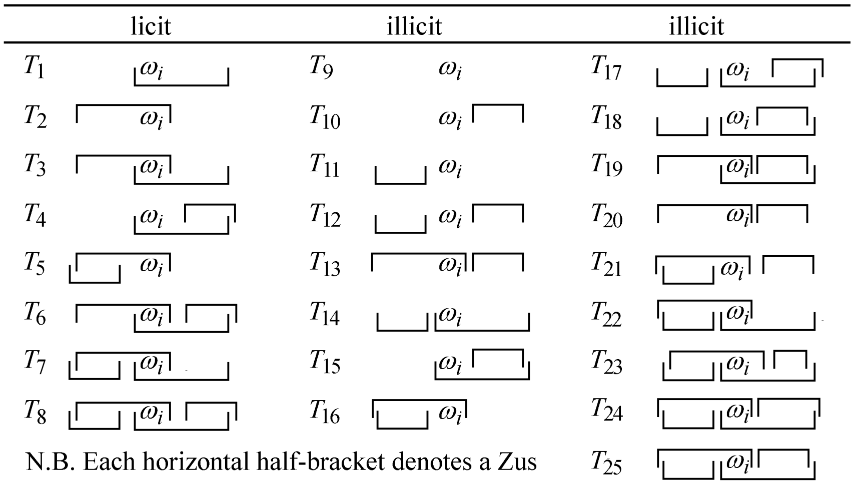
and they can be interpreted as follows:
[C1] says that ω_i *should* be the *first or last* component of some ZUS, or both.
[C2] says that ω_i *should not be squeezed in between two adjacent* ZUS $\boldsymbol{\omega}^*$ and $\boldsymbol{\omega}^{**}$, with $\boldsymbol{\omega}^*$ included in $\boldsymbol{\omega}^{**}$. It says, too, that if ω_i is *externally adjacent* to a ZUS $\boldsymbol{\omega}^*$, it should also be *internally adjacent* to another ZUS $\boldsymbol{\omega}^{**}$ which overlaps with $\boldsymbol{\omega}^*$ (but doesn't contain it!).

It is immediate that there exist exactly $25 = 5^2$ distinct *adjacency types* $T_1, T_2, \dots T_{25}$ and that only $8 = 3^2 - 1$ of them, namely $T_1, T_2, \dots T_8$ are allowed by the selection rule $[C1]+[C2]$. All *licit* and *illicit* adjacency types are listed in Table 4.1.

Let us show that $\boldsymbol{\omega}$ has at least *two licit components* ω_i and ω_j. Indeed, suppose that $\omega_1, \omega_2, \omega_3, \dots, \omega_{s-1}, \omega_s$ are *not allowed* by $[C1]+[C2]$. By induction on $p = 1, 2, \dots, s+1$, we check that this is possible only if:

(4.13a) *for each* p in $\{1, \dots, s+1\},\ \omega_p\ initiates$ a ZUS $\boldsymbol{\omega}^p$.

Table 4.1 The licit and illicit adjacency types



(4.13b) $\quad$ *for no* p in $\{1,\ldots,s+1\}$ *does* ω_p *terminate* a ZUS.

$$\boldsymbol{\omega}^1 \supset \boldsymbol{\omega}^2 \supset \boldsymbol{\omega}^3 \supset \ldots \boldsymbol{\omega}^s \supset \boldsymbol{\omega}^{s+1} \neq \varnothing \tag{4.13c}$$

which in turn is possible only if $2s+2 \leq r$. Therefore, at least one component ω_i with $i \leq r/2$ is licit and, by symmetry, at least another component ω_j with $j > r/2$ is licit too. (Sometimes, there are only two licit components, but sometimes there can be as many as $2d$).

Next, a linear combination $\eta = \omega_p + \ldots + \omega_q$ is said to be a *formal pole* of the sequence $\boldsymbol{\omega}$ if it can be obtained by breaking up some ZUS, as in (4.7); and the *formal multiplicity* $\begin{bmatrix}\eta\\ \boldsymbol{\omega}\end{bmatrix}$ of η in $\boldsymbol{\omega}$ is defined as being the number of ZUS which, when split, can produce η.

We now fix some *licit* component ω_i, which is automatically (by $[C1]$) a *formal pole* of $\boldsymbol{\omega}$, with a *formal multiplicity* $\begin{bmatrix}\omega_i\\ \boldsymbol{\omega}\end{bmatrix} = l \geq 1$. Associated with ω_i, there is a unique factorization of $\boldsymbol{\omega}$ of the form:

$$\begin{cases} \boldsymbol{\omega} = \mathbf{w}^{*,m}\mathbf{w}^{m}\mathbf{w}^{m,m+1}\mathbf{w}^{m+1}\ldots\mathbf{w}^{-1}\mathbf{w}^{-1,0}\mathbf{w}^{0}\mathbf{w}^{0,1}\mathbf{w}^{1} \\ \qquad \ldots\mathbf{w}^{n-1}\mathbf{w}^{n-1,n}\mathbf{w}^{n,*} \end{cases} \tag{4.14}$$

with *non-empty* sequences $\mathbf{w}^s$ and *possibly empty* sequences $\mathbf{w}^{*,m}$, $\mathbf{w}^{s,s+1}$, $\mathbf{w}^{n,*}$ such that:

$$\mathbf{w}^0 = (\omega_i) = (\text{one term only});\ \|\mathbf{w}^s\| = (-1)^s\omega_i \quad (\forall s) \tag{4.14a}$$

$$m \leq 0 \leq n \text{ and } n - m = l = \begin{bmatrix} \omega_i \\ \boldsymbol{\omega} \end{bmatrix} \geq 1 \quad (m, n \in \mathbf{Z}) \tag{4.14b}$$

$$\mathbf{w}^{[s,s+1]} \stackrel{\text{def}}{=} \mathbf{w}^s \mathbf{w}^{s,s+1} \mathbf{w}^{s+1} \text{ is a ZUS} \quad (\forall s \in [m, n[) \tag{4.14c}$$

$$\begin{cases} \mathbf{w}^{s,s+1} \text{ has zero-sum and factors into } l^{s,s+1} \\ \text{different ZUS (with } l^{s,s+1} \geq 0) \end{cases} \tag{4.14d}$$

Alongside the ZUS of the form $\mathbf{w}^{[s,s+1]}$, there may of course exist in $\boldsymbol{\omega}$ many other ZUS (which may or may not overlap with the $\mathbf{w}^{[s,s+1]}$ and among themselves) but these additional ZUS have no bearing on the formal multiplicity of the pole ω_i.

Now, the variance rule (3.33) may be rewritten as:

$$\begin{aligned} \mathrm{Carr}^{\boldsymbol{\omega}} \equiv \frac{1}{\omega_i} \Big\{ &+\mathrm{Carr}^{\boldsymbol{\omega}^1} - \mathrm{Carr}^{\boldsymbol{\omega}^2} + \sum \mathrm{Carr}^{\boldsymbol{\omega}^3} \mathrm{Carr}^{\boldsymbol{\omega}^4} \\ &- \sum \mathrm{Carr}^{\boldsymbol{\omega}^5} \mathrm{Carr}^{\boldsymbol{\omega}^6} \Big\} \end{aligned} \tag{4.15}$$

with:

$$\boldsymbol{\omega}^1 = (\ldots, \omega_{i-1} + \omega_i, \ldots);\ \boldsymbol{\omega}^2 = (\ldots, \omega_i + \omega_{i+1}, \ldots) \tag{4.15a}$$

$$\boldsymbol{\omega}^3 = \boldsymbol{\omega}^{3,4} \omega_i \boldsymbol{\omega}^{4,3} \text{ with } \boldsymbol{\omega} = \boldsymbol{\omega}^{3,4} \omega_i \boldsymbol{\omega}^4 \boldsymbol{\omega}^{4,3} \tag{4.15b}$$

$$\boldsymbol{\omega}^6 = \boldsymbol{\omega}^{6,5} \omega_i \boldsymbol{\omega}^{5,6} \text{ with } \boldsymbol{\omega} = \boldsymbol{\omega}^{6,5} \boldsymbol{\omega}^5 \omega_i \boldsymbol{\omega}^{5,6} \tag{4.15c}$$

(Take care to distinguish $\boldsymbol{\omega}^i$, $\boldsymbol{\omega}^{i,j}$ from $\mathbf{w}^s$, $\mathbf{w}^{s,s+1}$).

Reverting to (4.14), we may now calculate the *formal multiplicity* of ω_i within $\boldsymbol{\omega}^1, \boldsymbol{\omega}^2, \boldsymbol{\omega}^3, \boldsymbol{\omega}^4, \boldsymbol{\omega}^5, \boldsymbol{\omega}^6$. We find:

$$\begin{cases} \begin{bmatrix} \omega_i \\ \boldsymbol{\omega}^1 \end{bmatrix} = l - 1 (\text{resp. } l - 2) \text{ if } \omega_i \\ \qquad \text{has no (resp. 1 or 2) left-adjacent ZUS} \end{cases} \tag{4.16a}$$

$$\begin{cases} \begin{bmatrix} \omega_i \\ \boldsymbol{\omega}^2 \end{bmatrix} = l - 1 (\text{resp. } l - 2) \\ \qquad \text{if } \omega_i \text{ has no (resp. 1 or 2) right-adjacent ZUS} \end{cases} \tag{4.16b}$$

(4.16c)

$$\begin{bmatrix} \omega_i \\ \boldsymbol{\omega}^3 \end{bmatrix} = l - 2p;\ \begin{bmatrix} \omega_i \\ \boldsymbol{\omega}^4 \end{bmatrix} = 2p - 1;\ \begin{bmatrix} \omega_i \\ \boldsymbol{\omega}^3 \end{bmatrix} + \begin{bmatrix} \omega_i \\ \boldsymbol{\omega}^4 \end{bmatrix} = l - 1 \quad (2p \leq n)$$

Table 4.2 The decrease of formal multiplicity under a licit inductive step

Adjacent type	$\mathrm{Carr}^{\boldsymbol{\omega}}$	$\mathrm{Carr}^{\boldsymbol{\omega}^1}$	$\mathrm{Carr}^{\boldsymbol{\omega}^2}$	$\mathrm{Carr}^{\boldsymbol{\omega}^3}\mathrm{Carr}^{\boldsymbol{\omega}^4}$	$\mathrm{Carr}^{\boldsymbol{\omega}^5}\mathrm{Carr}^{\boldsymbol{\omega}^6}$
T_1	$\left[{\omega_i \atop \boldsymbol{\omega}}\right]=1$	$\left[{\omega_i \atop \boldsymbol{\omega}^1}\right]=0$	nothing	nothing	
T_2	$\left[{\omega_i \atop \boldsymbol{\omega}}\right]=1$	$\left[{\omega_i \atop \boldsymbol{\omega}^1}\right]=0$	$\left[{\omega_i \atop \boldsymbol{\omega}^2}\right]=0$	nothing	nothing
T_3	$\left[{\omega_i \atop \boldsymbol{\omega}}\right]=2$	$\left[{\omega_i \atop \boldsymbol{\omega}^1}\right]=0$	$\left[{\omega_i \atop \boldsymbol{\omega}^2}\right]=0$	nothing	nothing
T_4	$\left[{\omega_i \atop \boldsymbol{\omega}}\right]=l\geq 2$	$\left[{\omega_i \atop \boldsymbol{\omega}^1}\right]=l-1$	$\left[{\omega_i \atop \boldsymbol{\omega}^2}\right]=l-2$	$\left[{\omega_i \atop \boldsymbol{\omega}^3}\right]\div\left[{\omega_i \atop \boldsymbol{\omega}^4}\right]=l-1$	nothing
T_5	$\left[{\omega_i \atop \boldsymbol{\omega}}\right]=l\geq 2$	$\left[{\omega_i \atop \boldsymbol{\omega}^1}\right]=l-2$	$\left[{\omega_i \atop \boldsymbol{\omega}^2}\right]=l-1$	nothing	$\left[{\omega_i \atop \boldsymbol{\omega}^5}\right]+\left[{\omega_i \atop \boldsymbol{\omega}^6}\right]=l-1$
T_6	$\left[{\omega_i \atop \boldsymbol{\omega}}\right]=l\geq 3$	$\left[{\omega_i \atop \boldsymbol{\omega}^1}\right]=l-2$	$\left[{\omega_i \atop \boldsymbol{\omega}^2}\right]=l-2$	$\left[{\omega_i \atop \boldsymbol{\omega}^3}\right]\div\left[{\omega_i \atop \boldsymbol{\omega}^4}\right]=l-1$	nothing
T_7	$\left[{\omega_i \atop \boldsymbol{\omega}}\right]=l\geq 3$	$\left[{\omega_i \atop \boldsymbol{\omega}^1}\right]=l-2$	$\left[{\omega_i \atop \boldsymbol{\omega}^2}\right]=l-2$	nothing	$\left[{\omega_i \atop \boldsymbol{\omega}^5}\right]+\left[{\omega_i \atop \boldsymbol{\omega}^6}\right]=l-1$
T_8	$\left[{\omega_i \atop \boldsymbol{\omega}}\right]=l\geq 4$	$\left[{\omega_i \atop \boldsymbol{\omega}^1}\right]=l-2$	$\left[{\omega_i \atop \boldsymbol{\omega}^2}\right]=l-2$	$\left[{\omega_i \atop \boldsymbol{\omega}^3}\right]\div\left[{\omega_i \atop \boldsymbol{\omega}^4}\right]=l-1$	$\left[{\omega_i \atop \boldsymbol{\omega}^5}\right]+\left[{\omega_i \atop \boldsymbol{\omega}^6}\right]=l-1$

(4.16d)
$$\begin{bmatrix}\omega_i\\ \boldsymbol{\omega}^5\end{bmatrix}=2q-1;\begin{bmatrix}\omega_i\\ \boldsymbol{\omega}^6\end{bmatrix}=l-2q;\begin{bmatrix}\omega_i\\ \boldsymbol{\omega}^5\end{bmatrix}+\begin{bmatrix}\omega_i\\ \boldsymbol{\omega}^6\end{bmatrix}=l-1 \quad (2q\leq |m|)$$

if $\boldsymbol{\omega}^4$ (resp. $\boldsymbol{\omega}^5$) incorporates $2p$ (resp. $2q$) factors $\mathbf{w}^s$ (see (4.14)).

The total picture is summed up on Table 4.2.

