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Intertwined mappings

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Intertwined mappings [*]

Jean Ecalle [(1)], Bruno Vallet [(2)]

Abstract. — We show that, contrary to expectations, there exist pairs of formal and even analytic, non-commuting and non-elementary (neither algebraic nor algebraic-differential) mapping germs in *Diff*($\mathbb{C}$, 0) that are 'entwined' in a group relation $W(f, g) = id$. In the case of identity-tangent mappings, 'twins' exhibit, rather than analyticity, generic divergence , but of a particularly interesting sort: resurgent, accelero-summable, and with simple alien derivatives.

Résumé. — Nous montrons que, contrairement à une attente assez partagée, il existe dans *Diff*($\mathbb{C}$, 0) des paires (f, g) de difféos locaux *jumelés*, qui engendrent des groupes « intéressants », *i.e.* ni libres ni trop élémentaires (ils ne sont pas abéliens et ne se réduisent pas, même après éclatement, à des groupes d'homographies). De tels groupes sont dits *liés*. Nous ébauchons une classification des relations $W(f, g)$, nécessairement très sporadiques, qui les définissent. Dans le cas de difféos tangents à l'identité, les générateurs jumelés f, g sont génériquement divergents, mais résurgents, accéléro-sommables, et ils possèdent des dérivées étrangères remarquables.

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1. Introduction. Intertwined mappings and bound groups

1.1. Interwined mappings and bound groups

The mappings we shall be concerned with are mainly the *local diffeomorphisms* (or "*diffeos*" for short) of $\mathbb{C}$ (with either $x = 0$ or $z = \infty$ as fixed point) as well as their *formal counterparts*. More to the point, we shall be looking for "intertwined mappings", i.e. for pairs (f, g) of diffeos (or "*twins*") that generate a *'bound' group* $G\{f, g\}$, by which we mean a group that is

(i) *non-free*, i.e not isomorphic to the two-generator free group

(ii) *non-abelian*

(iii) *non-elementary*, i.e. not conjugate to a group of homographies.

The last requirement is there to rule out uninteresting situations. It means that no local change of variable, whether analytic or formal, unramified ($x \mapsto c\,x + \cdots; z \mapsto a\,z + \cdots$) or ramified ($x \mapsto c\,x^n + \cdots; z \mapsto a\,z^n + \cdots$) should turn *f and g simultaneously* into a pair of "homographies" or "möbius maps" $x \mapsto ax(1+b\,x)^{-1}$ (near $x=0$) or, if the fixed point be $z=\infty$, into a pair of "similitudes" $z \mapsto a\,z+b$.

In order to generate a *bound group* in the above sense, f and g must of course verify some composition relation :

$$
\begin{aligned}
W(f,g) &= id \qquad \text{with} && (1.1)\\
W(f,g) &= f^{\circ m_1}\circ g^{\circ n_1}\circ\cdots f^{\circ m_r}\circ g^{\circ n_r} && (m_i,n_i\in\mathbb{Z}^*; r\in\mathbb{N}^*)
\end{aligned}
$$

The point, however, is that most relations W force f and g to *commute* or "*nearly-commute*", i.e. to be simultaneously conjugate to homographies. In fact, as we shall see, only finely honed and fairly intricate relations W can lead to suitable solutions (f,g) and give rise to bound groups $G\{f,g\}$.

1.2. Motivations

(i) Originally, the question as to the existence of *bound groups* (specifically, for *analytic, identity-tangent local self-mappings of* $\mathbb{C}$) was raised in the early 90s by D. Cerveau, R. Moussu and others in connection with the classification of local analytic foliations on $\mathbb{C}^2$ and the holonomy of local differential equations (see [Ce]). The expectation, it seems, was that such groups *didn't exist*. In 1995, however, one of us came up with a first series of examples, namely those listed in the mid-part of §3 (mainly §3.3). But then the subject was provisionally laid to rest, and only recently taken up anew for systematic investigation.

(ii) Another way of looking at the same question ([Ce]) is by trying to extend the so-called *Tits alternative*. In its classical formulation, the Tits alternative applies to subgroups $G \subseteq Gl(n,\mathbb{C})$ of the linear group, and it states that
– *either* G is virtually solvable, i.e. it contains a finite solvable subgroup of finite index
– *or* it contains a free 2-generator subgroup, which is Zariski dense.
So a natural and important question is: does a similar alternative hold for *Diff*$(\mathbb{C},0)$ and its formal counterpart? For the latter, we shall show that the answer is no. For the former also, the answer is no, but the identity-tangent sub-case is still open.

(iii) Actually, bound groups of mappings are basic and natural objects in *their own right* and their study requires no elaborate apology. Indeed, in the one-operation-only context of group theory, "*being intertwined*" is for a pair of mappings (f,g) the closest equivalent one could think of to "*algebraic dependence*" for a pair of irrational numbers (a, b). Moreover, since twins (f, g) are so scarce, so thinly spread out in the landscape, their precise shape and properties are a matter of legitimate curiosity.

(iv) The search for *twins* (and more generally for *siblings*, ie *"overrelated"* systems of diffeomorphisms; see §6.7–8) is closely bound up with the topic of infinite-dimensional group *representations* ("into") and *realisations* ("onto"). Whereas finite-dimensional representations and realisations are quite adequate for finite groups, infinite groups (especially solvable ones) often call for infinite-dimensional representations, and in particuliar for realisations as special groups of mappings (local or global).

(v) Twins provide a striking illustration of the ubiquity of *resurgence* and the usefulness of *acceleration.* Indeed, as we shall show, although *identity-tangent twins* are generically (and probably always) divergent, they are always *resurgent* (in a suitable chart) and even *biresurgent*, i.e. resurgent with respect to two distinct "*critical times*". Therefore, to resum them, we must resort not only to the usual *Borel-Laplace scheme*, but also to one intermediary *acceleration transform.*

1.3. Reminders about local diffeos of $\mathbb{C}$

Rather than working directly with local diffeos f :

$$f : x \mapsto ax\{1 + \sum_{n \geqslant 1} a_n x^n\} \quad (x \sim 0; a \neq 0) \tag{1.2}$$

we shall give precedence to the associated *substitution operators* F, denoted by the same letters, but capitalised :

$$F : \varphi \mapsto F.\varphi := \varphi \circ f \quad (\forall \varphi \in \mathbb{C}[[x]]) \tag{1.3}$$

The reason is that operators afford greater flexibility : instead of one single operation for diffeos (namely the composition product $\circ$), operators can be subjected to four operations: addition; multiplication; the taking of the logarithm (or infinitesimal generator) and the Lie bracketing of infinitesimal generators. But one should be mindful of the order reversal: the operator product FG corresponds to the composition product $g \circ f$.

Diffeos (1.2) with a "*multiplier*" $a = 1$ are said to be *identity-tangent.* When a is $\neq 1$ but a unit root, f is said to be *pre-identity tangent* (since a suitable iterate is identity tangent). When neither is the case, f is declared *non-identity tangent*.

If f is identity-tangent, the substitution operator F is triangular with respect to the natural basis $\{x^n\}$ of $\mathbb{C}[[x]]$. Moreover, $F-1$ is triangular with a zero-diagonal, and so is $F_* := \log F$, but whereas F is an *automorphism* of the algebra $\mathbf{C}[[x]]$:

$$F(\varphi.\psi) \equiv F(\varphi).F(\psi) \quad (\forall \varphi, \psi \in \mathbf{C}[[x]]) \tag{1.4}$$

its logarithm F_* is a *derivation* and thus of the form:

$$F_* = f_*(x)\,\partial \quad \text{with} \quad f_*(x) = \sum_{n \geqslant 1} a_{*n}\, x^{n+1} \quad \text{and} \quad \partial = \partial_x. \tag{1.5}$$

The *tangency order* p ($\geqslant 2$) of an identity-tangent diffeo f is the index of the first non zero coefficient a_n in (1.2) or a_{*n} in (1.5), which amounts to the same, since the leading coefficients coincide in both series ($a_p = a_{*p}$).