Thus, we see that the formal multiplicity of ω_i in $\boldsymbol{\omega}^1$, $\boldsymbol{\omega}^2$, $(\boldsymbol{\omega}^3;\boldsymbol{\omega}^4)$ or $(\boldsymbol{\omega}^5;\boldsymbol{\omega}^6)$ is always less than in $\boldsymbol{\omega}$, but since we divide the right-hand side of (4.15) by ω_i, all we can say is that the *formal multiplicity* of ω_i *does not increase* under the *licit inductive step* (4.15). The same also holds for all poles η other than ω_i, but for a much simpler reason – namely, because the passage from $\boldsymbol{\omega}$ to $\boldsymbol{\omega}^1$, or $\boldsymbol{\omega}^2$, or $(\boldsymbol{\omega}^3;\boldsymbol{\omega}^4)$, or $(\boldsymbol{\omega}^5;\boldsymbol{\omega}^6)$ cannot increase the number of distinct ZUS capable of producing η by splitting. Indeed, these η-generating ZUS are also interlocked in a chain of type (4.14), except that (4.14a) must be replaced by:

$$\|\mathbf{w}^0\|=\eta;\|\mathbf{w}^s\|=(-1)^s\eta\,. \tag{4.17}$$

Therefore, any *contraction or split* of the total sequence $\boldsymbol{\omega}$ can only *lower*, or at most *leave unchanged, the total number of factors* $\mathbf{w}^s$ in the chain of η-generating ZUS.

We then apply the inductive scheme to each of the new terms on the right-hand side of (4.15), by choosing each time some new *licit component*. After $(r-1)$ inductive steps at most, we get $\mathrm{Carr}^{\boldsymbol{\omega}}$ expressed as a sum (4.8) and that sum involves only the *original formal poles* η of $\boldsymbol{\omega}$ (because of the selection rule [C1]) and that too with an *actual multiplicity* μ *no larger than their formal multiplicity* $\left[\begin{smallmatrix}\eta\\ \boldsymbol{\omega}\end{smallmatrix}\right]$.

Our selection rule [C1] + [C2] is thus an *algebraic, mechanical, and utterly simple device that keeps at bay, simultaneously, all illusory multiple poles.*

As for the bound (4.9), we observe that the right-hand side of (4.15) carries at most two linear terms $\mathrm{Carr}^{\boldsymbol{\omega}'}$, with $r(\boldsymbol{\omega}') = r-1$, and at most $(r-2)$ bilinear terms $\mathrm{Carr}^{\boldsymbol{\omega}''}\mathrm{Carr}^{\boldsymbol{\omega}'''}$, with $r(\boldsymbol{\omega}'')+r(\boldsymbol{\omega}''') = r$. Therefore, the iteration procedure, when it comes to a halt, has produced a total number of terms which cannot exceed the number $\mathrm{cat}(r)$ of maximal bracketings of a given, non-associative word consisting of r letters. That number $\mathrm{cat}(r)$ is none other than the "Catalan number", which is the middle term of the inequalities (4.9).

Remark 1. This bound for $\sum |n_p|$, though quite sufficient for our purpose, is rather coarse. In fact, the "extreme" examples which are to follow (See Ex. 2 and 3 *infra*) would seem to suggest that the best bound is not $\mathrm{cat}(r)$ but $\mathrm{cat}(r')$ with $r = 2r'$ or $r = 2r'-1$ depending on the parity.

Remark 2. Different licit choices of i usually (though not always) lead to different decompositions of $\mathrm{Carr}^{\boldsymbol{\omega}}$. This latitude in the choice of i both *explains* and *reflects* the non-existence of a canonical "best decomposition" (4.8) for $\mathrm{Carr}^{\boldsymbol{\omega}}$.

Remark 3. *"Non-repetition" of poles rather than "compensation".*
Let $\eta = \omega_i + \ldots + \omega_j$ $(i \le j)$ be a *formal pole* of the sequence $\boldsymbol{\omega}$, with its *formal multiplicity* $l = \left[\begin{smallmatrix}\eta\\ \boldsymbol{\omega}\end{smallmatrix}\right]$ and its chain (4.14) of generating ZUS. (Recall that (4.17) must replace (4.14a) whenever $i < j$). If we were to calculate $\mathrm{Carr}^{\boldsymbol{\omega}}$ by applying the induction (4.15) with *illicit indices* at each step, we would of course still get the correct answer, but the decomposition (4.8) might involve illusory poles $(\eta)^{-\mu_*}$ of order μ_* nearly as large as:

$$(4.18)\quad \mu_* = l + \sum l^{s,s+1} \quad \text{(with } m \le s < n \text{ and } l^{s,s+1} \text{ as in (4.14d)).}$$

Obviously, these *illusory poles*, or rather these *poles with illusorily high multiplicities*, would cancel each other within the sum (4.8). *However, what we witness here is not a true compensation phenomenon, because these phantom poles have no genuine existence: they are just will-o'-the-wisps, "optical illusions" conjured up by an ill-adapted calculational strategy.*

Remark 4. *"Flexible" versus "rigid" inductions.*
Actually, one may argue that these *illusory poles* and *illusory multiplicities* are not totally bereft of reality, since they have an insidious way of *smuggling themselves* into the *expressions* of $\mathrm{Carr}^{\boldsymbol{\omega}}$ and $\mathrm{Carr}^{\overset{<}{\boldsymbol{\omega}}}$ whenever the

calculations are conducted according to a *rigid induction*, which does not take into account the particular degeneracy pattern.

One such *rigid scheme*, which produces cart-loads of illusory poles, is the already mentioned composition formula:

$$\mathrm{I}^\bullet - \mathrm{Carr}^\bullet = \lim \mathrm{stat}_{n\to\infty}(\mathrm{I}^\bullet - \mathrm{Taa}^\bullet)^{\circ n} \tag{4.19}$$

with the elementary mould $\mathrm{Taa}^\bullet$ defined as in (3.7), (3.7*).

Another *rigid*, and for that very reason *flawed, calculation scheme* consists in applying the variance formula $\mathrm{var}_i \mathrm{Carr}^{\boldsymbol{\omega}} = (\ldots)$ systematically with the *first* or *last* i ($i = 1$ or $i = r(\boldsymbol{\omega})$).

Example 1. *Minimal degeneracy order* $(d(\boldsymbol{\omega}) = 1)$
For $r(\boldsymbol{\omega}) = r$ and the following ("minimal") degeneracy pattern:

$$0 = \omega_1 + \omega_2 + \ldots + \omega_r \tag{4.20}$$

we may apply the above inductive scheme with $i = 1$ at each step, and we find:

$$\mathrm{Carr}^{\boldsymbol{\omega}} = (-1)^{r-1}(\omega_1)^{-1}(\omega_1+\omega_2)^{-1}(\omega_1+\omega_2+\ldots+\omega_{r-1})^{-1} \tag{4.21}$$

Example 2. *Maximal degeneracy order* $(d(\boldsymbol{\omega}) = r(\boldsymbol{\omega})^{-1})$
For $r(\boldsymbol{\omega}) = r = 2r'$ and the following ("maximal") degeneracy pattern:

$$0 = \omega_1 + \omega_2 = \omega_2 + \omega_3 = \omega_3 + \omega_4 = \ldots = \omega_{r-1} + \omega_r \tag{4.22}$$

we have only *one* independent variable ω_1, since $\omega_i = (-1)^{i-1}\omega_1$, and the standard inductive scheme yields in this case:

$$\mathrm{Carr}^{\boldsymbol{\omega}} = -\frac{(2r'-2)!}{(r'-1)!r'!}(\omega_1)^{1-r} \quad (r = 2r') \tag{4.23}$$

The calculations leading to (4.23) may be given a concise and elegant form by considering the generating function:

$$\begin{cases} \mathrm{Carr}(t) \overset{\text{def}}{=} t^2 \mathrm{Carr}^{x,-x} + t^4 \mathrm{Carr}^{x,-x,x,-x} \\ \qquad\qquad\quad + t^6 \mathrm{Carr}^{x,-x,x,-x,x,-x} + \ldots \end{cases} \tag{4.24}$$

and by applying the induction rules to $\mathrm{Carr}(t)$ itself.

Example 3. *Intermediary degeneracy order* $(d(\boldsymbol{\omega}) = r(\boldsymbol{\omega})/2)$
For $r(\boldsymbol{\omega}) = r = 2r'$ and a degeneracy pattern:

$$0 = \omega_1 + \omega_r = \omega_2 + \omega_{r-1} = \omega_3 + \omega_{r-2} = \ldots = \omega_{r'} + \omega_{r'+1} \tag{4.25}$$

we have r' independent variables $\omega_1, \omega_2, \ldots, \omega_{r'}$, and the standard scheme yields a sum (4.8) which involves several terms, each with a coefficient $n_p = 1$. Moreover, the total number of these terms is exactly:

$$\mathrm{cat}(r') = \frac{(2r'-2)!}{(r'-1)!r'!} \tag{4.26}$$

It is thus equal to the *multiplicity* of the *only* term of (4.19). For instance we find:

$$\begin{cases} \mathrm{Carr}^{\omega_1,\omega_2,-\omega_2,-\omega_1} = -(\omega_1\omega_2\omega_{12})^{-1} \\ \qquad\qquad (\text{with } \omega_{12} \stackrel{\text{def}}{=} \omega_1 + \omega_2) \end{cases} \tag{4.27}$$

$$\begin{cases} \mathrm{Carr}^{\omega_1,\omega_2,\omega_3,-\omega_3,-\omega_2,-\omega_1} = (\omega_1\omega_2\omega_3\omega_{12}\omega_{23})^{-1} \\ \qquad\qquad -(\omega_1\omega_3\omega_{12}\omega_{23}\omega_{123})^{-1} \end{cases} \tag{4.28}$$

$$\begin{cases} \mathrm{Carr}^{\omega_1,\omega_2,\omega_3,\omega_4,-\omega_4,-\omega_3,-\omega_2,-\omega_1} = \textit{five terms} \\ \qquad\qquad (\text{because } \mathrm{cat}(4) = 5) \end{cases} \tag{4.29}$$

Example 4. *Juxtaposition of zero-sum, unbreakable sequences.*

If the degeneracy pattern is:

$$0 = \omega_1 + \ldots + \omega_{i_1} = \omega_{1+i_1} + \ldots + \omega_{i_2} = \ldots = \omega_{1+i_{s-1}} + \ldots + \omega_{i_s} \tag{4.30}$$

or, in other words, if $\boldsymbol{\omega}$ factors into a product $\boldsymbol{\omega}^1\boldsymbol{\omega}^2\ldots\boldsymbol{\omega}^s$ and *carries no other zero-sum, unbreakable sequences than the above* $\boldsymbol{\omega}^i$, then a simple induction, based on the standard scheme, yields:

$$\mathrm{Carr}^{\boldsymbol{\omega}} \equiv 0\ . \tag{4.31}$$

If, however, the first and last factor-sequences have sums different from 0 (i.e. $\|\boldsymbol{\omega}^1\| = -\|\boldsymbol{\omega}^s\| \neq 0$) and all others have zero-sums $(0 = \|\boldsymbol{\omega}^2\| = \|\boldsymbol{\omega}^3\|$ etc.), then $\mathrm{Carr}^{\boldsymbol{\omega}}$ is generally $\neq 0$, as the example (4.12) testifies.

Lemma 4.3. *Non-repetition of the denominators in* $\mathrm{Carr}^{\overset{<}{\boldsymbol{\omega}}}$.
For arborescent sequences $\overset{<}{\boldsymbol{\omega}}$ *of a given length* r *and of a fixed degeneracy pattern of order* d*, the correction mould* $\mathrm{Carr}^{\overset{<}{\boldsymbol{\omega}}}$*, as a rational function of its* $(r-d)$ *independent variables* ω_i*, has only poles of the form* $(\eta)^{-\mu}$*, with linear combinations* η *of the* ω_i *obtained by splitting unbreakable zero-sum subsequences* $(\boldsymbol{\omega}^*)^<$ *of* $\overset{<}{\boldsymbol{\omega}}$*:*

$$\eta = \|(\boldsymbol{\omega}')^<\| = -\|(\boldsymbol{\omega}'')^<\| \quad ((\boldsymbol{\omega}')^<(\boldsymbol{\omega}'')^< = (\boldsymbol{\omega}^*)^<) \tag{4.32}$$

and with a multiplicity μ *no larger than the number of unbreakable, zero-sum sequences* $(\boldsymbol{\omega}^*)^<$ *which, when split, can produce* η*. Moreover, although* *there is in general no canonical "best way" of decomposing* $\mathrm{Carr}^{\overset{<}{\boldsymbol{\omega}}}$*, there always exist decompositions of the form:*

$$(4.33)\quad \begin{cases} \mathrm{Carr}^{\overset{<}{\boldsymbol{\omega}}} = \sum_p n_p (\eta_{p,1})^{-\mu_{p,1}} (\eta_{p,2})^{-\mu_{p,2}} \dots (\eta_{p,s_p})^{-\mu_{p,s_p}} \\ \qquad\qquad \left(\sum_i \mu_{p,i} \equiv r-1 \right) \end{cases}$$

which involve only effective poles η*, with a multiplicity never exceeding their effective multiplicity* μ*; and with integral coefficients* n_p *bounded by:*

$$(4.34)\qquad \sum_p |n_p| \le (16)^r \quad (r = r(\boldsymbol{\omega})\ ; \quad n_p \in \mathbf{Z} - \{0\})$$

The proof is exactly like that of Lemma 4.2, except that it now relies on the "arborified" *variance rule* (4.6), subject only to the choice of licit indices ω_i, under selection rules $[\mathrm{C}\overset{<}{1}]$ + $[\mathrm{C}\overset{<}{2}]$ which are the literal arborification of [C1]+[C2]. As for the (admittedly very coarse) bound $(16)^r$ in (4.34) instead of $(4)^r$ in (4.9), it simply reflects the larger number of "bracketings" for trees than for fully-ordered sequences.

We might now wind up the proof of Proposition 4.1, but for greater clarity, we shall slightly rephrase our two key-lemmas. This rephrasing involves the *alternal mould* $\mathrm{Ta}^{\bullet}$ and its arborification $\mathrm{Ta}^{\overset{<}{\bullet}}$, whose definition (see Sect. 2) we recall:

$$(4.35)\qquad \mathrm{Ta}^{\omega_1,\dots,\omega_r} \overset{\mathrm{def}}{=} \mathrm{Sa}^{\omega_2,\dots,\omega_r} \overset{\mathrm{def}}{=} (\hat{\omega}_2 \hat{\omega}_3 \dots \hat{\omega}_r)^{-1}$$

$$(4.36)\qquad \mathrm{Ta}^{(\omega_1,\dots,\omega_r)^<} \overset{\mathrm{def}}{=} \mathrm{Sa}^{(\omega_2,\dots,\omega_r)^<} \overset{\mathrm{def}}{=} (\hat{\omega}_2 \hat{\omega}_3 \dots \hat{\omega}_r)^{-1}$$

if $\|\boldsymbol{\omega}\| = 0$ or $\| \overset{<}{\boldsymbol{\omega}} \| = 0$ *and* if $\overset{<}{\boldsymbol{\omega}}$ has only *one root* ω_1.
In all other cases we put:

$$(4.37)\qquad \mathrm{Ta}^{\boldsymbol{\omega}} = \mathrm{Ta}^{\overset{<}{\boldsymbol{\omega}}} = 0$$

Corollary of Lemma 4.2 and Lemma 4.3.
Let $\boldsymbol{\omega}$ *and* $\overset{<}{\boldsymbol{\omega}}$ *be two zero-sum sequences, totally ordered (resp. arborescent), and each with a given degeneracy pattern. The key-identities* (4.8) *and* (4.33) *actually express* $\mathrm{Carr}^{\boldsymbol{\omega}}$ *and* $\mathrm{Carr}^{\overset{<}{\boldsymbol{\omega}}}$ *as sums of elementary moulds* $\mathrm{Ta}^{\overset{<}{\boldsymbol{\eta}}}$ *indexed (in both cases) by non-repetitive (see below) sequences* $\overset{<}{\boldsymbol{\eta}}$ *obtained by suitably reordering the components* ω_i *of* $\boldsymbol{\omega}$ *or* $\overset{<}{\boldsymbol{\omega}}$

$$(4.38)\qquad \mathrm{Carr}^{\boldsymbol{\omega}} = \sum n \begin{pmatrix} \boldsymbol{\omega} \\ \overset{<}{\boldsymbol{\eta}} \end{pmatrix} \mathrm{Ta}^{\overset{<}{\boldsymbol{\eta}}} \quad (\text{with} \quad n \begin{pmatrix} \bullet \\ \bullet \end{pmatrix} \in Z)$$

$$\mathrm{Carr}^{\overset{<}{\boldsymbol{\omega}}} = \sum n \begin{pmatrix} \overset{<}{\boldsymbol{\omega}} \\ \overset{<}{\boldsymbol{\eta}} \end{pmatrix} \mathrm{Ta}^{\overset{<}{\boldsymbol{\eta}}} \quad (\text{with} \quad n \begin{pmatrix} \bullet \\ \bullet \end{pmatrix} \in Z) \tag{4.39}$$

The integral-valued coefficients $n \begin{pmatrix} \bullet \\ \bullet \end{pmatrix}$ *satisfy, of course, (4.9) and (4.34).*

All the arborescent sequences $\overset{<}{\boldsymbol{\eta}}$ *have* $\eta_1 = \omega_1$ *as their single root, and they respect the order of* $\boldsymbol{\omega}$ *or* $\overset{<}{\boldsymbol{\omega}}$ *in as far as they do not alter the internal order of each uncut sequence* $\boldsymbol{\omega}^*$ *or* $(\boldsymbol{\omega}^*)^<$ *of* $\boldsymbol{\omega}$ *or* $\overset{<}{\boldsymbol{\omega}}$ *("uncut" of course means "uncut by the zero-sum unbreakable sequences" of* $\boldsymbol{\omega}$ *or* $\overset{<}{\boldsymbol{\omega}}$*). All the partial sums* $\overset{<_i}{\boldsymbol{\eta}} = \sum_{i \le j} \eta_j$ *of* $\overset{<}{\boldsymbol{\eta}}$ *are true poles of* $\mathrm{Carr}^{\boldsymbol{\omega}}$ *or* $\mathrm{Carr}^{\overset{<}{\boldsymbol{\omega}}}$*.*

Furthermore, the sequences $\overset{<}{\boldsymbol{\eta}}$ *are non-repetitive in the sense that:*

$$\overset{<_i}{\boldsymbol{\eta}} \neq \overset{<_j}{\boldsymbol{\eta}} \quad \textit{whenever} \quad i < j \quad \textit{within} \quad \overset{<}{\boldsymbol{\eta}}. \tag{4.40}$$

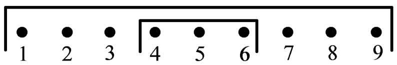
Lastly, for any fixed licit induction, like the first-licit-choice or last-licit-choice induction (see below), all the terms of the decomposition (4.38) and (4.39) are mechanically and unambiguously defined.

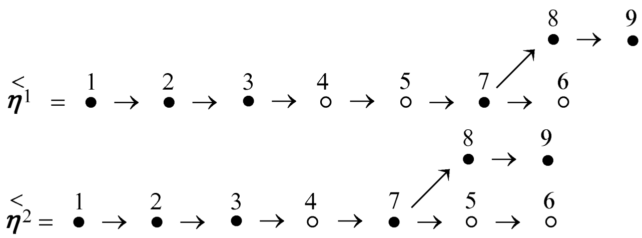
Thus, although there is no "canonical best way" of expanding $\mathrm{Carr}^{\boldsymbol{\omega}}$ and $\mathrm{Carr}^{\overset{<}{\boldsymbol{\omega}}}$ as sums of "simple elements", there exist two symmetrical expansions that come very close to it. The *first-licit-choice* (resp. *last-licit-choice*) *induction*, of course, consists in taking, at each inductive step, the first (resp. last) *licit* index i (which usually differ from the first and last indices in absolute terms).

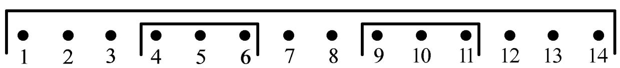
We observe that the above statement clearly *implies* Lemma 4.2 and 4.3 and, conversely, that Lemma 4.2 and 4.3 essentially contain the above "corollary", because any *licit inductive scheme automatically* produces an arborescent structure on the components ω_i; *automatically* respects the internal order of all uncut subsequences; and *automatically precludes repetitions of the* $\hat{\eta}_i$ for pairs of *comparable* indices (i, j). Indeed, if (4.40) was violated, the *actual multiplicity* μ of the poles η would often *exceed* their formal multiplicity $\begin{bmatrix} \boldsymbol{\eta} \\ \boldsymbol{\omega} \end{bmatrix}$, contrary to what we have established when proving the Lemma 4.2.

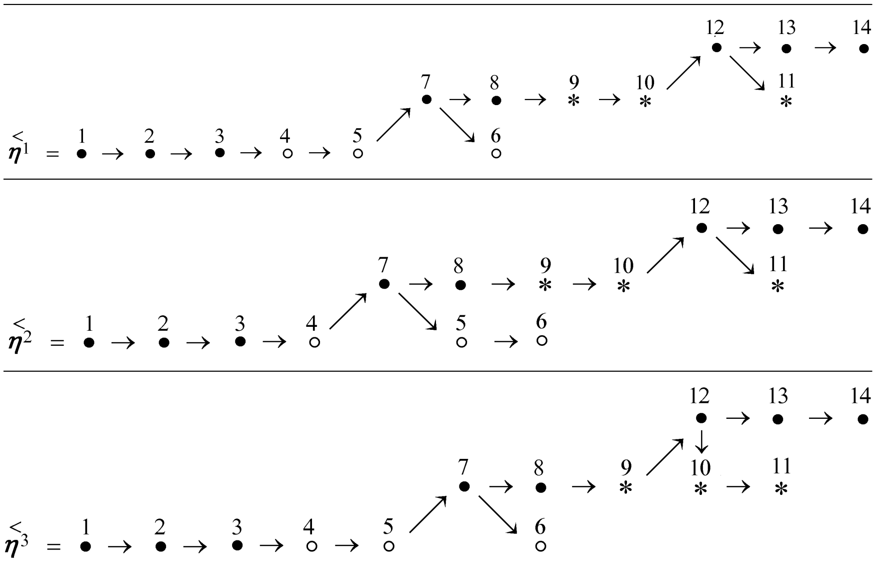
Actually, the lemmas 4.2 and 4.3 and their above corollary are simply *two different ways of looking at the same decomposition* of $\mathrm{Carr}^{\boldsymbol{\omega}}$ and $\mathrm{Carr}^{\overset{<}{\boldsymbol{\omega}}}$, but the corollary could also be established, *directly and rather simply, under any licit inductive scheme*. But rather than inflicting on the hapless reader a repetition of the induction drudgery (which followed the Lemma 4.2 and constituted its proof), we prefer to show how the *last-licit-choice induction* works on five typical examples, some of which involve multiple overlappings. In each case, the components ω_i which stem from the same

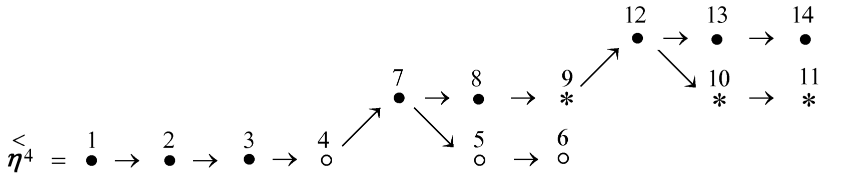
ZUS (other than the full sequence $\boldsymbol{\omega}$) are marked by some distinctive sign (like $\circ, *, \mathrm{etc}\ldots$) to facilitate their visual identification and to show how the *repetition-preventing mechanism* works. Of course, i stands everywhere for ω_i, and the forward-going arrows denote the arborescent order on $\overset{<}{\boldsymbol{\eta}}$.

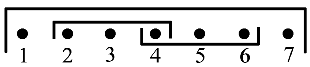
Example 5.  $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_9)$
with the following two ZUS: $\boldsymbol{\omega}$ itself (unmarked); $(\omega_4, \omega_5, \omega_6)$ (marked $\circ$).
We find $\mathrm{Carr}^{\boldsymbol{\omega}} = -\mathrm{Ta}^{\overset{<}{\boldsymbol{\eta}}{}^1} - \mathrm{Ta}^{\overset{<}{\boldsymbol{\eta}}{}^2}$ with:



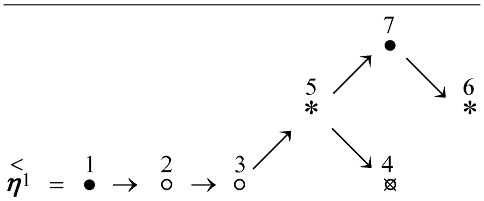
Example 6. 
$\boldsymbol{\omega} = (\omega_1, \ldots, \omega_{14})$
with the following three ZUS: $\boldsymbol{\omega}$ itself (unmarked); $(\omega_4, \omega_5, \omega_6)$ (marked $\circ$); $(\omega_9, \omega_{10}, \omega_{11})$ (marked $*$).
We find $\mathrm{Carr}^{\boldsymbol{\omega}} = n_1 \mathrm{Ta}^{\overset{<}{\boldsymbol{\eta}}{}^1} + n_2 \mathrm{Ta}^{\overset{<}{\boldsymbol{\eta}}{}^2} + n_3 \mathrm{Ta}^{\overset{<}{\boldsymbol{\eta}}{}^3} + n_4 \mathrm{Ta}^{\overset{<}{\boldsymbol{\eta}}{}^4}$ with: $n_1 = n_2 = n_3 = n_4 = 1$ and:

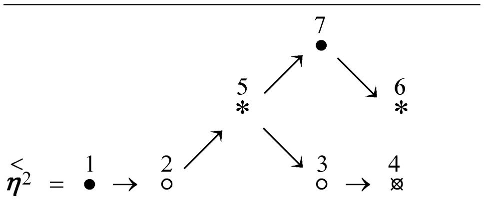


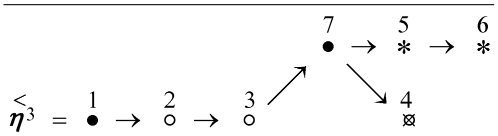


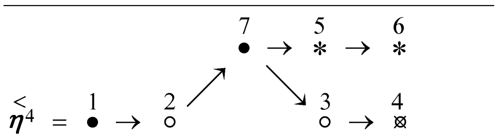
Example 7.  $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_7)$ with the following three overlapping ZUS: $\boldsymbol{\omega}$ itself (unmarked); $(\omega_2, \omega_3, \omega_4)$ (marked $\circ$); $(\omega_4, \omega_5, \omega_6)$ (marked $*$).

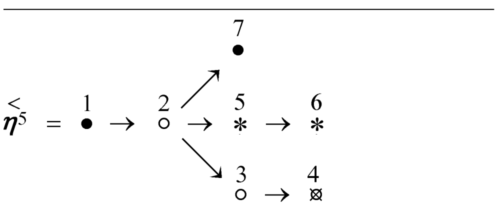
We find $\mathrm{Carr}^{\boldsymbol{\omega}} = n_1 \mathrm{Ta}^{\overset{<}{\boldsymbol{\eta}}{}^1} + \ldots + n_5 \mathrm{Ta}^{\overset{<}{\boldsymbol{\eta}}{}^5}$ with: $n_1 = n_2 = n_3 = n_4 = 1$; $n_5 = -1$ and:

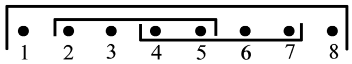




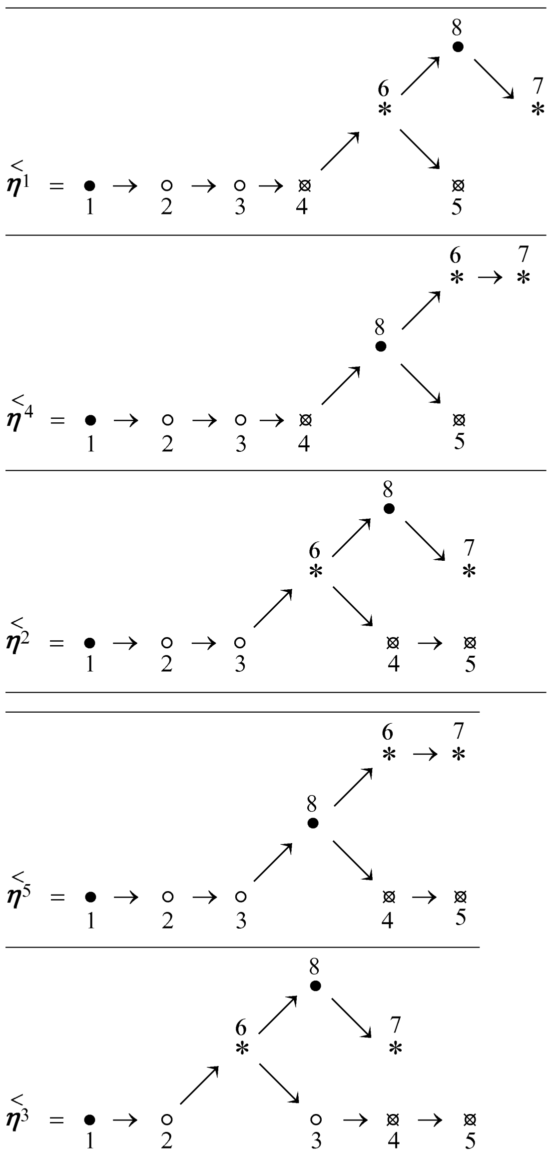


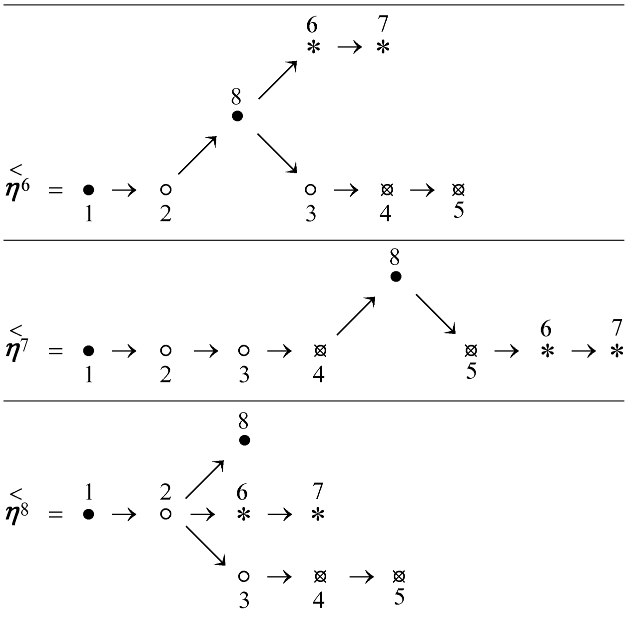


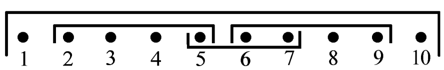


Example 8.  $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_8)$
with the following three overlapping ZUS: $\boldsymbol{\omega}$ itself (unmarked); $(\omega_2, \ldots, \omega_5)$ (marked $\circ$); $(\omega_4, \ldots, \omega_7)$ (marked $*$).