If f has tangency order p, then under some (formal, unramified) change of coordinate it can be brought to the normal form:

$$f^{nor}(x) \equiv x\,\{1 - \frac{1}{p}\, x^p + (\frac{1}{2}\,\frac{1+p}{p^2} - \frac{\alpha_*}{p})\, x^{2p} + \cdots\} \tag{1.6}$$

$$F^{nor} \equiv \exp(F_*^{nor}) \equiv \text{postcomposition by } f^{nor} \tag{1.7}$$

$$F_*^{nor} \equiv -\frac{1}{p}\, x^{p+1}\,(1 + \alpha_*\, x^p)^{-1}\,\partial \tag{1.8}$$

with a well-defined *iteration residue* $\alpha_* \in \mathbb{C}$. As a consequence, a further (ramified) change of coordinate :

$$x \mapsto z := x^{-p} + \alpha_*\, \log(x^{-p}) \tag{1.9}$$

will turn f into the plain *unit shift* $z \mapsto z+1$ with the fixed point moving from $x = 0$ to $z = \infty$.

If f is *pre-identity-tangent*, its multiplier is a primitive unit root $a = \exp(2\pi i\, \frac{p_{**}}{p_*})$ of some order p_*. So the iterate f^{p_*} is identity-tangent, and f itself can be normalised to:

$$F^{nor} = R\,.\,\exp(-\frac{1}{p}\, x^{p+1}\,(1 + \alpha_*\, x^p)^{-1})\,\partial) \tag{1.10}$$

with a well-defined iteration residue α_* , a tangency order p divisible by p_* , and with R denoting the rotation operator of angle $-2\,\pi\, \frac{p_{**}}{p_*}$:

$$R\,.\,\varphi := \varphi\,(x\,.\,\exp(-2\pi i\, \frac{p_{**}}{p_*})) \tag{1.11}$$

Lastly, if f is *non-identity-tangent*, it can be linearised, at least formally, i.e. normalised to $f^{nor}(x) = a\,x$.

Throughout the first half of this investigation (right through §5), we shall be concerned with *formal twins*, without any growth restriction on the coefficients a_n in (1.2). Only in the second half shall we look for *analytic twins* or, failing that, for *resurgent ones.* But right now, we must briefly mention how analyticity affects the main constructs pertaining to diffeos.

For non-identity-tangent analytic diffeos f, the normalising (i.e. linearising) change of coordinate (unique up to a linear map $x \mapsto c\,x$) is always analytic, except if the multiplier a of f is of module 1 *and* Liouvillian, which in this context means that it violates A.D. Bryuno's diophantine condition. In that case, the normalising map is generically divergent and non-resummable.

For identity-tangent (or pre-identity-tangent) analytic diffeos, on the other hand, both the normalising change of coordinate and the infinitesimal generator f_* are (generically) *divergent* but (always) *resurgent* (with critical time $z = x^{-p}$) and *Borel-Laplace resummable.* Such diffeos, on top of their two formal invariants (the tangency order p and the iteration residue α_*) possess a countable infinity of independent *analytic invariants*, each of which depends *holomorphically* on f (see [E2]).

All these results carry over, with only slight changes, to the context of higher-dimensional local diffeos (i.e. diffeos of $\mathbb{C}_0^\nu$), but there is one respect in which one-dimensional diffeos stand out: for them, the set of all iterates f^w of *complex* order w always coincides with the *centraliser* of f, that is, with the set of all diffeos g that commute with f.

1.4. Free groups and algebras

Let $\boldsymbol{Al(a,b)}$ (resp. $\boldsymbol{Lie(a,b)}$ denote the associative (resp. Lie) algebra freely generated by two non-commuting symbols $\boldsymbol{a}$ and $\boldsymbol{b}$, with the natural immersion $\boldsymbol{Lie(a,b)} \subset \boldsymbol{Al(a,b)}$. Further, let $\boldsymbol{Gr(A,B)}$ be the free non-abelian group generated by $\boldsymbol{A}$ and $\boldsymbol{B}$. By substituting $(e^{\boldsymbol{a}}, e^{\boldsymbol{b}})$ for $(\boldsymbol{A}$,$\boldsymbol{B})$, we may regard $\boldsymbol{Gr(A,B)}$ as a multiplicative subset of $\overline{\boldsymbol{Al}}\boldsymbol{(a,b)}$ and $\log(\boldsymbol{Gr(A,B)})$ as a subset of $\overline{\boldsymbol{Lie}}\boldsymbol{(a,b)}$, where $\overline{\boldsymbol{Al}}$ and $\overline{\boldsymbol{Lie}}$ denote the natural completion of $\boldsymbol{Al}$ and $\boldsymbol{Lie}$ obtained by allowing infinite sums.

A *polynomial function* on $\boldsymbol{Gr(A,B)}$ is a (scalar-valued) function P of the form:

$$P(\boldsymbol{A}^{m_1}\,\boldsymbol{B}^{n_1}\ldots\boldsymbol{A}^{m_r}\,\boldsymbol{B}^{n_r}) \quad\equiv\quad P(r \parallel m_1,n_1,\ldots,m_r,n_r) \qquad (1.12)$$
$$\textit{with} \quad r\in\mathbb{N}\ ;\ m_i,n_i\in\mathbb{Z})$$

with polynomial dependence on the variables m_i and n_i and the natural connection condition:

$$P(r \parallel \ldots,m_j,0,m_{j+1},\ldots)\equiv P(r-1 \parallel \ldots,m_j+m_{j+1},\ldots) \qquad (1.13)$$

$$P(r \parallel \ldots,n_j,0,n_{j+1},\ldots)\equiv P(r-1 \parallel \ldots,n_j+n_{j+1},\ldots) \qquad (1.14)$$

A subset $\boldsymbol{H}$ of $\boldsymbol{Gr(A,B)}$ is said to be of (polynomial) codimension d if it is the zero locus of d independent polynomial functions on $\boldsymbol{Gr(A,B)}$:

$$\boldsymbol{H}=\{\boldsymbol{W}\in\boldsymbol{Gr(A,B)}\quad ;\quad P_1(\boldsymbol{W})=P_2(\boldsymbol{W})=\cdots P_d(\boldsymbol{W})=0\} \quad (1.15)$$

If, for each monomial $c=\boldsymbol{a}^{\mu_1}\,\boldsymbol{b}^{\nu_1}\cdots\boldsymbol{a}^{\mu_s}\,\boldsymbol{b}^{\nu_s}$ in $\boldsymbol{Al(a,b)}$ and each element $\boldsymbol{W(A,B)}=\boldsymbol{A}^{m_1}\,\boldsymbol{B}^{n_1}\cdots\boldsymbol{A}^{m_s}\,\boldsymbol{B}^{n_s}$ in $\boldsymbol{Gr(A,B)}$, we define $P_c(\boldsymbol{W})$ as the (rational) coefficient of c in the natural expansion of $\boldsymbol{W(e^a,e^b)}$ then the aggregate of all these P_c clearly constitutes a *complete* but *non-free* set of polynomial functions on $\boldsymbol{Gr(A,B)}$. But if we let $\boldsymbol{\gamma}$ run through some basis (eg Lyndon basis) of $\boldsymbol{Lie(a,b)}$ and define $P^*_{\boldsymbol{\gamma}}(\boldsymbol{W})$ as the coefficient of $\log(\boldsymbol{W(e^a,e^b)})$ along the basis vector $\boldsymbol{\gamma}$, then these $P^*_{\boldsymbol{\gamma}}$ yield a set of polynomial functions on $\boldsymbol{Gr(A,B)}$ that is still *complete* but *free* as well.