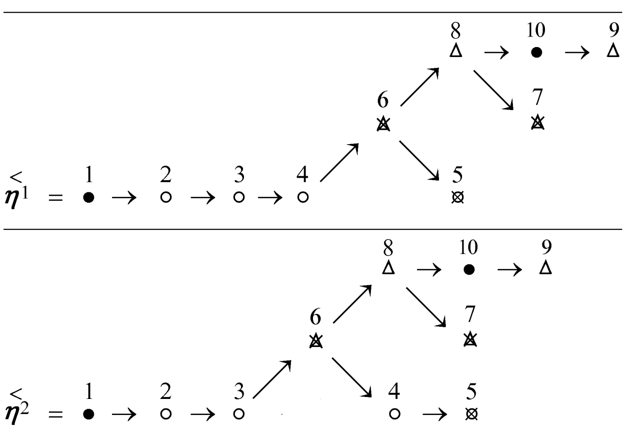
We find $\mathrm{Carr}^{\boldsymbol{\omega}} = n_1 \mathrm{Ta}^{\overset{<_1}{\boldsymbol{\eta}}} + \ldots + n_8 \mathrm{Ta}^{\overset{<_8}{\boldsymbol{\eta}}}$ with: $n_1 = \ldots = n_6 = 1$; $n_7 = n_8 = -1$ and:

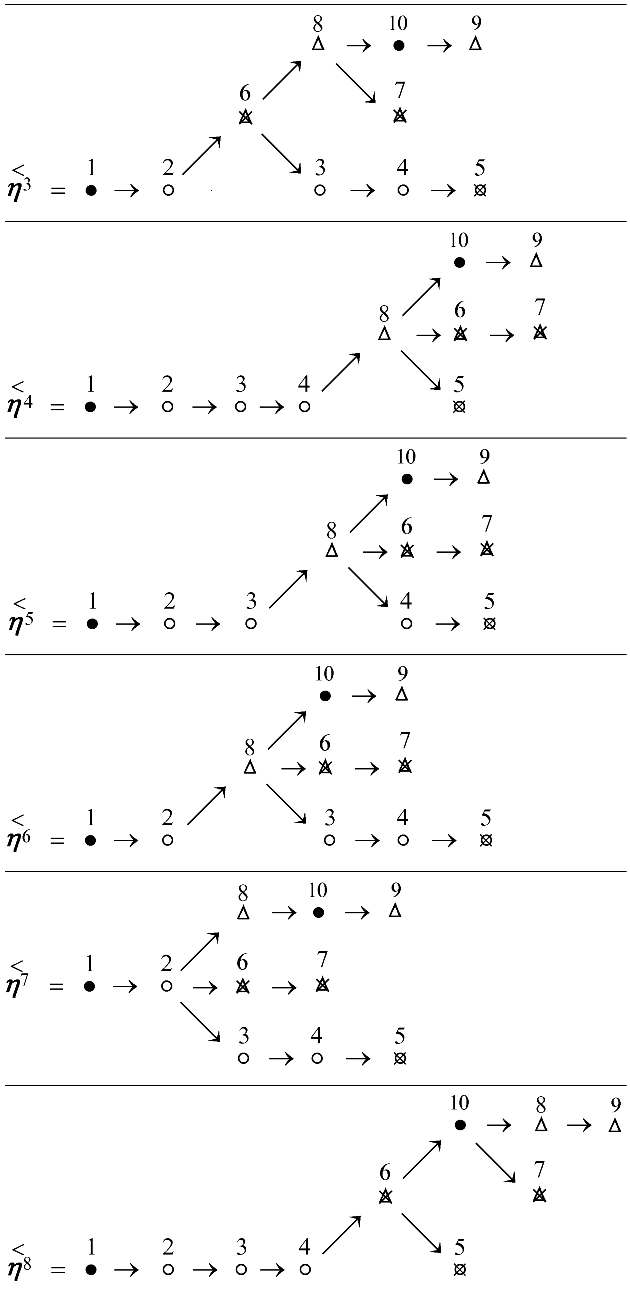


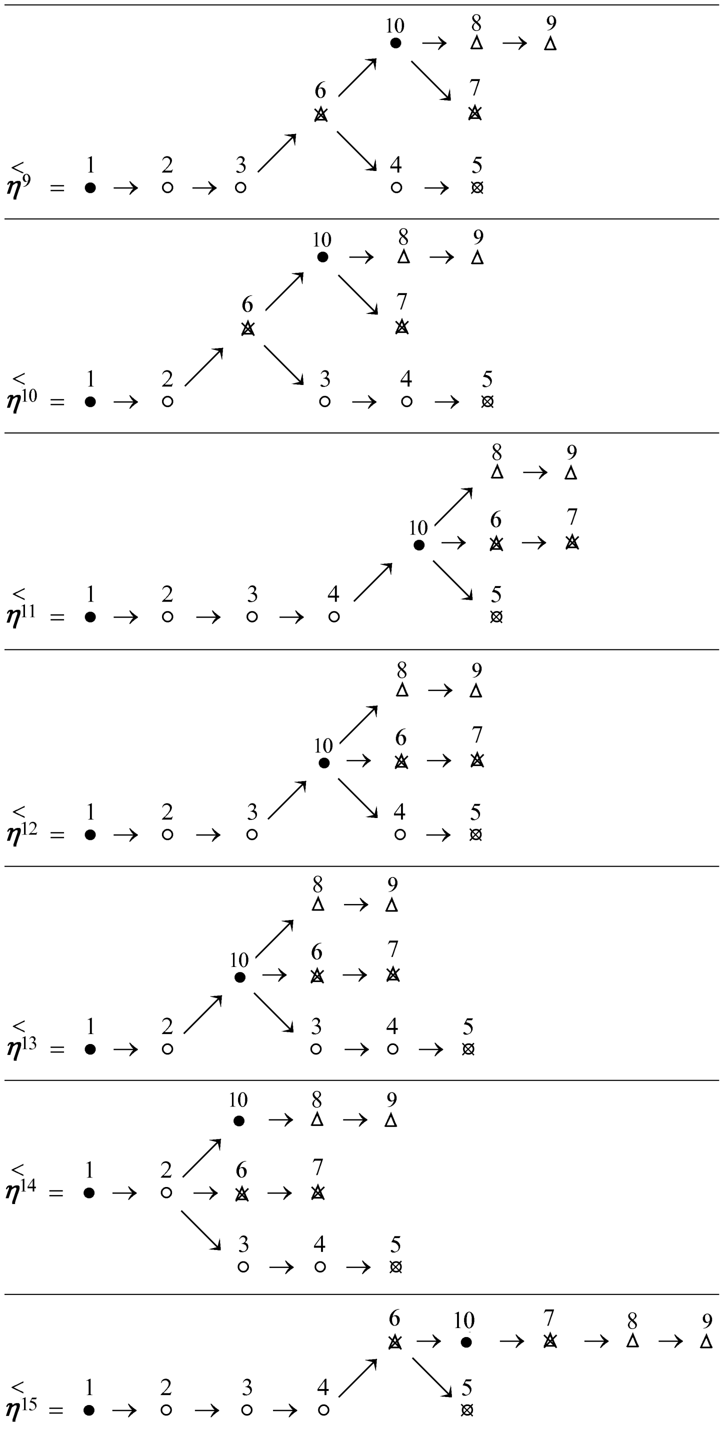


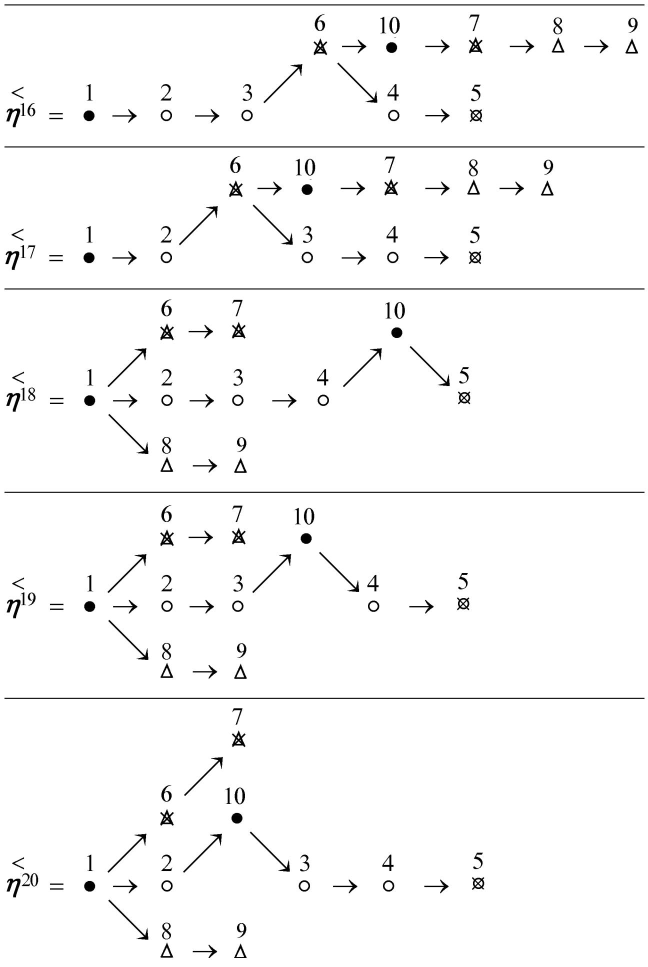
Example 9.  $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_{10})$ with the following overlapping three ZUS: $\boldsymbol{\omega}$ itself (unmarked); $(\omega_2, \ldots, \omega_5)$ (marked $\circ$); $(\omega_5, \omega_6, \omega_7)$ (marked $*$); $(\omega_6, \ldots, \omega_9)$ (marked Δ).

We find $\mathrm{Carr}^{\boldsymbol{\omega}} = n_1 \mathrm{Ta}^{\overset{<}{\boldsymbol{\eta}}{}^1} + \ldots + n_{20} \mathrm{Ta}^{\overset{<}{\boldsymbol{\eta}}{}^{20}}$ with: $n_1 = \ldots = n_6 = -1$; $n_7 = 1$; $n_8 = \ldots = n_{13} = -1$; $n_{14} = 1$; $n_{15} = \ldots = n_{20} = 1$, and:






$\overset{<}{\eta}{}^{9}$ =
$\overset{<}{\eta}{}^{10}$ =
$\overset{<}{\eta}{}^{11}$ =
$\overset{<}{\eta}{}^{12}$ =
$\overset{<}{\eta}{}^{13}$ =
$\overset{<}{\eta}{}^{14}$ =
$\overset{<}{\eta}{}^{15}$ =



These nine examples will suffice for illustration.

We can now pick up the thread and conclude the proof of Proposition 4.1. The non-repetition of poles, as guaranteed by the Lemmas 4.2 and 4.3, together with the estimates (4.9) and (4.34) on the number of terms, makes it plain that $\mathrm{Carr}^{\boldsymbol{\omega}}$ and $\mathrm{Carr}^{\overset{<}{\boldsymbol{\omega}}}$ verify the standard Brjuno estimates (see [B]):

(4.41)
$$\begin{cases} |\mathrm{Carr}^{\omega_1,\ldots,\omega_r}| \leq (c_{1^*})^{q_1+\ldots+q_r} \\ \quad \Big(\omega_i = \langle n_i, \lambda \rangle \ ; \ q_i = |n_i| = \sum n_{ij}\Big) \end{cases}$$

(4.41*)
$$\begin{cases} |\mathrm{Carr}^{(\omega_1,\ldots,\omega_r)^<}| \leq (c_1)^{q_1+\ldots+q_r} \\ \quad \Big(\omega_i = \langle n_i, \lambda \rangle \ ; \ q_i = |n_i| = \sum n_{ij}\Big) \end{cases}$$

quite simply because, according to (4.38) and (4.39), $\mathrm{Carr}^{\boldsymbol{\omega}}$ and $\mathrm{Carr}^{\overset{<}{\boldsymbol{\omega}}}$ are sums of *elementary terms* $\mathrm{Ta}^{\overset{<}{\boldsymbol{\eta}}}$, each of which verifies the standard Brjuno estimates (due to the non-repetitiveness (4.40)) and also because the *total number* of such terms is bounded by (4.9) or (4.34).

The *mould estimates* (4.41), (4.41*) may now be paired with the comould part of the expansions:

$$(4.42) \qquad \|B_{n_1,\ldots,n_r}\|_{U,V} \leq r!(c_{2^*})^{q_1+\ldots+q_r}\|B_{n_1}\|_{U,V}\ldots\|B_{n_r}\|_{U,V}$$

$$(4.42^*) \qquad \|B_{(n_1,\ldots,n_r)^<}\|_{U,V} \leq (c_2)^{q_1+\ldots+q_r}\|B_{n_1}\|_{U,V}\ldots\|B_{n_r}\|_{U,V}$$

Since the vector field X is assumed to be analytic, we may, for each given $\varepsilon > 0$, ensure that:

$$(4.43) \qquad \|B_{n_i}\|_{U,V} \leq \varepsilon \quad (\forall\, n_i)$$

merely by choosing V small enough, and then U small enough within V. Moreover, there exists a constant c_3 (depending on the dimension ν alone) such that $(c_3)^q$ be larger than the number of arborescent sequences $(n_1,\ldots,n_r)^<$ of total "weight" $\sum|n_i|$ less than q (see [E.3], pp. 94–95)

Piecing all this information together, we get:

$$(4.44) \qquad \sum|\mathrm{Carr}^{\overset{<}{\bullet}}|\|B_{n_1}\|_{U,V} \leq \sum_{1\leq q}(\varepsilon c_0 c_1 c_2 c_3)^q$$

which establishes the *normal convergence* of the arborified expansions and, *a fortiori*, the analyticity of the *correction* and *corrected form*. The same, however, does not hold true for the non-arborified expansion $\sum\mathrm{Carr}^\bullet B_\bullet$, due to the unremovable factor $r!$ in the estimates (4.42). Actually, by looking at proper subseries of $\sum\mathrm{Carr}^\bullet B_\bullet$ for each term of which the estimates (4.41) and (4.42) are sharp, one immediately concludes that the mould expansion

$$\sum\mathrm{Carr}^\bullet B_\bullet$$

in its raw, non-arborified form, is *generically divergent in norm* (except, of course, in the trivial case when all eigenvalues λ_i are located in one complex halfplane with boundary passing through the origin, because in this case there appears in the estimates (4.30) a factor $1/r!$ which exactly offsets the factor $r!$ of (4.42)).