1.5. Campbell-Hausdorff type formulae

We shall make constant use of the notations:

$$\overline{\boldsymbol{b}}\,\boldsymbol{a}:=[\boldsymbol{b}\,,\,\boldsymbol{a}]:=\boldsymbol{b}\,\boldsymbol{a}-\boldsymbol{a}\,\boldsymbol{b} \qquad (1.16)$$

$$\overline{\overline{\boldsymbol{B}}}\,\boldsymbol{A}:=\{\boldsymbol{B}\,,\,\boldsymbol{A}\}:=\boldsymbol{B}^{-1}\,\boldsymbol{A}^{-1}\,\boldsymbol{B}\,\boldsymbol{A} \qquad (1.17)$$

We shall also require the Campbell-Hausdorff formula :

$$\log(e^{\boldsymbol{b}}\,e^{\boldsymbol{a}})=\boldsymbol{b}+\boldsymbol{a}+\frac{1}{2}\,[\boldsymbol{b},\boldsymbol{a}]+\frac{1}{12}\,[\boldsymbol{b},[\boldsymbol{b},\boldsymbol{a}]]+\frac{1}{12}\,[\boldsymbol{a},[\boldsymbol{a},\boldsymbol{b}]]+\ldots \qquad (1.18)$$

along with these two variants:

$$\log\{e^{\boldsymbol{b}},e^{\boldsymbol{a}}\}:=\log(e^{-\boldsymbol{b}}\,e^{-\boldsymbol{a}}\,e^{\boldsymbol{b}}\,e^{\boldsymbol{a}})=[\boldsymbol{b},\boldsymbol{a}]-\frac{1}{2}\,[\boldsymbol{b}+\boldsymbol{a},[\boldsymbol{b},\boldsymbol{a}]]+\ldots \qquad (1.19)$$

$$\log\{e^{\boldsymbol{a_r}},\{e^{\boldsymbol{a_{r-1}}}\cdots\{e^{\boldsymbol{a_2}},e^{\boldsymbol{a_1}}\}\cdots\}\} \tag{1.20}$$
$$=\overline{\boldsymbol{a}}_{\boldsymbol{r}}\,\overline{\boldsymbol{a}}_{\boldsymbol{r}-1}\cdots\overline{\boldsymbol{a}}_{\boldsymbol{2}}\,\boldsymbol{a_1}-\frac{1}{2}\sum_{j=1}^{r}\overline{\boldsymbol{a}}_{\boldsymbol{r}}\,\overline{\boldsymbol{a}}_{\boldsymbol{r-1}}\cdots\overline{\boldsymbol{a}}_{\boldsymbol{j}}^{\boldsymbol{2}}\cdots\overline{\boldsymbol{a}}_{\boldsymbol{2}}\,\boldsymbol{a_1}+\ldots$$

where the first term in the sum $\sum$ (corresponding to $j=1$) has to be interpreted as $\overline{\boldsymbol{a}}_{\boldsymbol{r}}\cdots\overline{\boldsymbol{a}}_{\boldsymbol{3}}\,\overline{\boldsymbol{a}}_{\boldsymbol{1}}\,\overline{\boldsymbol{a}}_{\boldsymbol{2}}\,\boldsymbol{a_1}$ or, equivalently, as $-\overline{\boldsymbol{a}}_{\boldsymbol{r}}\cdots\overline{\boldsymbol{a}}_{\boldsymbol{3}}\,\overline{\boldsymbol{a}}_{\boldsymbol{1}}^{\boldsymbol{2}}\,\boldsymbol{a_2}$.

1.6. General notations

Boldface symbols shall be reserved almost exclusively for *free structures* (groups, associative algebras, Lie algebras) and their elements. *Plain symbols* shall be used for *bound structures*, such as the group $\tilde{\mathcal{G}}$ (resp. $\mathcal{G}$) of all formal (resp. analytic) local diffeomorphisms of $\mathbb{C}$. As a rule, the twiddle $\sim$ shall signal *formalness* (as opposed to analyticity) in series, diffeos, differential equations etc. But in practice, to avoid unnecessary clumsiness, the twiddle will mark only sets (such as $\tilde{\mathcal{G}}$) and seldom their elements (f, g etc.), except when strictly indispensible, for instance whenever we must carefully distinguish between a divergent formal power series $\tilde{\varphi}$ and its (not necessarily unique) sums φ obtained by various resummation procedures.

As already pointed out, diffeomorphisms shall be denoted by *small letters* f, g, etc, while the corresponding substitution operators F, G, etc, shall always be *capitalised.* For expediency, both will usually be referred to as "diffeos".

1.7. Outline of the paper

The paper's first half is devoted to the construction of *formal twins.* After some group-theoretic reminders (§2), we go through a series of easy-to-construct examples (§3) illustrative of the main types of (identity-tangent) twins. Then, after a Lie-theoretic excursion with some novel material (§4), we attempt a *reasonable* systematisation (§§5, 6), as opposed to an *exhaustive* one which, as we shall see, is completely out of reach. For each of the leading twin types, we carefully indicate the simplest defining relations W, i.e. those with lowest (polynomial) codimension or with other low "complexity indices". We then sketch a natural generalisation: bound groups with more than two generators ("siblings".) [1]

[1] A section devoted to yet another generalisation – bound groups consisting of higher-dimensional diffeos – was scrapped for lack of space.

In the next section (§7) we investigate the *analytic nature* of the twins constructed thus far. Since the solutions (F,G) of any equation $W(F,G)=1$ are defined upto a simultaneous conjugacy:

$$(F,G) \mapsto (H^{-1}\,F\,H, H^{-1}\,G\,H) \tag{1.21}$$

the pertinent question is not, of course, whether the general solution converges, but rather: how simple can the pair (F,G) be made in a suitable chart? Or again: how simple does F (resp. G) become after its twin G (resp. F) has been normalised? For non-identity tangent twins, it is rather easy (by applying the theory of the so-called "sandwich equation") to show the generic *analyticity* of well-chosen representatives (F,G). For identity-tangent twins, the position is exactly the reverse: we establish their generic divergence *and* resurgence. It would even seem that identity-tangent twins are *always* divergent and *always* resurgent, but a regular proof (especially of the former) appears to be a long way off.

The last, very sketchy section (§8) discusses twins in the larger setting of transseries and analysable functions (as opposed to power series and analytic or resurgent germs) and broaches a number of side-issues, such as the ordering of free groups. [2]

2. Some group theory. Alternators and word factorisation. Periodic automorphisms

In this section, $\boldsymbol{Gr}$ and $\boldsymbol{Lie}$ will stand for $\boldsymbol{Gr(A,B)}$ and $\boldsymbol{Lie(a,b)}$, i.e. will denote the two-generator free group resp. Lie algebra.

2.1. Filtrations and gradations on $\boldsymbol{Gr}$ and $\boldsymbol{Lie}$

The algebra $\boldsymbol{Lie}$ has an elementary gradation into homogeneous components

$$\boldsymbol{Lie} = \bigoplus_{d\geqslant 1} \boldsymbol{Lie}^{[d]} = \bigoplus_{d_i\geqslant 1} \boldsymbol{Lie}^{[d_1,d_2]} \tag{2.1}$$

with $\boldsymbol{Lie}^{[d]}$ (resp. $\boldsymbol{Lie}^{[d_1,d_2]}$) spanned by multicommutators of global degree d (resp. of degree d_1 and d_2 in $\boldsymbol{a}$ and $\boldsymbol{b}$). The corresponding dimensions are

(2) Another such side-issue, originally meant for inclusion in this paper, has been removed for lack of space and redirected to [EV2]. It asks: what is the "proper" and most "comprehensive" notion of *analyticity* on *free algebras* (Lie or associative)? We rephrase the question so as to give it a clear-cut meaning. Then we solve it and come up with a startling, quite counter-intuitive answer.

given by classical formulas:

$$lie(d) := dim\, \boldsymbol{Lie}^{[d]} := \frac{1}{d} \sum_{\delta|d} \mu(\delta)\, 2^{d/\delta} \tag{2.2}$$

$$lie(d_1, d_2) := dim\, \boldsymbol{Lie}^{[d_1,d_2]} := \frac{1}{d_1 + d_2} \sum_{\delta|d_i} \frac{((d_1 + d_2)/\delta)\,!}{(d_1/\delta)\,!\,(d_2/\delta)\,!} \tag{2.3}$$

$\mu(\bullet)$ being the classical Möbius function.[3] Going with that graduation we have the filtration:

$$\boldsymbol{Lie} = \bigcup_d \boldsymbol{Lie}^{\{d\}} = \bigcup_{d_i} \boldsymbol{Lie}^{\{d_1,d_2\}} \tag{2.4}$$

$$\boldsymbol{Lie}^{\{d\}} = \bigoplus_{d' \geqslant d} \boldsymbol{Lie}^{[d']} \quad ; \quad \boldsymbol{Lie}^{\{d_1,d_2\}} = \bigoplus_{d'_i \geqslant d_i} \boldsymbol{Lie}^{[d'_1,d'_2]} \tag{2.5}$$

with the obvious commutation inclusions.