Remark 5. As always, arborification owes its success to the fact that it removes the factor $r!$ in the comould estimates, without reintroducing it in the mould estimates. The removal of $r!$ in the comoulds is the *natural* consequence of the coarborification transform defined in (2.26), but the non-appearance of $r!$ in the moulds is much less "automatic": it is due to *specific* identities which our moulds *happen to verify*, and which say, roughly speaking, that these moulds essentially retain their shape after arborification: in the present instance, $\mathrm{Carr}^{\overset{<}{\bullet}}$ verifies a "variance rule" very similar to that of $\mathrm{Carr}^{\bullet}$ (compare (4.5) with (3.33)).

Remark 6. The crux of the proof of Proposition 4.1, however, lies in the phenomenon of "cancellation", which applies both *before* and *after* arborification, and has to do with the "non-repetition" of denominators in $\mathrm{Carr}^{\bullet}$ (and $\mathrm{Carr}^{\overset{<}{\bullet}}$), which sets this mould apart from most others, like $\mathrm{Nal}^{\bullet}$ and all prenormalizing moulds $\mathrm{Pran}^{\bullet}$ etc.

Remark 7. *The work of L.H. Eliasson on small denominators.*

Professor L.H. Eliasson has been one of the first analysts to make a systematic use of "arborification" in the study of *small denominator problems aggravated by resonance*, and he has proven a number of difficult theorems in this field. However, despite our common, and clearly inevitable, recourse to "trees", we must say that Eliasson's methods are vastly different from ours. The main difference lies in this: whereas we majorize expressions like $\mathrm{Carr}^{\boldsymbol{\omega}}$ by expressing them, via a purely algebraic mechanism, as finite sums of elements which, taken separately, possess obvious, or at least well-known, bounds (modulo the work of Siegel and Brjuno), Professor Eliasson seems to be under the impression that the poles of "abnormally high order" are "*actually there*", and so he must resort to a *very complex and subtle*, but *non-algebraic machinery of "sign compensations"* to keep some control on *the sums of these poles*. Indeed, until now, the consensus among the experts on *resonance-cum-small-denominators* seems to be that no purely algebraic mechanism can account for the phenomenon of "compensation".

We, however, have shown that there does exist a *very simple*, and *quite fascinating, algebraic mechanism* (at least in the case of $\mathrm{Carr}^{\boldsymbol{\omega}}$, $\mathrm{Carr}^{\overset{<}{\boldsymbol{\omega}}}$, $\mathrm{Cerr}^{\boldsymbol{\omega}}$, $\mathrm{Cerr}^{\overset{<}{\boldsymbol{\omega}}}$, but also, we suspect, in many similar instances) and that this completely *does away with the need for any form of compensation*. Indeed, *once* $\mathrm{Carr}^{\boldsymbol{\omega}}$ *has been purged of its illusory poles, there is nothing left to compensate*, and it is our firm belief that no significant compensation occurs among the terms of the sums (4.8) = (4.38) or (4.33) = (4.39). By "significant compensation" we mean one which could make the difference between *divergence* and *convergence*. Of course, by a "clever" choice of some specially suited *licit induction*, one might perhaps manage to reduce somewhat the constants c_{1^*} and c_1 in (4.41) and (4.41*) but that is totally immaterial.

The main bonus we get from our *algebraic method* (apart from pure conceptual insight), seems to be the weaker diophantine assumptions (Brjuno's instead of Siegel's) under which we are able to prove the convergence of the correction.

Remark 8. *Ecalle's "compensators" and the Liouvillian small denominators.*

To preclude any terminological confusion, let us point out that the notion of "compensation", as used by Professor Eliasson, bears no relation to Ecalle's notion of "compensation", which belongs to an altogether different context, namely that of *Liouvillian* or *non-diophantine spectra,* and serves the purpose of *ramified-seriable* linearization (see above Sect. 1, (F2); and also [E.2]). "Compensation" in Ecalle' s sense involves the so-called "*compensators*" of the form:

$$z^{\sigma_0,\sigma_1,\ldots,\sigma_r} \stackrel{\text{def}}{=} \sum_{i=0}^{r} z^{\sigma_i} \prod_{j\neq i}(\sigma_i - \sigma_j)^{-1} \quad (z \in \mathbf{C}_\bullet; \sigma_i \in \mathbf{R}^+)$$

When the exponents σ_i are pair-wise different, but very close to each other, the coefficient in front of the power z^{σ_i} (which have a genuine, individualized existence) become very large and yet, due to a *true phenomenon of compensation*, the finite sum on the right-hand side of (11.36) remains bounded.

Algebraic complements. Elementary linearizability criteria

We append to this section a few lemmas which shed some additional light on the structure of $\mathrm{Carr}^\bullet$ and $\overset{<}{\mathrm{Carr}}{}^\bullet$ and, above all, lead to very simple *linearizability criteria* for resonant vector fields.

Let $\mathrm{Carr}^\bullet$ be as in (3.15) and $\mathrm{Taa}^\bullet$ as in (3.7) (3.7*).

Lemma 4.4. *The alternal mould* $\mathrm{Carr}^\bullet$ *can be calculated from the non-alternal mould* $\mathrm{Taa}^\bullet$ *by:*

$$\mathrm{I}^\bullet - \mathrm{Carr}^\bullet = \mathrm{stat} \lim_{n\to+\infty} (\mathrm{I}^\bullet - \mathrm{Taa}^\bullet)^{\circ n} \tag{4.45}$$

This corresponds to (3.17) with the choice (3.18**), the legitimacy of which was left open in Sect. 3 (unlike that of the choices (3.18) or (3.18*)).

A short but indirect proof of (4.45) consists in deducing this identity from the induction rule (3.33). Let us sketch a direct, though somewhat lengthier proof, based on the identity:

$$M_n^{\boldsymbol{\omega}} = N_n^{\boldsymbol{\omega}} \quad \text{if } r(\boldsymbol{\omega}) \leq n \tag{4.46}$$

with

$$M_n^\bullet \stackrel{\text{def}}{=} (\mathrm{I}^\bullet - \mathrm{Nal}_*^\bullet)^{\circ n} \ ; \quad N_n^\bullet \stackrel{\text{def}}{=} (\mathrm{I}^\bullet - \mathrm{Taa}^\bullet)^{\circ n} \tag{4.46*}$$

for a mould $\mathrm{Nal}_*^\bullet$ defined by $\mathrm{Nal}_*^{\boldsymbol{\omega}} \equiv \mathrm{Nal}^{\boldsymbol{\omega}}$ (resp. 0) if $\|\boldsymbol{\omega}\| = 0$ (resp. $\neq 0$). See (3.3) and (3.18*). Since, as mentioned in Sect. 3, $M_n^\bullet$ coincides with $\mathrm{I}^\bullet - \mathrm{Carr}^\bullet$ for $r(\bullet) \leq n$, (4.46) clearly implies (4.45).

Let us first calculate $M_n^{\boldsymbol{\omega}}$ for sequences $\boldsymbol{\omega}$ admitting only *one maximal factorization* $\mathbf{a}\boldsymbol{\omega}^1\boldsymbol{\omega}^2\ldots\boldsymbol{\omega}^s\mathbf{b}$ as in (3.3), and that too with factors $\boldsymbol{\omega}^i$ of length $r_i \geq 2$ and $\mathbf{a} = \mathbf{b} = \varnothing$.

$$(4.46^{**}) \quad \begin{cases} \boldsymbol{\omega} = \boldsymbol{\omega}^1\boldsymbol{\omega}^2\ldots\boldsymbol{\omega}^s \text{ with } \|\boldsymbol{\omega}^1\| = \ldots = \|\boldsymbol{\omega}^s\| = 0\ ; \\ \boldsymbol{\omega}^1, \boldsymbol{\omega}^2, \ldots, \boldsymbol{\omega}^s \neq \varnothing \end{cases}$$

($\boldsymbol{\omega}^i$ unbreakable; $r(\boldsymbol{\omega}^i) \geq 2$).

Using the convenient short-hand:

$$\mathrm{Nal}_*^{\boldsymbol{\omega}^1} = s^1;\ \mathrm{Nal}_*^{\boldsymbol{\omega}^1,\boldsymbol{\omega}^2} = s^{1,2};$$

$$(4.47) \qquad \mathrm{Nal}_*^{\boldsymbol{\omega}^1,0,\boldsymbol{\omega}^2,\boldsymbol{\omega}^3,0,\boldsymbol{\omega}^4} = s^{1,0,2,3,0,4}\ ;\ \text{etc.}$$

and applying the rules (2.3) for mould composition, we get the following expressions for $M_n^{\boldsymbol{\omega}}$ when $n \geq s$:

$$(4.48^1) \qquad M_n^{\boldsymbol{\omega}^1} = -s^1$$

$$(4.48^2) \qquad M_n^{\boldsymbol{\omega}^1,\boldsymbol{\omega}^2} = -s^{1,2} + s^{0,2}s^1 + s^{1,0}s^2$$

$$(4.48^3) \quad \begin{cases} M_n^{\boldsymbol{\omega}^1,\boldsymbol{\omega}^2,\boldsymbol{\omega}^3} = -s^{1,2,3} + s^{0,2,3}s^1 + s^{1,0,3}s^2 + s^{1,2,0}s^3 \\ \qquad -s^{0,0,3}s^1s^2 - s^{0,2,0}s^1s^3 - s^{1,0,0}s^2s^3 \\ \qquad +s^{0,3}s^{1,2} + s^{1,0}s^{2,3} - s^{0,3}s^{0,2}s^1 \\ \qquad -s^{0,3}s^{1,0}s^2 - s^{1,0}s^{2,0}s^3 - s^{1,0}s^{0,3}s^2 \end{cases}$$

etc. . . .

By Lemma 3.1 we can express everything in terms of the derivatives $(\mathrm{daa})^{n_i}(\mathrm{Taa}^{\omega_i})$ which, for short, we write: $D^{n_i}(s^i)$ $(n_i \geq 0;\ i \neq 0)$.

Thus we find:

$$(4.49^2) \qquad M_n^{\boldsymbol{\omega}^1,\boldsymbol{\omega}^2} = -D(s^1s^2) + s^1D(s^2) + s^2D(s^1) \equiv 0$$

$$(4.49^3) \quad \begin{cases} M_n^{\boldsymbol{\omega}^1,\boldsymbol{\omega}^2,\boldsymbol{\omega}^3} = -(1/2)D^2(s^1s^2s^3) + (1/2)s^1D^2(s^2s^3) \\ \qquad +(1/2)s^2D^2(s^1s^3) + (1/2)s^3D^2(s^1s^2) \\ \qquad -(1/2)s^1s^2D^2(s^3) - (1/2)s^1s^3D^2(s^2) \\ \qquad -(1/2)s^2s^3D^2(s^1) + D(s^1s^2)D(s^3) \\ \qquad +D(s^1)D(s^2s^3) - s^1D(s^2)D(s^3) \\ \qquad -s^2D(s^1)D(s^3) - s^3D(s^1)D(s^2) \\ \qquad -s^2D(s^1)D(s^3) \\ \qquad \equiv 0 \end{cases}$$

Let us show that the cancellation goes on, and that we have:

$$M_n^{\boldsymbol{\omega}^1,\boldsymbol{\omega}^2,\dots,\boldsymbol{\omega}^s} \equiv 0 \quad \text{for } n \geq s \geq 2\ . \tag{4.49$^{\rm n}$}$$

In view of the formal equivalence between Newton's binomial formula for $(a+b)^n$ and Leibniz's rule for calculating $D^n(ab)$, proving (4.49$^{\rm n}$) is the same as proving the identity:

$$0 \equiv A_K \stackrel{\text{def}}{\equiv} \sum_{J\subset K} (-1)^{\text{card}(J)} (a_J)^{\text{card}(K)-1} \quad (\text{with } \text{card}(K) = n \geq 2)$$

(4.50$^{\rm n}$)

for any sequence $a_1, a_2, a_3 \dots$ of real variables, and with the partial sums:

$$a_J \stackrel{\text{def}}{\equiv} \sum_{j\in J} a_j \quad (\text{if } J \text{ is a finite subset of } \mathbf{N}). \tag{4.51}$$

Thus, for $K = \{1,2\}$ or $\{1,2,3\}$, (4.50$^{\rm n}$) is obviously true since it reads:

$$0 = A_{\{1,2\}} = (a_1 + a_2) - (a_1) - (a_2) \tag{4.50^2}$$

$$\begin{cases} 0 = A_{\{1,2,3\}} = -(a_1+a_2+a_3)^2 + (a_1+a_2)^2 \\ \qquad +(a_1+a_3)^2 + (a_2+a_3)^2 - (a_1)^2 - (a_2)^2 - (a_3)^2\ . \end{cases} \tag{4.50^3}$$

The truth of (4.50$^{\rm n}$) follows by induction. Indeed, for any k in K we have:

$$\partial_{a_k} A_k \equiv (1 - \text{card}(K)) A_{K-\{k\}} \tag{4.52}$$

so that A_K is necessarily constant, and in fact $\equiv 0$ if card$(K) \geq 2$.

This establishes (4.49$^{\rm n}$) for $n \geq 2$ and shows in effect that, for sequences $\boldsymbol{\omega}$ of type (4.46**), when we express $M_n^{\boldsymbol{\omega}}$ as a polynomial of the variables:

$$D^{n_i}(s^i) \stackrel{\text{def}}{=} D^{n_i}(\text{Nal}_*^{\omega_i}) \quad ; \quad D^0(s^i) \stackrel{\text{def}}{=} s^i \tag{4.53}$$

all the "derivatives" of order $n_i \geq 1$ cancel out. The same cancellation also takes place for general series $\boldsymbol{\omega}$ admitting *several* maximal factorizations $\mathbf{a}\boldsymbol{\omega}^1 \dots \boldsymbol{\omega}^s\mathbf{b}$ (with empty or non-empty factors $\mathbf{a}$ and $\mathbf{b}$), as can be seen by repeating, for each such factorization, the above argument, based on the identities

$$0 = A_K \quad \text{for card}(K) \geq 2.$$

So what is left in the expression of $M_n^{\boldsymbol{\omega}}$ is the sum of all "derivative-free" terms. But this is *exactly* what we get when calculating $N_n^{\boldsymbol{\omega}}$, with $N_n^{\boldsymbol{\omega}}$

defined as the n-th self-iterate of $\mathrm{I}^\bullet - \mathrm{Taa}^\bullet$. This establishes (4.46) and proves Lemma 4.4.