As far as the algebra $\boldsymbol{Lie}$ is concerned, the gradation is the primary notion and the filtration rather derivative. But with the group $\boldsymbol{Gr}$ the position is exactly the reverse. Moreover, we seem to have the choice between two alternative, *a priori* non-equivalent definitions for the filtration on $\boldsymbol{Gr}$. According to the *first definition*, $\boldsymbol{Gr}^{\{d\}}$ (resp. $\boldsymbol{Gr}^{\{d_1,d_2\}}$) is simply the set of all $\boldsymbol{W(A,B)}$ in $\boldsymbol{Gr}$ whose natural Lie image

$$\boldsymbol{w(a,b)} := \log \boldsymbol{W(e^a, e^b)} \tag{2.6}$$

lies in $\boldsymbol{Lie}^{\{d\}}$ (resp. $\boldsymbol{Lie}^{\{d_1,d_2\}}$). This defines a filtration, i.e.

$$\begin{aligned} \{\boldsymbol{Gr}^{\{d'\}}, \boldsymbol{Gr}^{\{d''\}}\} &\subset \boldsymbol{Gr}^{\{d'+d''\}} \\ \{\boldsymbol{Gr}^{\{d'_1,d'_2\}}, \boldsymbol{Gr}^{\{d''_1,d''_2\}}\} &\subset \boldsymbol{Gr}^{\{d'_1+d''_1,d'_2+d''_2\}} \end{aligned} \tag{2.7}$$

because the Campbell-Hausdorff formula (1.19) for brackets involves, on its right-hand side, only terms of degree $\geqslant 1$ in $\boldsymbol{a}$ and $\boldsymbol{b}$.

Then we have a second definition (provisionnally distinguished by double braces) according to which $\boldsymbol{Gr}^{\{\{d\}\}}$ (resp. $\boldsymbol{Gr}^{\{\{d_1,d_2\}\}}$) is the subgroup generated by all multicommutators of alternance $\geqslant d$ (resp. $\geqslant (d_1, d_2)$). We speak here of "alternance" rather than "degree" to preclude any confusion with the "degree" of a word $\boldsymbol{W(A,B)}$. A multicommutator of alternance d is of course one with exactly d arguments (which we may take to be $\boldsymbol{A}^{\pm 1}$

[(3)] Recall that $\mu(n) = (-1)^s$ (*resp* 0) if n is *quadrat-frei* and a product of s distinct primes (resp otherwise).

or $\boldsymbol{B}^{\pm 1}$) and a multicommutator of alternance (d_1, d_2) is one with exactly d_1 arguments $\boldsymbol{A}^{m_i}$ (or $\boldsymbol{A}^{\pm 1}$) and d_2 arguments $\boldsymbol{B}^{n_i}$ (or $\boldsymbol{B}^{\pm 1}$). The reason why we may assume all exponents m_i and n_i to be ± 1 is that by repeated use of the Witt-Hall identities:

$$\{\boldsymbol{A},\boldsymbol{B}\}\,\{\boldsymbol{B},\boldsymbol{A}\} = 1 \tag{2.8}$$

$$\{\boldsymbol{A},\boldsymbol{B}\,\boldsymbol{C}\} = \{\boldsymbol{A},\boldsymbol{C}\}\,\{\boldsymbol{A},\boldsymbol{B}\}\,\{\{\boldsymbol{A},\boldsymbol{B}\},\boldsymbol{C}\} \tag{2.9}$$

$$\{\boldsymbol{A}\,\boldsymbol{B},\boldsymbol{C}\} = \{\boldsymbol{A},\boldsymbol{C}\}\,\{\{\boldsymbol{A},\boldsymbol{C}\},\boldsymbol{B}\}\{\boldsymbol{B},\boldsymbol{C}\} \tag{2.10}$$

we may break up any multicommutator into a product of multicommutators with equal (or greater) alternance, but with arguments of the form $\boldsymbol{A}^{\pm 1}$ or $\boldsymbol{B}^{\pm 1}$.

Clearly, the $\boldsymbol{Gr}^{\{\{d\}\}}$ and $\boldsymbol{Gr}^{\{\{d_1,d_2\}\}}$ define a new filtration on $\boldsymbol{Gr}$, i.e. the inclusions (2.7) extend to the double-braced $\boldsymbol{Gr}^{\{\{\cdots\}\}}$. But in fact:

Proposition 2.1. — *The two natural filtrations on* $\boldsymbol{Gr}$ *coincide:*

$$\boldsymbol{Gr}^{\{d\}} = \boldsymbol{Gr}^{\{\{d\}\}} \tag{2.11}$$

$$\boldsymbol{Gr}^{\{d_1,d_2\}} = \boldsymbol{Gr}^{\{\{d_1,d_2\}\}} \tag{2.12}$$

Corollary 2.2. — *The quotients:*

$$\boldsymbol{Gr}^{[d]} := \boldsymbol{Gr}^{\{d\}}/\boldsymbol{Gr}^{\{d+1\}} \tag{2.13}$$

$$\boldsymbol{Gr}^{[d_1,d_2]} := \boldsymbol{Gr}^{\{\{d_1,d_2\}\}}/(\boldsymbol{Gr}^{\{\{d_1+1,d_2\}\}}\,.\,\boldsymbol{Gr}^{\{\{d_1,d_2+1\}\}}) \tag{2.14}$$

make sense (since the groups right of the slash are distinguished subgroups of those left of the slash) and define abelian groups $\boldsymbol{Gr}^{[d]}$ *and* $\boldsymbol{Gr}^{[d_1,d_2]}$ *that are isomorphic to* $\boldsymbol{Lie}^{[d]}$ *and* $\boldsymbol{Lie}^{[d_1,d_2]}$ *(or rather to their additive submodules spanned by multicommutators).*

Proof. — The inclusions $\boldsymbol{Gr}^{\{\cdots\}} \supset \boldsymbol{Gr}^{\{\{\cdots\}\}}$ are trivial, since formula (1.20) carries on its right-hand side only multicommutators of degree $\geqslant 1$ in *each* of the variables $\mathbf{a_i}$. The reverse inclusions, however, are by no means obvious. The one involving the "global" filtration, namely $\boldsymbol{Gr}^{\{d\}} \subset \boldsymbol{Gr}^{\{\{d\}\}}$, is a classical result by P. Hall (see [L.S]),which involves much more than dabbling with the Witt-Hall identities (2.8), (2.9), (2.10). Fortunately, it is but a short step from this inclusion to the more precise inclusion $\boldsymbol{Gr}^{\{d_1,d_2\}} \subset \boldsymbol{Gr}^{\{\{d_1,d_2\}\}}$. In fact, all it takes is proving the following:

Lemma 2.3 (Word factorisation). — *For each* $\boldsymbol{W(A,B)}$ *in* $\boldsymbol{Gr}^{\{d\}}$ *(*$d \geqslant 1$*) and each pair* $(d_1(\bullet), d_2(\bullet))$ *of 'complementary' permutations of* $\{1,\ldots,d-1\}$ *(i.e. permutations such that* $d_1(i)+d_2(i)\equiv d$*), there exists a (non-unique) factorisation:*

$$\boldsymbol{W(A,B)} \quad = \quad \boldsymbol{W_1(A,B)\,W_2(A,B)\cdots W_{r-1}(A,B)} \qquad (2.15)$$
$$\textit{with} \quad \boldsymbol{W_i} \in \boldsymbol{Gr}^{\{\{d_1(i),d_2(i)\}\}}$$

The argument being much the same for all permutations, let us check this for $d_1(i)\equiv i$. Due to the inclusion $\boldsymbol{Gr}^{\{d\}} \subset \boldsymbol{Gr}^{\{\{d\}\}}$ and by repeated use of the Witt-Hall identities, we can produce a factorisation $\boldsymbol{W} = \boldsymbol{V_1\,V_2\cdots V_s}$ with elementary factors $\boldsymbol{V_j}$ of alternance (d'_j, d''_j) in $(\boldsymbol{A},\boldsymbol{B})$ with $d'_j + d''_j \geqslant d$. Set $\delta = \inf d'_j$ and write $\boldsymbol{W} = \boldsymbol{P\,Q\,V_{j_1}\,R}$, with $\boldsymbol{P} = \boldsymbol{V_1\,V_2\cdots V_{j_1}}$ regrouping all factors $\boldsymbol{V_j}$ (if any) with alternance δ in $\boldsymbol{A}$ *and* already in front position; with $\boldsymbol{V_{j_1}}$ denoting the first factor (if any) with $\boldsymbol{A}$-alternance δ but *not* in front position; and with $\boldsymbol{Q}$ regrouping all remaining factors. This leads to a new factorisation:

$$\boldsymbol{W} = \boldsymbol{P\,V_{j_1}\,\{V_{j_1}, Q^{-1}\}\,Q\,R} \qquad (2.16)$$

with a factor $\{\boldsymbol{V_{j_1}}, \boldsymbol{Q}^{-1}\}\,\boldsymbol{Q}$ which in turn may be broken up into elementary factors $\boldsymbol{V'_j}$ of $\boldsymbol{A}$-alternance $> 2\,\delta$. By repeating the process for the remaining factors $\boldsymbol{V_{j_1}}, \boldsymbol{V_{j_2}}, etc.(iftherebesuch) of \boldsymbol{A} - alternance \delta$ in $\boldsymbol{Q}$, we can move all these factors into front position; and the all factors with $\boldsymbol{A}$-alternance $\delta+1$, $\delta+2$, etc.

This proves the lemma, which we may now apply to words $\boldsymbol{W(A,B)}$ in $\boldsymbol{Gr}^{\{d_1,d_2\}}$, in which case the decomposition (2.15) will reduce to only one non-trivial factor, that single factor being in $\boldsymbol{Gr}^{\{\{d_1,d_2\}\}}$. So $\boldsymbol{Gr}^{\{d_1,d_2\}} \subset \boldsymbol{Gr}^{\{\{d_1,d_2\}\}}$, which completes the proof of Proposition 2.1. The corollary about the structure of $\boldsymbol{Gr}^{[d]}$ and $\boldsymbol{Gr}^{[d_1,d_2]}$ is then an easy consequence. □

2.2. Finite criteria for alternance (d_1, d_2)

Each subgroup $\boldsymbol{Gr}^{\{d\}}$ has *finite codimension* in $\boldsymbol{Gr}$ (see §1.4). In fact, its codimension is exactly $lie(2)+\cdots+lie(d-1)$. But, save for $(d_1,d_2)=(1,1)$, each subgroup $\boldsymbol{Gr}^{\{d_1,d_2\}}$ has *infinite codimension.* Indeed, if we go by the (first) definition of $\boldsymbol{Gr}^{\{d_1,d_2\}}$, checking that a given word $\boldsymbol{W(A,B)}$ is in $\boldsymbol{Gr}^{\{d_1,d_2\}}$ implies checking infinitely many "polynomial" identities $P(\boldsymbol{W}) = 0$ of type (1.12).

Fortunately, there exist criteria involving a finite number of steps. First, observe that any $\boldsymbol{W(A,B)}$ in $\boldsymbol{Gr}^{\{1,1\}}$ may be written in a unique way as finite products of factors $\boldsymbol{A}_n := \boldsymbol{B}^n\,\boldsymbol{A}\,\boldsymbol{B}^{-n}$ or $\boldsymbol{B}_m := \boldsymbol{A}^m\,\boldsymbol{B}\,\boldsymbol{A}^{-m}$:

$$\boldsymbol{W(A,B)} = \boldsymbol{W}_1(\boldsymbol{A}_{n_1},\boldsymbol{A}_{n_2},\ldots,\boldsymbol{A}_{n_{s_1}}) \qquad (n_i \in I \subset \mathbf{Z}) \quad (2.17)$$

$$\boldsymbol{W(A,B)} = \boldsymbol{W}_2(\boldsymbol{B}_{m_1},\boldsymbol{B}_{m_2},\ldots,\boldsymbol{B}_{m_{s_1}}) \quad (m_i \in J \subset \mathbf{Z}) \quad (2.18)$$

Lemma 2.4. — *$\boldsymbol{W(A,B)}$ is in $\boldsymbol{Gr}^{\{d_1,d_2\}}$ $\big((d_1,d_2) > (1,1)\big)$ if and only if :*

$$\log \boldsymbol{W}_1(e^{\boldsymbol{a}_{n_1}},\ldots,e^{\boldsymbol{a}_{n_{s_1}}}) = \mathcal{O}(\boldsymbol{a}^{d_1}) \quad (2.19)$$

$$\log \boldsymbol{W}_2(e^{\boldsymbol{b}_{m_1}},\ldots,e^{\boldsymbol{b}_{m_{s_2}}}) = \mathcal{O}(\boldsymbol{b}^{d_2}) \quad (2.20)$$

The symbol $\mathcal{O}$ means that the right-hand sides are of total degree $\geqslant d_1$ (or d_2) in the $\boldsymbol{a}_{n_i}$ (or $\boldsymbol{b}_{m_j}$) regarded as free independent variables. The lemma directly follows from the fact that within the algebra $\overline{\boldsymbol{Lie}}\,\{\boldsymbol{a},\boldsymbol{b}\}$ (i.e. in the 'closure' derived from $\boldsymbol{Lie}\,\{\boldsymbol{a},\boldsymbol{b}\}$ by allowing infinite sums), the subalgebras $\boldsymbol{Lie}\,\{\boldsymbol{a}_{n_1},\ldots,\boldsymbol{a}_{n_{s_1}}\}$ or $\boldsymbol{Lie}\,\{\boldsymbol{b}_{m_1},\ldots,\boldsymbol{b}_{m_{s_2}}\}$ *finitely* generated by a *finite* number of distinct elements $\boldsymbol{a}_n$ or $\boldsymbol{b}_m$:

$$\boldsymbol{a}_n := e^{n\,\boldsymbol{b}}\,\boldsymbol{a}\,e^{-n\,\boldsymbol{b}} = (\exp(n\,\overline{\boldsymbol{b}}))\,\boldsymbol{a} \quad (n \in \mathbf{Z}) \quad (2.21)$$

$$\boldsymbol{b}_m := e^{m\,\boldsymbol{a}}\,\boldsymbol{b}\,e^{-m\,\boldsymbol{a}} = (\exp(m\,\overline{\boldsymbol{a}}))\,\boldsymbol{b} \quad (m \in \mathbf{Z}) \quad (2.22)$$

are themselves *free*. So, checking that $\boldsymbol{W(A,B)}$ has alternance (d_1,d_2) involves only a finite number of steps which, however, steeply increases with s_1 and s_2.

2.3. Low-complexity alternators. Measures of word complexity

Let us apply the above criterion to words of the form:

$$\boldsymbol{W(A,B)} = \prod_{j=1}^{r} \boldsymbol{B}^{n_j}\,\boldsymbol{A}\,\boldsymbol{B}^{n'_j}\,\boldsymbol{A}^{-1} = \boldsymbol{B}^{n_1}\,\boldsymbol{B}_1^{n'_1}\cdots\boldsymbol{B}^{n_r}\,\boldsymbol{B}_1^{n'_r} \quad (n_j,n'_j \in \mathbb{Z}^*) \quad (2.23)$$

with indices j regarded as elements of $\mathbb{Z}_r := \mathbb{Z}\,/\,r\,\mathbb{Z}$. For $r=3$, we find that $\boldsymbol{W(A,B)}$ has alternance $\geqslant (1,3)$ iff:

$$\sum n_j = \sum n'_j = 0 \quad \text{and} \quad n'_j/n_{j+2} = \lambda = \text{const} \in \mathbb{Q} \quad (\forall j) \quad (2.24)$$

and alternance $(2,3)$ iff $\lambda = 1$.