Now, for any sequence $\overset{<}{\boldsymbol{\omega}} = (\omega_1, \ldots, \omega_r)^<$ with an arborescent order on it (see Sect. 2), we define the usual forward sums $\hat{\boldsymbol{\omega}}_i$:

(4.54)
$$\hat{\boldsymbol{\omega}}_i = \sum \omega_j \quad \text{for all } j \geq i \text{ relative to the arborescent order of } \overset{<}{\boldsymbol{\omega}}$$

Next, if $\overset{<}{\boldsymbol{\omega}}$ has exactly one "root" or "least element" ω_1, we put:

$$\text{(4.55)} \qquad \begin{cases} \mathrm{Taaa}^{\overset{<}{\boldsymbol{\omega}}} \overset{\text{def}}{=} (\hat{\boldsymbol{\omega}}_2)^{-1}(\hat{\boldsymbol{\omega}}_3)^{-1} \ldots (\hat{\boldsymbol{\omega}}_r)^{-1} \\ \qquad \text{if } \hat{\boldsymbol{\omega}}_1 = 0 \textit{ and } \hat{\boldsymbol{\omega}}_2 \hat{\boldsymbol{\omega}}_3 \ldots \hat{\boldsymbol{\omega}}_r \neq 0 \end{cases}$$

$$\text{(4.55*)} \qquad \begin{cases} \mathrm{Taaa}^{\overset{<}{\boldsymbol{\omega}}} \overset{\text{def}}{=} 0 \\ \qquad \text{if } \hat{\boldsymbol{\omega}}_1 \neq 0 \textit{ or } \hat{\boldsymbol{\omega}}_2 \hat{\boldsymbol{\omega}}_3 \ldots \hat{\boldsymbol{\omega}}_r = 0 \end{cases}$$

If $\overset{<}{\boldsymbol{\omega}}$ has more than one "root", we also put $\mathrm{Taaa}^{\overset{<}{\boldsymbol{\omega}}} = 0$.
Lastly, for $r(\overset{<}{\boldsymbol{\omega}}) = 0$ or 1, we put:

$$\text{(4.55**)} \qquad \mathrm{Taaa}^{\varnothing} = 0;\ \mathrm{Taaa}^{\omega_1} = 0 \ \text{ if } \ \omega_1 \neq 0;\ \mathrm{Taaa}^{0} = 1.$$

It is easy to check that $\mathrm{Taaa}^{\overset{<}{\boldsymbol{\omega}}}$ is "*almost*", *but not exactly*, the arborification of $\mathrm{Taa}^\bullet$. Indeed, the identity:

$$\text{(4.56)} \qquad \mathrm{Taaa}^{\overset{<}{\boldsymbol{\omega}}} = \sum_{\boldsymbol{\omega}} \mathrm{sh}\begin{pmatrix} \overset{<}{\boldsymbol{\omega}} \\ \boldsymbol{\omega} \end{pmatrix} \mathrm{Taa}^{\boldsymbol{\omega}} \qquad \text{(see Sect. 2)}$$

holds true only for sequences $\overset{<}{\boldsymbol{\omega}}$ such that:

$$\text{(4.57)} \qquad \hat{\boldsymbol{\omega}}_i \neq \omega_i \qquad \text{for each "branching index" } i.$$

(i is dubbed a "branching index" if it has more than one immediate successor in $\overset{<}{\boldsymbol{\omega}}$).

Lemma 4.5. *Despite the fact that* $\mathrm{Taaa}^{\overset{<}{\bullet}}$ *is not the arborification of* $\mathrm{Taa}^\bullet$ *(or, for that matter, of any other mould), the relation* (4.45) *nonetheless possesses an "arborescent" counterpart, namely:*

$$\text{(4.58)} \qquad \mathrm{I}^{\overset{<}{\bullet}} - \mathrm{Carr}^{\overset{<}{\bullet}} = \mathrm{stat} \lim_{n \to +\infty} (\mathrm{I}^{\overset{<}{\bullet}} - \mathrm{Taaa}^{\overset{<}{\bullet}})^{\circ n}$$

The composition $D^{\overset{<}{\bullet}} = A^{\overset{<}{\bullet}} \circ C^{\overset{<}{\bullet}}$ implicit in (4.58) is the natural extension of the mould composition as defined in (2.3):

$$\text{(4.59)} \qquad D^{\overset{<}{\boldsymbol{\omega}}} = \sum A^{(\|\boldsymbol{\omega}^1\|, \ldots, \|\boldsymbol{\omega}^s\|)^<} C^{(\boldsymbol{\omega}^1)^<} \ldots C^{(\boldsymbol{\omega}^s)^<}$$

Here, we break up, in all possible ways, the arborescent sequence $\overset{<}{\boldsymbol{\omega}}$ into a "product" of connected, non-empty subsequences $(\boldsymbol{\omega}^i)^<$. Each factor $(\boldsymbol{\omega}^i)^<$ inherits its arborescent order from $\overset{<}{\boldsymbol{\omega}}$, and so does the sequence $\overset{<}{\boldsymbol{\eta}} = (\eta_1, \dots, \eta_s)$ formed with the sums $\eta_i = \|\boldsymbol{\omega}^i\|$. The composition law (4.59) is clearly associative, and there are two distinct reasons why it deserves to be called the *natural* arborification of the mould composition (2.3).

First, if $A^{\overset{<}{\bullet}}$ and $C^{\overset{<}{\bullet}}$ happen to be the arborification (2.24) of some moulds $A^\bullet$ and $C^\bullet$ (which is not always the case, as with $\mathrm{Taaa}^{\overset{<}{\bullet}}$), then $D^{\overset{<}{\bullet}}$ as defined by the composition (4.59) happens to be the arborification of $D^\bullet$ as defined by the composition (2.3).

Second, if we fix a spectrum λ and a vector field $Y = \sum B_n$ as in (2.6); then construct $B_{\overset{<}{\bullet}}$ from $B_\bullet$ as in (2.25) (2.26); and lastly define the *action* of $M^{\overset{<}{\bullet}}$ on vector fields by:

$$\mathrm{Act}^{M^{\overset{<}{\bullet}}}\left(\sum B_n\right) = \sum M^{\overset{<}{\bullet}} B_{\overset{<}{\bullet}} \tag{4.60}$$

we obtain a new vector field if and only if $M^{\overset{<}{\boldsymbol{\omega}}} = 0$ whenever $\overset{<}{\boldsymbol{\omega}}$ has more than one root (since $B_{\overset{<}{\mathbf{n}}}$ as a differential operator has an order equal to the number of roots in $\overset{<}{\mathbf{n}}$). Moreover, *composing the action* amounts to *composing the arborescent moulds*:

$$\mathrm{Act}^{M^{\overset{<}{\bullet}}}\mathrm{Act}^{N^{\overset{<}{\bullet}}} = \mathrm{Act}^{(M^{\overset{<}{\bullet}} \circ N^{\overset{<}{\bullet}})} \tag{4.61}$$

but the verification of (4.61), which we leave to the reader, demands greater care than that of its prototype (2.3). Indeed, (4.61) would fail if we tampered ever so slightly with the definition (2.26) of $B_{\overset{<}{\bullet}}$.

So much for the *meaning* of Lemma 4.5. Now to its *proof*. As with Lemma 4.4, there is a short but indirect proof which deduces (4.58) from the straightforward arborification of the induction rule (4.5). Let us sketch a more direct proof, which runs parallel to that of Lemma 4.4.

To do this, we require the *arborified version* of the identity (3.3). It involves on the right-hand side all maximal factorizations of $\overset{<}{\boldsymbol{\omega}}$:

$$\overset{<}{\boldsymbol{\omega}} = (\mathbf{a})^<(\boldsymbol{\omega}^1)^<(\boldsymbol{\omega}^2)^< \dots (\boldsymbol{\omega}^s)^<(\mathbf{b}^1)^<(\mathbf{b}^2)^< \dots (\mathbf{b}^t)^< \tag{4.62}$$

with *one initial* factor $(\mathbf{a})^<$; with a *connected cluster* of middle factors $(\boldsymbol{\omega}^i)^<$; and (unlike in (3.3)) with any number of *end factors* $(\mathbf{b}^i)^<$. The effect of the "derivations" $(\mathrm{daa})^{s-1}/(s-1)!$ on $\mathrm{Sa}^{\overset{<}{\mathbf{a}}}$, $\mathrm{Ta}^{\overset{<}{\boldsymbol{\omega}}}$, $\mathrm{Sa}^{\overset{<}{\mathbf{b}}}$, are still defined as in (3.3) (relative to all the subordinated, fully-ordered sequences) but with a slight modification in the case of $\mathrm{S}^{\overset{<}{\mathbf{a}}}$: we must modify the *order* on the *arborescent* sequence $\overset{<}{\mathbf{a}}$ so as to turn it into an *antiarborescent* sequence $\overset{<}{\mathbf{a}}_*$ (i.e., whereas each element in $\overset{<}{\mathbf{a}}$ has at most one predecessor, each element in $\overset{<}{\mathbf{a}}_*$ has at most one successor) by *retaining the order* on the (necessarily unique) branch of $\overset{<}{\mathbf{a}}$ that borders on the *connected cluster* of the middle factors $(\boldsymbol{\omega}^i)^<$, and by *reversing the order* on all the other branches of $\overset{<}{\mathbf{a}}$ which do not border on that cluster. Once we are equipped with this arborescent version of (3.3), we can check the *cancellation* of all derivatives in $M_n^{\overset{<}{\bullet}}$ in much the same way as we did with $M_n^{\bullet}$ (see (4.46*)).

Lemma 4.6. *Linearizability criterion.*
If for any resonant vector field, formal or not:

$$X = X^{\mathrm{lin}} + \sum B_n \tag{4.63}$$

we define the "deviations" X^{aa} and X^{aaa} by:

$$X^{\mathrm{aa}} = \sum \mathrm{Taa}^{\bullet} B_{\bullet} \quad ; \quad X^{\mathrm{aaa}} = \sum \mathrm{Taaa}^{\overset{<}{\bullet}} B_{\overset{<}{\bullet}} \tag{4.64}$$

then the equivalences hold:

$$\{X \text{ formally linearizable}\} \Leftrightarrow \{X^{\mathrm{aa}} = 0\} \Leftrightarrow \{X^{\mathrm{aaa}} = 0\} \tag{4.65}$$

Corollary of Lemma 4.6. *Characterization of X^{carr}.*
The correction X^{carr} is characterized by each one of the relations:

$$(X - X^{\mathrm{carr}})^{\mathrm{aa}} = 0 \tag{4.66}$$

$$(X - X^{\mathrm{carr}})^{\mathrm{aaa}} = 0. \tag{4.66*}$$

Proof of Lemma 4.6.
If we define:

$$X^{\mathrm{aad}} \overset{\mathrm{def}}{=} X - X^{\mathrm{aa}} = X^{\mathrm{lin}} + \sum (\mathrm{I}^{\bullet} - \mathrm{Taa}^{\bullet}) B_{\bullet} \tag{4.67}$$

$$X^{\mathrm{aaad}} \overset{\mathrm{def}}{=} X - X^{\mathrm{aaa}} = X^{\mathrm{lin}} + \sum (\mathrm{I}^{\overset{<}{\bullet}} - \mathrm{Taa}^{\overset{<}{\bullet}}) B_{\overset{<}{\bullet}} \tag{4.67*}$$

then (4.58) in combination with (4.61) implies that the following sequence converges towards X^{carrd}:
(4.68)

$$X \longrightarrow X^{\mathrm{aad}} \longrightarrow X^{\mathrm{aad}^2} \longrightarrow X^{\mathrm{aad}^3} \longrightarrow \dots \overset{\lim}{\longrightarrow} X^{\mathrm{carrd}} = X - X^{\mathrm{carr}}$$

(4.68*)

$$X \longrightarrow X^{\mathrm{aaad}} \longrightarrow X^{\mathrm{aaad}^2} \longrightarrow X^{\mathrm{aaad}^3} \longrightarrow \ldots \stackrel{\lim}{\longrightarrow} X^{\mathrm{carrd}} = X - X^{\mathrm{carr}}$$

On the other hand, (4.58) also implies:

(4.69)

$$(X - X^{\mathrm{carr}}) - (X - X^{\mathrm{carr}})^{\mathrm{aa}} = (X - X^{\mathrm{carr}}) - (X - X^{\mathrm{carr}})^{\mathrm{aaa}} = X - X^{\mathrm{carr}}$$

which is exactly (4.66) and (4.66*).

Now, if X is linearizable, we have $X^{\mathrm{carr}} = 0$, and in that case (4.69) says that:

$$X^{\mathrm{aa}} = X^{\mathrm{aaa}} = 0$$

Conversely, if we assume $X^{\mathrm{aa}} = 0$ or $X^{\mathrm{aaa}} = 0$, then $X^{\mathrm{aad}} = X$ or $X^{\mathrm{aaad}} = X$ and so all terms in the sequences (4.68) and (4.68*) including the limit X^{carrd}, are equal to X, which means that X is formally linearizable.

Remark: The linearizability criteria (4.65) are remarkable in so far as X^{aa} and X^{aaa} are defined by the utterly elementary moulds $\mathrm{Taa}^{\bullet}$ and $\mathrm{Taa}^{\overset{<}{\bullet}}$ (compare for instance their definitions with the calculation procedures for $\mathrm{Carr}^{\bullet}$ or $\mathrm{Nal}^{\bullet}$ or $\mathrm{Ray}^{\bullet}$). However, of the two, only X^{aaa} is a vector field, so that verifying $X^{\mathrm{aaa}} = 0$ reduces to verifying $X^{\mathrm{aaa}} x_i = 0$ for each variable x_i ($i = 1, 2, \ldots \nu$). The operator X^{aa}, on the other hand, is neither a vector field nor an automorphism. Therefore, in order to check that $X^{\mathrm{aa}} = 0$, it seems *a priori necessary* to check that $X^{\mathrm{aa}}.x^m = 0$ for all monomials x^m, and not just for the single variables x_i. This somewhat detracts from the usefulness of X^{aa} for *providing linearizability criteria*, but does not prevent it from *providing very convenient obstructions to linearizability* !

5. Analyticity of the corrected form (for diffeos)

Proposition 5.1. *Let F be a local, analytic, resonant, torsion-less diffeo whose spectrum $l = (l_1, \ldots, l_\nu)$ meets Brjuno's diophantine condition* (1.9). *Then the correction F^{cerr} and corrected form*

$$F^{\mathrm{cerrd}} \stackrel{\mathrm{def}}{=} (F^{\mathrm{cerr}})^{-1} F^{\mathrm{lin}}$$

are analytic, and their arborified mould expansions:

$$\text{(5.1)} \qquad F^{\mathrm{cerr}} = \sum \mathrm{Cerr}^{\overset{<}{\bullet}} B_{\underset{\bullet}{<}} \quad ; \quad F^{\mathrm{cerrd}} = \sum \mathrm{Cerrd}^{\overset{<}{\bullet}} B_{\underset{\bullet}{<}} \quad ;$$

are normally convergent.