Similarly, for $r=4$, the word $\boldsymbol{W(A,B)}$ has alternance $\geqslant (1,4)$ iff:

$$\begin{aligned} n_j+n_{j+2} &= n'_j+n'_{j+2}=0 \qquad (\forall j) \qquad (2.25)\\ \text{and} \quad n_j+n_{j+1}/n'_j &= \tau = \text{const} \in \mathbb{Q} \qquad (\forall j) \end{aligned}$$

and alternance $(2,4)$ iff $\tau=-1$. In the case $r=3, \lambda=1$, we may take: $(n_1,n_2,n_3)=(n'_2,n'_3,n'_1)=(-1,-1,2)$ in which case the $(2,3)$-alternator $\boldsymbol{W(A,B)}$ has a fairly simple commutator factorisation:

$$\boldsymbol{W(A,B)} = \{\{\boldsymbol{A},\boldsymbol{B}\},\{\boldsymbol{B}\,\{\boldsymbol{B}^{-1},\boldsymbol{A}\}\} \qquad (2.26)$$

but otherwise the *simplest* words with alternance (d_1,d_2) tend to be much simpler than their *simplest* expressions in terms of multicommutators (of the same alternance).

There exist of course many ways of measuring the "complexity" of a word $\boldsymbol{W(A,B)} = \prod(\boldsymbol{A}^{m_i}\,\boldsymbol{B}^{n_i})$, but we shall use only the following four complexity indices:

(i) the *length*, global or partial:

$$l(W) = \sum \mid m_i \mid + \sum \mid n_i \mid$$

$$l_{\mathbf{A}}(W) = \sum \mid m_i \mid \quad ; \quad l_{\mathbf{B}}(W) = \sum \mid n_i \mid$$

(ii) the *number of alternating monomials* $\boldsymbol{A}^{m_i}$ or $\boldsymbol{B}^{m_i}$

(iii) the *variance* $var(\boldsymbol{W})$ (resp. $var_{\mathbf{A}}(\boldsymbol{W})$ or $var_{\mathbf{B}}(\boldsymbol{W})$), defined as the number of distinct pairs $(\check{m}_i,\check{n}_i)$ (resp. as the number of distinct values assumed separately by $\check{m}_i$ or $\check{n}_i$) whereby $\check{m}_j := m_1+\cdots m_j$ and $\check{n}_j := n_1+\cdots n_j$

(iii) the *alternance* $alt(\boldsymbol{W})$ (resp. $(alt_{\mathbf{A}}(\boldsymbol{W}), alt_{\mathbf{B}}(\boldsymbol{W})))$ defined as the smallest d (resp. (d_1,d_2)) such that $\boldsymbol{W}$ be in $\boldsymbol{Gr}^{\{d\}}$ (resp. $\boldsymbol{Gr}^{\{d_1,d_2\}}$).

In the usual graphical representation of words, the first three complexity indices are immediate to detect, but not so the alternance, which doesn't 'meet the eye', at least when $d_1+d_2 \geqslant 3$.

2.4. Periodic automorphisms of *Gr*

Let $\boldsymbol{Aut}(\boldsymbol{Gr})$ denote the group of all automorphisms of $\boldsymbol{Gr}$. It is known (Nielsen's theorem) to be generated by three elementary automorphisms:

$$(\boldsymbol{A},\boldsymbol{B}) \mapsto (\boldsymbol{B},\boldsymbol{A})\ ;\ (\boldsymbol{A},\boldsymbol{B}) \mapsto (\boldsymbol{A}^{-1},\boldsymbol{B})\ ;\ (\boldsymbol{A},\boldsymbol{B}) \mapsto (\boldsymbol{A}\,\boldsymbol{B},\boldsymbol{A}) \qquad (2.27)$$

For the sequel, we need to know, up to conjugacy, all the *finite* subgroups of $\boldsymbol{Aut}(\boldsymbol{Gr})$ and in particular all its *periodic* (unipotent) elements.

Lemma 2.5 (Periodic automorphisms of $\boldsymbol{Gr}$). — *They necessarily have order* 1,2,3,4, *the number of distinct conjugacy classes being respectively* 1,4,1,1. *Each involution (order 2) is conjugate either to* $(\boldsymbol{A},\boldsymbol{B}) \mapsto (\boldsymbol{B},\boldsymbol{A})$ *or* $(\boldsymbol{A},\boldsymbol{B}) \mapsto (\boldsymbol{A}^{-1},\boldsymbol{B})$ *or* $(\boldsymbol{A},\boldsymbol{B}) \mapsto (\boldsymbol{A}^{-1},\boldsymbol{B}^{-1})$ *or* $(\boldsymbol{A},\boldsymbol{B}) \mapsto (\boldsymbol{B}^{-1},\boldsymbol{A}^{-1})$. *Each automorphism of order* 3 *is conjugate to* $(\boldsymbol{A},\boldsymbol{B}) \mapsto (\boldsymbol{B}^{-1},\, \boldsymbol{A}\,\boldsymbol{B}^{-1})$. *Each automorphism of order* 4 *is conjugate to* $(\boldsymbol{A},\boldsymbol{B}) \mapsto (\boldsymbol{B}^{-1},\boldsymbol{A})$.

For a proof, see [L.S.], prop.4.6, p25, and the references thereafter.

Now that we have the list of all periodic automorphisms, it is an easy matter to construct all non-cyclic finite subgroups of $\boldsymbol{Aut}(\boldsymbol{Gr})$. There are four of them (upto conjugacy), namely the two abelian groups of order 4:

$$\boldsymbol{Aut}_1 \equiv \{\boldsymbol{I},\boldsymbol{I}_1,\boldsymbol{I}_2,\boldsymbol{I}_3\} \quad \text{(two generators)} \tag{2.28}$$

$$\boldsymbol{Aut}_2 \equiv \{\boldsymbol{I},\boldsymbol{S}_1,\boldsymbol{S}_2,\boldsymbol{I}_3\} \quad \text{(two generators)} \tag{2.29}$$

and the two non-abelian groups of order 6 and 8:

$$\boldsymbol{Aut}_3 \equiv \{\boldsymbol{I},\boldsymbol{S}_1,\boldsymbol{J},\boldsymbol{J}^2,\boldsymbol{S}_1\,\boldsymbol{J},\boldsymbol{J}\,\boldsymbol{S}_1\} \quad \text{(two generators)} \tag{2.30}$$

$$\boldsymbol{Aut}_4 \equiv \{\boldsymbol{I},\boldsymbol{I}_1,\boldsymbol{I}_2,\boldsymbol{I}_3,\boldsymbol{S}_1,\boldsymbol{S}_2,\boldsymbol{K}_1\,\boldsymbol{K}_2\} \quad \text{(two gen., e.g. } \boldsymbol{I}_1,\ \boldsymbol{S}_1) \tag{2.31}$$

whose elements $\boldsymbol{P}$ are listed below, in terms of their action $\boldsymbol{P} : \boldsymbol{W} \mapsto \boldsymbol{W}^{\boldsymbol{P}}$on words:

$$\boldsymbol{W}^{\boldsymbol{I}(\boldsymbol{A},\boldsymbol{B})=\boldsymbol{W}(\boldsymbol{A},\boldsymbol{B})}, \quad \boldsymbol{W}^{\boldsymbol{I}_1}(\boldsymbol{A},\boldsymbol{B}) = \boldsymbol{W}(\boldsymbol{A}^{-1},\boldsymbol{B}) \tag{2.32}$$

$$\boldsymbol{W}^{\boldsymbol{I}_2}(\boldsymbol{A},\boldsymbol{B}) = \boldsymbol{W}(\boldsymbol{A},\boldsymbol{B}^{-1}), \quad \boldsymbol{W}^{\boldsymbol{I}_3}(\boldsymbol{A},\boldsymbol{B}) = \boldsymbol{W}(\boldsymbol{A}^{-1},\boldsymbol{B}^{-1}) \tag{2.33}$$