Since we are now dealing with diffeos, the relevant arborification is of course "symmetrel arborification" (or "erborification") as defined in Sect. 2, with due allowances for *component contractions*. That aside, the proof of

Proposition 5.1 exactly mirrors that of Proposition 4.1 and like with vector fields, everything rests on the *cancellation* or rather *non-appearance* of *multiple poles*. That in turn is proved exactly as with vector fields, but by resorting to the variance rules (3.35), (3.36), (3.37), (3.38), instead of (3.33). The lemmas 4.1, 4.2 and 4.3 have their literal counterpart; the avoidable and unavoidable poles or "denominators" are the same as with fields; and the rule which, at each inductive step, permits certain indices i in ver_i and prohibits others, is the same as in Sect. 4 with var_i.

6. Analytic linearizability of the corrected form. Explicit conjugacy

Proposition 6.1. *Explicit conjugacy for vector fields.*
Let $X = X^{\mathrm{lin}} + \sum B_n$ *be a resonant, analytic, local vector field with a spectrum fulfilling Brjuno's diophantine condition* (1.9). *Then the corrected form* X^{carrd} *is analytically linearizable. Moreover, there exists an explicit analytic conjugacy:*

$$X^{\mathrm{carrd}} = \Theta_{\mathrm{carr}} X^{\mathrm{lin}} \Theta_{\mathrm{carr}}^{-1} \tag{6.1}$$

with mould expansions:

$$\Theta_{\mathrm{carr}} = \sum \mathrm{Scarr}^{\bullet} B_{\bullet} \tag{6.2}$$

$$\Theta_{\mathrm{carr}}^{-1} = \sum \mathrm{invScarr}^{\bullet} B_{\bullet} \tag{6.3}$$

which after arborification become normally convergent. These expansions involve symmetral, mutually inverse moulds $\mathrm{Scarr}^{\bullet}$ *and* $\mathrm{invScarr}^{\bullet}$*, which are inductively defined by a system patterned on that of the "royal form", namely:*

$$\mathrm{Scarr}^{\bullet} \times \mathrm{invScarr}^{\bullet} = 1^{\bullet} \tag{6.4}$$

$$\nabla_{\boldsymbol{\omega}} \mathrm{Scarr}^{\bullet} = -\mathrm{Scarr}^{\bullet} \times \mathrm{Carrd}^{\bullet} \tag{6.5}$$

$$\textit{lan Scarr}^{\bullet} = \mathrm{Carra}^{\bullet} \times \mathrm{Scarr}^{\bullet} \tag{6.6}$$

with the mould derivation lan which simply multiplies $M^{\boldsymbol{\omega}}$ by the number $r^*(\boldsymbol{\omega})$ of non-vanishing components ω_i in $\boldsymbol{\omega}$*; with the customary alternal mould* $\mathrm{Carrd}^{\bullet} = \mathrm{I}^{\bullet} - \mathrm{Carr}^{\bullet}$*; and with a new alternal mould* $\mathrm{Carra}^{\bullet}$ *which is determined by the above system under addition of the usual condition:*

$$\mathrm{Carra}^{\boldsymbol{\omega}} = 0 \quad \text{if } \|\boldsymbol{\omega}\| = 0 \quad \textit{("complementarity" to } \mathrm{Carrd}^{\bullet}\textit{)} \tag{6.7}$$

Comments and sketch of proof.

By itself, the existence of an analytic linearization of X^{carrd} is a consequence of the analyticity of X^{carrd}, as proved in Sect. 4, and of classical theorems about the analytic prenormalization of certain exceptional resonant objects. Indeed, Brjuno gave in [B], (pp173–174) conditions (bearing on the *prenormal forms*) for a resonant field to be analytically prenormalizable. For general resonant fields these conditions are rarely fulfilled, but they automatically are in the case of a *formally linearizable* (and *analytic*) field like X^{carrd}, whose prenormal forms are necessarily $\equiv 0$.

So the real point of the above proposition is the "canonical" (though chart-dependent) conjugacy (6.1) with its explicit expansions (6.2), (6.3). The existence of such a canonical analytic conjugacy, which once again rests on the cancellation of multiple small denominators, can in no way be taken for granted because, as we shall prove towards the end of this section, although each *prenormalization induces an explicit linearization of the corrected form*, these induced linearizations are generically divergent, due precisely *to the persistence in them* of *multiple small denominators*.

Both $\mathrm{Scarr}^{\boldsymbol{\omega}}$ and $\mathrm{invScarr}^{\boldsymbol{\omega}}$ admit decompositions of type (4.8), with coefficients n_p bounded by (4.9) as in the case of $\mathrm{Carr}^{\boldsymbol{\omega}}$. The only, very slight, difference lies in the nature of the poles η and their multiplicities μ. In the case of $\mathrm{Scarr}^{\boldsymbol{\omega}}$ (resp. $\mathrm{invScarr}^{\boldsymbol{\omega}}$), to the poles η defined in (4.7) we must add the poles of the form:

$$\eta = \check{\boldsymbol{\omega}}_i = \omega_1 + \ldots + \omega_i \quad (\text{resp. } \eta = \hat{\boldsymbol{\omega}}_i = \omega_i + \ldots + \omega_r) \tag{6.8}$$

and each pole η has a multiplicity μ defined as in Lemma 4.2 (we recall that μ was the number of decompositions (4.7)) except that one must *augment* μ *by one unit* if η happens to be also of the form $\check{\boldsymbol{\omega}}_i$ (resp. $\hat{\boldsymbol{\omega}}_i$).

Moreover, to establish these assertions, one *does not need* to duplicate the reasoning of Sect. 4 and to write down the *variance rules* for $\mathrm{Scarr}^{\bullet}$ and $\mathrm{invScarr}^{\bullet}$ (i.e. the effect on them of the operators var_i), but *merely to plug the results established in Sect. 4 for* $\mathrm{Carr}^{\bullet}$ (see Lemma 4.2) *into the induction* (6.4), (6.5), (6.6).

We shall merely give a few examples, for various degeneracy patterns, and with the usual abbreviations ($\omega_{ij} = \omega_i + \omega_j$, etc.)

Example 1: If $\omega_3 + \omega_4 = 0$, then:

$$\mathrm{Scarr}^{\omega_1,\omega_2,\omega_3,\omega_4} = -(\omega_1\omega_{12}\omega_3\omega_{123})^{-1}$$

Example 2: If $\omega_2 + \omega_3 + \omega_4 + \omega_5 = 0$; $\omega_3 + \omega_4 = 0$, then:

$$\mathrm{Scarr}^{\omega_1,\ldots,\omega_5} = +(\omega_1^2\omega_{12}\omega_3\omega_{123})^{-1} - (\omega_1^2\omega_2\omega_3\omega_{23})^{-1}$$

Example 3: If $0 = \omega_1 + \ldots + \omega_6 = \omega_2 + \ldots + \omega_5 = \omega_3 + \omega_4$, then:

$$\begin{aligned}\mathrm{Scarr}^{\omega_1,\ldots,\omega_6} &= -(\omega_1^2\omega_{12}\omega_{123}\omega_2\omega_3)^{-1} - (\omega_1^2\omega_{12}\omega_{123}\omega_{23}\omega_3)^{-1} \\ &\quad +(\omega_1^2\omega_{12}^2\omega_{123}\omega_3)^{-1} - 1/2(\omega_1^2\omega_{12}^2\omega_{123}^2)^{-1}\end{aligned}$$

Exemple 4: If $\boldsymbol{\omega}$ has no degeneracy at all, then:

$$\mathrm{Scarr}^{\boldsymbol{\omega}} = \mathrm{Sa}^{\boldsymbol{\omega}} = \prod_{i=1}^{r}(-\check{\boldsymbol{\omega}}_i)^{-1} \qquad (r = r(\boldsymbol{\omega}))$$

Example 5: If $\omega_1 + \ldots + \omega_r = 0$, then:

$$\mathrm{Scarr}^{\boldsymbol{\omega}} = \left(\prod_{i=1}^{r-1}(-\check{\boldsymbol{\omega}}_i)^{-1}\right)\left(\sum(\check{\boldsymbol{\omega}}_i)^{-1}\right)$$

Example 6: If $r = 2r'$ and $0 = \omega_1+\omega_2 = \omega_3+\omega_4 = \ldots = \omega_{r-1}+\omega_r = 0$, then:

$$\mathrm{Scarr}^{\boldsymbol{\omega}} = (-1)^{r'}\frac{(2r')!}{4^{r'}(r'!)(r'!)}(\omega_1)^{-2}(\omega_3)^{-2}\ldots(\omega_{r-1})^{-2}$$

Proposition 6.2. *Explicit conjugacy for diffeos.*

Let $F = (1 + \sum B_n)F^{\mathrm{lin}}$ *be a resonant, analytic, local diffeo with a spectrum fulfilling Brjuno's diophantine condition* (1.9)*. Then the corrected form* F^{cerrd} *is analytically linearizable. Moreover, there exists an explicit analytic conjugacy:*

$$F^{\mathrm{cerrd}} = \Theta_{\mathrm{cerr}}F^{\mathrm{lin}}\Theta_{\mathrm{cerr}}^{-1} \tag{6.24}$$

with mould expansions:

$$\Theta_{\mathrm{cerr}} = \sum \mathrm{Scerr}^{\bullet}B_{\bullet} \tag{6.25}$$

$$\Theta_{\mathrm{cerr}}^{-1} = \sum \mathrm{invScerr}^{\bullet}B_{\bullet} \tag{6.26}$$

which after erborification (see below) becomes normally convergent. These expansions involve symmetrel, mutually inverse moulds $\mathrm{Scerr}^{\bullet}$ *and* $\mathrm{invScerr}^{\bullet}$*, which are inductively defined by a system patterned on that of the "royal form", namely:*

$$\mathrm{Scerr}^{\bullet} \times \mathrm{invScerr}^{\bullet} = 1\ . \tag{6.27}$$

$$e^{\nabla\boldsymbol{\omega}}\mathrm{Scerr}^{\bullet} = \mathrm{Scerr}^{\bullet} \times \mathrm{invCerrd}^{\bullet} \tag{6.28}$$

$$\mathrm{len}\ \mathrm{Scerr}^{\bullet} = \mathrm{Cerre}^{\bullet} \times \mathrm{Scerr}^{\bullet} \tag{6.29}$$

with a mould derivation len *analogous to* lan*; with the customary symmetrel (not alternel) mould* $\mathrm{Cerrd}^\bullet$*; and with a new alternel (not symmetrel!) mould* $\mathrm{Cerre}^\bullet$ *which is determined by the above system under addition of the usual condition:*

$$(6.30) \qquad \mathrm{Cerre}^{\boldsymbol{\omega}} = 0 \quad \text{if } \|\boldsymbol{\omega}\| = 0 \quad \textit{(“complementarity” to Cerrd}^\bullet\textit{)}$$

Complement. Prenormalization of X and linearization of X^{carrd}

Any *continuous prenormalization* $X = \Theta_{\mathrm{pran}} X^{\mathrm{pran}} \Theta_{\mathrm{pran}}^{-1}$ with the moulds $\mathrm{Spran}^\bullet$, $\mathrm{invSpran}^\bullet$ and $\mathrm{Pran}^\bullet$ that go with it, automatically induces an *explicit* (formal) *linearization*

$$X^{\mathrm{carrd}} = \Theta_{\mathrm{prac}} X^{\mathrm{lin}} \Theta_{\mathrm{prac}}^{-1}$$

of the corrected form X^{carrd}, with the expansions:

$$(6.31) \qquad \Theta_{\mathrm{prac}} = \sum \mathrm{Sprac}^\bullet B_\bullet \ ; \quad \Theta_{\mathrm{prac}}^{-1} = \sum \mathrm{invSprac}^\bullet B_\bullet$$

$$(6.32) \quad \mathrm{Sprac}^\bullet \stackrel{\mathrm{def}}{=} \mathrm{Spran}^\bullet \circ \mathrm{Carrd}^\bullet \ ; \quad \mathrm{invSprac}^\bullet \stackrel{\mathrm{def}}{=} \mathrm{invSpran}^\bullet \circ \mathrm{Carrd}^\bullet$$

for the simple reason that $\mathrm{Pran}^\bullet \circ \mathrm{Carrd}^\bullet = 0$.

All these *induced linearization* of X^{carrd}, however, suffer from the same basic flaw, and differ from the “canonical” one of Proposition 6.1. Indeed:

Proposition 6.3. *(Inadequacy of the induced linearizations).*
Each induced linearization involves generically divergent automorphisms Θ_{prac} *and* $\Theta_{\mathrm{prac}}^{-1}$*, due to the persistance in the mould expansions* (6.31)*, both before and after arborification, of multiple powers (of arbitrarily high order) of the small denominators.*

Corollary of Proposition 6.3. *The “canonical” linearization of* X^{carrd} *constructed in Proposition* 6.1 *is not induced by the “royal” prenormalization nor, for that matter, by any other continuous prenormalization.*

7. Examples. The simplest non-trivial case

Elementary objects

Fix a resonant spectrum $\lambda = \{\lambda_1, \ldots, \lambda_\nu\}$ and a set $\{\omega_1, \ldots, \omega_s\}$ of pairwise distinct complex numbers. Clearly, the space $L_{\{\omega_1,\ldots,\omega_s\}}$ spanned by all homogeneous vector fields B_n of degree n with $\langle n, \lambda\rangle \in \{\omega_1, \ldots, \omega_s\}$

is a Lie algebra in three cases only, namely for the sets $\{0\}$, $\{0, \omega_1\}$ and $\{0, \omega_1, \omega_2\}$ with $\omega_1 + \omega_2 = 0$. Vector fields X with spectrum λ and components B_n in $L_{\{0\}}$ are trivial in every respect (at least for *one* degree of resonance). Vector fields with components in $L_{\{0,\omega_1\}}$ are more interesting, in the sense of having non-trivial analytic moduli, holomorphic invariants, divergent associated objects, etc. but their *correction* X^{carr} is trivial, since it reduces to the series of *all resonant components*. Vector fields with components in $L_{\{0,\omega_1,\omega_2\}}$ are the most promising ones from this paper's point of view: they have both non-trivial moduli *and* non-trivial corrections; and yet they are elementary enough to permit a fully explicit description of the principal associated objects, beginning with the correction.