$$\boldsymbol{W}^{\boldsymbol{S}_1}(\boldsymbol{A},\boldsymbol{B}) = \boldsymbol{W}(\boldsymbol{B},\boldsymbol{A}), \quad \boldsymbol{W}^{\boldsymbol{S}_2}(\boldsymbol{A},\boldsymbol{B}) = \boldsymbol{W}(\boldsymbol{B}^{-1},\boldsymbol{A}^{-1}) \tag{2.34}$$

$$\boldsymbol{W}^{\boldsymbol{K}_1}(\boldsymbol{A},\boldsymbol{B}) = \boldsymbol{W}(\boldsymbol{B}^{-1},\boldsymbol{A}), \quad \boldsymbol{W}^{\boldsymbol{K}_2}(\boldsymbol{A},\boldsymbol{B}) = \boldsymbol{W}(\boldsymbol{B},\boldsymbol{A}^{-1}) \tag{2.35}$$

$$\boldsymbol{W}^{\boldsymbol{J}}(\boldsymbol{A},\boldsymbol{B}) = \boldsymbol{W}(\boldsymbol{B}^{-1},\boldsymbol{A}\,\boldsymbol{B}^{-1}), \quad \boldsymbol{W}^{\boldsymbol{J}^2}(\boldsymbol{A},\boldsymbol{B}) = \boldsymbol{W}(\boldsymbol{B}\,\boldsymbol{A}^{-1},\boldsymbol{A}^{-1}) \tag{2.36}$$

$$\boldsymbol{W}^{\boldsymbol{S}_1\,\boldsymbol{J}}(\boldsymbol{A},\boldsymbol{B}) = \boldsymbol{W}(\boldsymbol{A}\,\boldsymbol{B},\boldsymbol{B}^{-1}), \quad \boldsymbol{W}^{\boldsymbol{J}\,\boldsymbol{S}_1}(\boldsymbol{A},\boldsymbol{B}) = \boldsymbol{W}(\boldsymbol{A}^{-1}\,\boldsymbol{B},\boldsymbol{A}^{-1}) \tag{2.37}$$

Note that the composition law follows from the rule $(\boldsymbol{W}^{\boldsymbol{P}_1})^{\boldsymbol{P}_2} = \boldsymbol{W}^{(\boldsymbol{P}_1\,\boldsymbol{P}_2)}$ (beware of the proper order). Thus:

$$\boldsymbol{I}_1^2 = \boldsymbol{I}_2^2 = \boldsymbol{I}_3^2 = \boldsymbol{S}_1^2 = \boldsymbol{S}_2^2 = \boldsymbol{J}^3 = \boldsymbol{K}_1^4 = \boldsymbol{K}_2^4 = \boldsymbol{I} \tag{2.38}$$

$$\boldsymbol{K}_1^2 = \boldsymbol{K}_2^2 = \boldsymbol{I}_3, \quad \boldsymbol{I}_{\sigma(1)}\,\boldsymbol{I}_{\sigma(2)}\,\boldsymbol{I}_{\sigma(3)} = \boldsymbol{I}, \qquad \sigma \in \mathit{Perm}\{1,2,3\} \tag{2.39}$$

$$\boldsymbol{K}_1 = \boldsymbol{I}_1\,\boldsymbol{S}_1 = \boldsymbol{S}_2\,\boldsymbol{I}_1 = \boldsymbol{S}_1\,\boldsymbol{I}_2 = \boldsymbol{S}_2\,\boldsymbol{I}_2 \tag{2.40}$$

$$\boldsymbol{K}_2 = \boldsymbol{S}_1\,\boldsymbol{I}_1 = \boldsymbol{I}_1\,\boldsymbol{S}_2 = \boldsymbol{I}_2\,\boldsymbol{S}_1 = \boldsymbol{S}_1\,\boldsymbol{I}_1 \tag{2.41}$$

3. Identity-tangent twins: twenty typical examples

3.1. Collapsors and divisors

Our aim in this section is to produce examples of identity-tangent twins that cover all the main possible situations and yet remain easy to construct. But this ease of construction comes at a price: it usually entails severe complexity. It also involves highly non-generic relations $W(F,G)=1$, with the corresponding words $\boldsymbol{W(A,B)}$ never ranging over subsets of finite codimension in $\boldsymbol{Gr\{A,B\}}$. For truly generic examples and finite codimensions, we shall have to wait for §5.

As in most of this paper, we shall have to work simultaneously in the four structures:

$$\left\{\begin{array}{ccccccc} \boldsymbol{W(A,B)} & \in & \boldsymbol{Gr\{A,B\}} & \longrightarrow & \tilde{\mathcal{G}}_0 & \ni F \\ & & \downarrow & & \downarrow\uparrow & \\ \boldsymbol{w(a,b)} & \in & \overline{\boldsymbol{Lie}}\boldsymbol{\{a,b\}} & \longrightarrow & \tilde{\mathcal{L}}_0 & \ni F_* \end{array}\right. \tag{3.1}$$

$\tilde{\mathcal{G}}_0$ is the group of formal, identity-tangent diffeos $f : x \mapsto x\,\{1+\sum a_n\,x^n\}$, usually denoted by the corresponding substitution operators F. $\tilde{\mathcal{L}}_0$ is the Lie algebra of $\tilde{\mathcal{G}}_0$, with its natural basis elements:

$$l_n := x^{n+1}\,\partial := x^{n+1}\,\frac{d}{dx} \qquad (n=1,2,\ldots) \tag{3.2}$$

The map $F \mapsto F_* := \log F$ from $\tilde{\mathcal{G}}_0$ into $\tilde{\mathcal{L}}_0$ is one-to-one, but the injective map $\boldsymbol{W(A,B)} \mapsto \boldsymbol{w(a,b)} = \log \boldsymbol{W(e^a,e^b)}$ from $\boldsymbol{Gr\{A,B\}}$ into $\overline{\boldsymbol{Lie}}\boldsymbol{\{a,b\}}$ is of course far from surjective. Among the maps from $\overline{\boldsymbol{Lie}}\boldsymbol{\{a,b\}}$ into $\tilde{\mathcal{L}}_0$, deserving of special attention are the *graded morphisms* (i.e. which respect the natural gradation of both algebras) and are necessarily of the form:

$$(\boldsymbol{a},\boldsymbol{b}) \mapsto (l_p,l_q) \qquad (p,q\in\mathbb{N}^*) \tag{3.3}$$

Perhaps the quickest way to produce twins in $\tilde{\mathcal{G}}_0$ is to consider multicommutators $\boldsymbol{W(A,B)}$ (with alternance $(d_1,d_2)>(1,1)$) whose *corner component* $\boldsymbol{w_0(a,b)}$ (defined as the homogeneous component of $\boldsymbol{w(a,b)}$ that lies in $\boldsymbol{Lie}^{[d_1,d_2]}$) has the following three properties:

(i) $\boldsymbol{w_0(a,b)}\neq 0$.

(ii) It is a *collapsor*, i.e. it belongs to the kernel of some (or all) graded morphisms of $\overline{\boldsymbol{Lie}}$ into $\tilde{\mathcal{L}}_0$:

$$w_0(l_p,l_q)=0 \quad \text{for some (resp. all) } p,\ q \text{ in } \mathbb{N}^*\times\mathbb{N}^* \tag{3.4}$$

(iii) Its *divisor* D is $\neq 0$ for all t in $\mathbf{N}^*$. The divisor $D(t) \equiv D(p,q;t)$ is a polynomial in t characterised (for all p, q such that $w_0(l_p, l_q) = 0$) by the identity:

$$\begin{aligned} w_0(l_p + \epsilon_1\, l_{p+t}, l_q + \epsilon_2\, l_{q+t}) &\equiv \\ &(\epsilon_1\,(q-t) - \epsilon_2\,(p-t))\, D(t)\, l_{p\,d_1+q\,d_2+t} + o(\epsilon_1, \epsilon_2) \end{aligned} \quad (3.5)$$

which makes good sense, since the right-hand of (3.5) is *a priori* of the form:

$$(\epsilon_1\, D_1(t) - \epsilon_2\, D_2(t))\, l_{p\,d_1+q\,d_2+t} + o(\epsilon_1, \epsilon_2)$$

but necessarily vanishes when we take:

$$\epsilon_1\, l_{p+t} = \epsilon\,[l_t, l_p] = \epsilon\,(p-t)\, l_{p+t}$$

$$\epsilon_2\, l_{q+t} = \epsilon\,[l_t, l_q] = \epsilon\,(q-t)\, l_{q+t}$$

Divisors owe their name to the fact that, when solving the equation $W(F,G) = 1$ through coefficient identification, one has to divide by $D(t)$ at the t-th inductive step. And the '*collapse*' (3.4) is necessary to get the induction started.