So we will turn to these vector fields, and to the corresponding diffeos, for an illustration of our main conclusions, by verifying in this simple instance *the convergence of what should converge (correction) and the divergence of what may diverge (nilpotent part; trimmed form, etc.).*

Elementary vector fields and their correction

Let us investigate resonant analytic vector fields X and Y of the form:

$$X = (a_0(x^m) + x^{m^1} a_1(x^m) + x^{m^2} a_2(x^m))X^{\text{lin}} = X^{\text{lin}} + \dots \tag{7.1}$$

$$a_0(x^m), a_1(x^m), a_2(x^m) \in \mathbf{C}\{x^m\} \; ; \; a_0(0) = 1 \tag{7.1*}$$

$$Y = X + X^{\text{res}} = X^{\text{lin}} + \dots \tag{7.2}$$

with a common linear part:

$$X^{\text{lin}} = Y^{\text{lin}} = \sum \lambda_i x_i \partial_{x_i} \tag{7.3}$$

with at least one degree of resonance and homogeneous components in $L_{\{0,\omega_1,\omega_2\}}$:

$$\langle m, \lambda \rangle = 0 \; ; \; m = (m_1, \dots, m_\nu) \; ; \; (m_i \geq 0) \; ; \; m^1 + m^2 = m \; . \tag{7.4}$$

$$\langle m^1, \lambda \rangle = \omega_1 \; ; \; \langle m^2, \lambda \rangle = \omega_2 \; ; \; \omega_1 + \omega_2 = 0 \; , \; \omega_1 \neq 0 \neq \omega_2 \tag{7.5}$$

and, inside Y, one homogeneous component X^{res} that does not annihilate the resonant monomial x^m:

$$X^{\text{res}} \cdot x^m = -(x^m)^2 \; ; \; X^{\text{res}} = x^m \sum \tau_i x_i \partial_{x_i} \; ; \; \langle m, \tau \rangle = -1 \; . \tag{7.6}$$

The coefficients a_0, a_1, a_2 are convergent power series of the resonant monomial x^m, and since $a_1(0) \equiv 1$, the linear part of both fields is indeed given by (7.3).

For instance, in dimension $\nu = 2$, we may take:

$$(\lambda_1, \lambda_2) = (1, -1)\ ;\ x^m = x_1 x_2 ; x^{m^1} = x_1\ ;\ x^{m^2} = x_2 \tag{7.7}$$

$$X^{\mathrm{lin}} = x_1 \partial_{x_1} - x_2 \partial_{x_2}\ ;\ X^{\mathrm{res}} = x_1 x_2 \{\tau_1 x_1 \partial_{x_1} + \tau_2 x_2 \partial_{x_2}\}\ ;\ \tau_1 + \tau_2 = -1 \tag{7.8}$$

Now, if we put:

$$U_0 = X^{\mathrm{lin}}\ ;\ U_1 = x^{m^1} X^{\mathrm{lin}}\ ;\ U_2 = x^{m^2} X^{\mathrm{lin}} \tag{7.9}$$

we have the obvious Lie brackets:

$$\bar{X} \cdot U_i \overset{\mathrm{def}}{=} [X, U_i] = A_i^0 U_0 + A_i^1 U_1 + A_i^2 U_2 \tag{7.10}$$

with coefficients A_j^i as follows (we write a_i for $a_i(x^m)$):

$$A_0^0 = 0\ ; A_0^1 = -\omega_1 a_1\ ;\ A_0^2 = -\omega_2 a_2 \tag{7.10*}$$

$$A_1^0 = 2\omega_1 x^m a_2\ ;\ A_1^1 = \omega_1 a_0\ ;\ A_1^2 = 0 \tag{7.10**}$$

$$A_2^0 = 2\omega_2 x^m a_1\ ;\ A_2^1 = 0\ ;\ A_2^2 = \omega_2 a_0\ . \tag{7.10***}$$

The matrix $A = \{A_j^i\}$ has as its characteristic polynomial:

$$\det(A - t\ \mathrm{Id}) = -t^3 - (4x^m a_1 a_2 - a_0 a_0) t^2\ . \tag{7.11}$$

When X reduces to its linear part X^{lin}, we have $a_0 \equiv 1$, $a_1 \equiv a_2 \equiv 0$, and so in this case:

$$\det(A - t\ \mathrm{Id}) = -t^3 + t^2\ . \tag{7.12}$$

Now, if a general X of the form (7.1) is conjugate to X^{lin}, it is so *within* the algebra $L_{\{0,\omega_1,\omega_2\}}$, and therefore the two characteristic polynomials (7.11) and (7.12) must coincide, yielding the condition:

$$a_0 a_0 = 1 + 4x^m a_1 a_2 \quad (\text{with } a_i = a_i(x^m) \in \mathbf{C}\{x\})\ . \tag{7.13}$$

This, however, means that the corrections of the fields X and Y as given by (7.1) and (7.2) are respectively:

$$X^{\mathrm{carr}} = (a_0(x^m) + \alpha_0(x^m)) X^{\mathrm{lin}} \tag{7.14}$$

$$Y^{\mathrm{carr}} = X^{\mathrm{carr}} + X^{\mathrm{res}} \tag{7.15}$$

with a *convergent power series* $\alpha_0 = \alpha_0(x^m)$ given by:

$$(7.16) \quad \alpha_0 = (1 + 4x^m a_1 a_2)^{1/2} = 1 + \sum_{r \geq 1} 2 \cdot (-1)^{p+1} \mathrm{Cat}_p (a_1 a_2 x^m)^p$$

with the Catalan numbers:

$$(7.17) \qquad \mathrm{Cat}_p = \frac{(2p-2)!}{(p-1)!p!} \quad (p \in \mathbf{N}) \; .$$

It is a nice exercise to check that the general formula:

$$(7.18) \qquad X^{\mathrm{carr}} = \sum \mathrm{Carr}^{\boldsymbol{\omega}} B_{\boldsymbol{\omega}} = \sum \frac{1}{r(\boldsymbol{\omega})} \mathrm{Carr}^{\boldsymbol{\omega}} B_{[\boldsymbol{\omega}]}$$

tallies exactly with (7.16) (7.17).

Indeed, in view of (7.9) and of the opposite homogeneousness of $\mathrm{Carr}^{\boldsymbol{\omega}}$ and $B_{\boldsymbol{\omega}}$, we can assume $\{\omega_1, \omega_2\} = \{1, -1\}$, and (7.18) transforms into:

$$(7.19) \qquad 2(-1)^{p+1} \mathrm{Cat}_p = \sum_{\varepsilon_i = \pm 1} \mathrm{Carr}^{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2p}} \mathit{\Gamma}_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2p}}$$

$$(7.19^*) \qquad 2(-1)^{p+1} \mathrm{Cat}_p = (2p)^{-1} \sum_{\varepsilon_i = \pm 1} \mathrm{Carr}^{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2p}} \mathit{\Gamma}_{[\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2p}]}$$

with:

$$(7.20) \quad \mathit{\Gamma}_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2p}} \stackrel{\mathrm{def}}{=} \varepsilon_1 (\varepsilon_1 + \varepsilon_2)(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \ldots (\varepsilon_1 + \ldots + \varepsilon_{2p-1})$$

$$(7.20*) \quad \mathit{\Gamma}_{[\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2p}]} \stackrel{\mathrm{def}}{=} (\varepsilon_1 - \varepsilon_2)(\varepsilon_1 + \varepsilon_2 - \varepsilon_3) \ldots (\varepsilon_1 + \ldots + \varepsilon_{2p-1} - \varepsilon_{2p}) \; .$$

With the help of the generating induction (3.33) for $\mathrm{Carr}^{\bullet}$, one can then verify the identities (7.19) and (7.19*) with Cat_p as in (7.17). Note that the sum (7.19*) involves far fewer terms than (7.19), since the only non-zero contributions to (7.19*) come from sequences $\{\varepsilon_i\}$ such that:

$$(7.20^{**}) \qquad 0 = \varepsilon_1 + \varepsilon_2 = \varepsilon_3 + \varepsilon_4 = \ldots = \varepsilon_{2p-1} + \varepsilon_{2p}$$

So much for the *corrections* X^{carr}, Y^{carr} *and their convergence* in this elementary case. The other associated objects relative to X are also convergent, but those relative to Y are generically divergent, and always resurgent (in the variable $z = (x^m)^{-1}$). For the *intrinsical objects* (Y^{nal}, $\mathit{\Theta}_{\mathrm{nar}}$, etc.) we have *canonical resurgence*, governed by the Bridge Equation (1.19), with only two *acting alien derivations*, namely $\mathit{\Delta}_{\omega_1}$ and $\mathit{\Delta}_{\omega_2}$, and the two holomorphic invariants A_{ω_1} and A_{ω_2} that go with them. But the continuous prenormal forms are also generically divergent, *and* resurgent, with their own special resurgence lattices (see [E.V.2]).

Elementary diffeos and their correction

The natural counterparts of X^{lin}, X^{res}, X, Y are diffeos F^{lin}, F^{res}, F, G of the form:

$$(7.21) \qquad F^{\text{lin}} \cdot x^m = x^m$$

$$(7.21^*) \qquad F^{\text{lin}} \cdot x^{m^1} = e^{\omega_1} x^{m^1}$$

$$(7.21^{**}) \qquad F^{\text{lin}} \cdot x^{m^2} = e^{\omega_2} x^{m^2}$$

$$(7.22) \qquad F^{\text{res}} \cdot x^m = x^m (1 - x^m)^{-1}$$

$$(7.22^*) \qquad F^{\text{res}} \cdot x^{m^1} = x^{m^1} (1 - x^m)^{\tau_1}$$

$$(7.22^{**}) \qquad F^{\text{res}} \cdot x^{m^2} = x^{m^2} (1 - x^m)^{\tau_2}$$

with $\tau_1 + \tau_2 = -1$ as earlier on

$$(7.23) \qquad F \cdot x^m = x^m$$

$$(7.23^*) \qquad F \cdot x^{m^1} = (a x^{m^1} + b)(c x^{m^1} + d)^{-1}$$

$$(7.23^{**}) \quad F \cdot x^{m^2} = x^m / (F \cdot x^{m^1}) = (d x^{m+m^2} + c x^{2m})(b x^{m^2} + a x^m)^{-1}$$

with, of course, analytic coefficients:

$$(7.23^{***}) \qquad a, b, c, d \in \mathbf{C}\{x^m\} \; ; \; x^{m^1} x^{m^2} = x^m$$

and lastly:

$$(7.24) \qquad G = F^{\text{res}} F \; .$$

Now, the correction F^{cerr} is necessarily of the form:

$$(7.25) \qquad F^{\text{cerr}} \cdot x^m = x^m$$

$$(7.25^*) \qquad F^{\text{cerr}} \cdot x^{m^1} = \gamma x^{m^1}$$

$$(7.25^{**}) \qquad F^{\text{cerr}} \cdot x^{m^2} = (1/\gamma) x^{m^2}$$

for a suitable power series $\gamma = \gamma(x^m)$. Moreover, the corrected form $F^{\mathrm{cerrd}} = (F^{\mathrm{corr}})^{-1}F$ is characterized by being conjugate to F^{lin}. Therefore F^{cerrd}, which acts on x^{m^1} like the Möbius map:

$$x^{m^1} \longrightarrow ((a/\gamma)x^{m^1} + b)((c/\gamma)x^{m^1} + d)^{-1} \tag{7.26}$$

must be conjugate to the homothety induced by F^{lin}:

$$x^{m^1} \longrightarrow e^{\omega_1}x^{m^1} \tag{7.27}$$

which yields the condition:

$$e^{\omega_1}(a + d\gamma)^2 - (1 + e^{\omega_1})^2(ad - bc)\gamma \equiv 0\ . \tag{7.28}$$

This second order equation admits *one* acceptable solution $\gamma(x^m) = 1 + \ldots$, which is a convergent power series of x^m whenever a, b, c, d are themselves so, i.e. when the diffeo F is analytic. But knowing F^{cerr} we know G^{cerr} also:

$$G^{\mathrm{cerr}} = F^{\mathrm{res}}F^{\mathrm{cerr}}\ . \tag{7.29}$$

Thus we see that the corrections of both F and G are convergent, in full agreement with the general theorems, but in contrast with the other objects associated with G, such as G^{nel}, G^{prem} etc., which are generically *divergent* and *resurgent*. Amongst these, the intrinsic objects, such as G^{nel}, Θ_{ner} etc $\ldots$, display canonical resurgence, and satisfy the Bridge Equation, with acting alien derivations $\Delta_{\boldsymbol{\omega}}$ and holomorphic invariants $A_{\boldsymbol{\omega}}$ whose indices ω are of the form:

$$\omega = 2\pi i k\ (k \neq 0)\ \textit{or}\ \omega = \omega_1 + 2\pi i k\ \textit{or}\ \omega = \omega_2 + 2\pi i k\ (k \in \mathbf{Z})\ . \tag{7.30}$$

As for the objects F^{nel}, F^{prem} associated with F, they are of course convergent, due to the absence in them of the crucial factor F^{res}.

8. Conclusion

For conciseness, this recapitulation mentions vector fields only, but of course each of the coming remarks extends to the diffeomorphisms as well.

The bulk of this paper was devoted to the *correction* X^{carr} of resonant vector fields X. Despite its outward similarity to most other objects associated with vector fields, such as the nilpotent part X^{nal} and the continuous prenormal forms X^{pran}, which are generically divergent and resurgent, the correction turns out to possess completely different properties: it *converges* under the sole diophantine condition introduced by A.D. Brjuno to prove the analytic linearizability of non-resonant vector fields. Indeed, one may view the convergence of the *corrected form* $X^{\mathrm{carrd}} := X - X^{\mathrm{carr}}$ and its *analytic linearizability* as the natural extension to the resonant case of the classical Siegel-Brjuno-Rüssmann linearizability theorem.

There is no doubt that the *correction* X^{carr} is a very important object, as borne out by the remarkable *variance rules* (3.33) which it verifies, and by the non-occurrence in it of *highly multiple poles*. It would certainly be rash to dismiss X^{carr} on the mere ground that it lacks geometric significance. In fact, the correction *becomes a geometric notion* as soon as we regard it as depending, not on X alone, but on the pair (X, X^{lin}). See Sect. 1, Remark 3. Moreover, the *correction* is not an isolated curiosity or freak of nature: the system of Lie-bracket equations of which X^{carr} is the solution, is merely the simplest instance of a large class of similar systems, whose common feature is a "*miraculous*" *non-occurence of multiple small denominators*. And even though X^{carr}, when looked upon as a function of X alone, is non-intrinsic, its *vanishing* (i.e. $X^{\mathrm{carr}} = 0$) carries intrinsic significance, and provides us with one of the simplest available criteria for the linearizability of X.

A noteworthy features of our proofs is their constant use of *moulds*, and the prominent part which specific *mould identities*, of a distinctly algebraic or combinatorial nature, play, even when it comes to proving results about convergence/divergence.

One question left partially unanswered in this paper is: just how "complex" is the *correction*? For instance, what exactly is the nature of X^{carr} when X is a polynomial vector field? One would expect *endless analytic continuability* with *isolated singularities* (like in the semi-elementary examples of Sect. 7), but where precisely do these singularities lie?

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Addition (March 1997). We are told that Eliasson's paper [El.1] has in the meantime appeared in Math. Phys. Elec. J. Vol. **2** (1996), paper 7.