Now, a cheap (if wasteful) way to get words $\boldsymbol{W}$ with those three properties is to take them of the form $\boldsymbol{W} = \{\boldsymbol{U}, \boldsymbol{V}\}$, where $\boldsymbol{U}$ and $\boldsymbol{V}$ are themselves multicommutators with *non-proportional* corner components u_0, v_0 and with distinct (resp. identical) alternance which generally leads to twins with a fixed (resp. free) tangency ratio p/q.

The actual expression of the *divisors* shall of course involve the *structure constants* $T_\bullet$ of the algebra $\mathcal{L}_0$:

$$[l_{n_r} \cdots [l_{n_2}, l_{n_1}] \cdots] \equiv T_{n_1,n_2,\ldots,n_r}\, l_{n_1+\cdots n_r} \quad (l_n \equiv x^{n+1}\,\partial) \quad (3.6)$$

which are explicitely given by:

$$T_{n_1,n_2,\ldots,n_r} = (n_1 - n_2)\,(n_1 + n_2 - n_3) \cdots (n_1 + \cdots + n_{r-1} - n_r) \quad (3.7)$$

We shall also make frequent use of the identity:

$$\sum_i (t - n_i)\, T_{n_1,\ldots,n_i+t,\ldots,n_r} \equiv (t - (n_1 + \cdots + n_r))\, T_{n_1,\ldots,n_r} \quad (3.8)$$

which immediately results from applying the derivation $ad(l_t)$ to both sides of (3.6).

3.2. Normal-conormal form of twins. Connectors

Since knowing a pair of twins (F, G) is the same as knowing their infinitesimal generators (F_*, G_*), we shall often work with the latter, for the greater flexibility they afford. And since these pairs are defined up to a common conjugacy, we shall privilege "normal-conormal forms", which normalise *one* of the twins to F_*^{nor} (resp. G_*^{nor}) while the other assumes a rigidly fixed conormal form G_*^{conor} (resp. F_*^{conor}). Actually, for complete rigidity we have to demand that F_*^{conor} and G_*^{conor} should contain no term of the form l_{p+q} if F_* starts with l_p and G_* with l_q.

In the most typical examples, these namely with exactly one parameter (other than the ratio p/q, which is discrete), we shall find that:

$$F_*^{nor} = f_*^{nor}(x)\,\partial \;=\; (1 - \alpha\,\alpha_*\,x^p)^{-1}\,(\frac{\alpha}{p}\,x^{1+p}\,\partial) \tag{3.9}$$

$$G_*^{conor} = g_*^{conor}(x)\,\partial \;=\; (1 + \beta\,\beta_*\,x^q + \sum_{m,n\geqslant 0\,,\,m+n\geqslant 2} \alpha_{(m,n)}\,\alpha^m\,\beta^n\,x^{m\,p+n\,q})\,(\frac{\beta}{q}\,x^{1+q}\,\partial) \tag{3.10}$$

$$F_*^{conor} = f_*^{conor}(x)\,\partial \;=\; (1 + \alpha\,\alpha_*\,x^p + \sum_{m,n\geqslant 0\,,\,m+n\geqslant 2} \beta_{(m,n)}\,\alpha^m\,\beta^n\,x^{m\,p+n\,q})\,(\frac{\alpha}{p}\,x^{1+p}\,\partial) \tag{3.11}$$

$$G_*^{nor} = g_*^{nor}(x)\,\partial \;=\; (1 - \beta\,\beta_*\,x^q)^{-1}\,(\frac{\beta}{q}\,x^{1+q}\,\partial) \tag{3.12}$$

with a countable infinity of invariants:

$$(\alpha,\beta)\ ,\ (\alpha_*,\beta_*)\ ,\ (p,q)\ ,\ \{\alpha_{(m,n)}\,;m,n\in\mathbb{N}^*\}\ \mathit{or}\ \{\beta_{(m,n)}\,;m,n\in\mathbb{N}^*\} \tag{3.13}$$

Another object of central importance, on account both of its invariance and (anti)symmetry in (F, G), is the connector $H^{nor} = \exp H_*^{nor}$ which conjugates the normal to the conormal forms:

$$F^{conor} = H^{nor}\,F^{nor}\,(H^{nor})^{-1} \tag{3.14}$$

$$G^{conor} = (H^{nor})^{-1}\,G^{nor}\,H^{nor} \tag{3.15}$$

but which may also be constructed directly from any solution (F, G) by setting:

$$F = H_1\,F^{nor}\,H_1^{-1} \qquad ; \qquad G = H_2\,G^{nor}\,H_2^{-1} \tag{3.16}$$

$$H^{nor} = H_2^{-1}\,H_1 \tag{3.17}$$

The *connector* H^{nor} is uniquely defined, provided we normalise it by imposing the absence in H^{nor}_* of terms l_p and l_q. In the most typical (one-continuous parameter) examples, we shall see that the connector assumes the form:

$$H^{nor}_* = (1 + \sum_{\substack{m,n \geqslant 0 \\ m+n \geqslant 0}} \gamma_{(m,n)}\, \alpha^m\, \beta^n\, x^{m\,p+n\,q})\, ((\frac{1}{p} - \frac{1}{q})\, x\, \partial) \tag{3.18}$$

In this case, the secondary invariants $\{\alpha_{(m,n)}\}$ or $\{\beta_{(m,n)}\}$ or $\{\gamma_{(m,n)}\}$, together with the primary invariants (α, β) *and* (α_*, β_*) *and* (p, q), constitute a *complete* and *free* set of joint invariants for the pair (F, G). The three sets of secondary invariants are clearly equivalent. Indeed for $m + n \geqslant 2$:

$$\alpha_{(m,n)} = (\frac{1}{p} - \frac{1}{q})\,(m\,p + n\,q - p)\,\gamma_{(m,n)} + \text{ earlier terms} \tag{3.19}$$

$$\beta_{(m,n)} = (\frac{1}{q} - \frac{1}{p})\,(m\,p + n\,q - q)\,\gamma_{(m,n)} + \text{ earlier terms} \tag{3.20}$$

$$\alpha_{(m,n)} = -\frac{(m\,p + n\,q - p)}{(m\,p + n\,q - q)}\,\beta_{(m,n)} + \text{ earlier terms} \tag{3.21}$$

with the condition $m + n \geqslant 2$ excluding the terms:

$$\alpha_{(1,0)} = \alpha_* \;\; ; \;\; \alpha_{(0,1)} = 0 \;\; ; \;\; \beta_{(1,0)} = 0 \;\; ; \;\; \beta_{(0,1)} = \beta_*$$

and ensuring that the leading factors in (2.19), (2.20), (2.21) be $\neq 0, \infty$.

The invariants (α, β) in the first pair of *primary invariants* have a separate existence only if we restrict ourselves to identity-tangent conjugacies. If we accept conjugacies with a multiplier $k'(0) \neq 1$, then there remains only one invariant, namely the ratio:

$$c = \alpha^{q_*}\, \beta^{-p_*} \quad \text{with } p_*/q_* = p/q \text{ and } p_*,\, q_* \text{ coprime.} \tag{3.22}$$

Similarly, the tangency orders (p, q) lose their invariance if we accept *ramified conjugacies* and there only remains the pair (p_*, q_*) of coprime numbers proportional to (p, q). On the other hand, the third pair (α_*, β_*), as well as all the secondary invariants $\{\alpha_{(m,n)}\}$ $\{\beta_{(m,n)}\}$, possess *strong invariance*, since they remain unaffected even by non-identity-tangent or ramified conjugacies.

3.3. Main types of twin-begetting relators $\boldsymbol{W(A,B)}$

Our first series of examples (3.1 through 3.9) illustrates two basic dichotomies:
– the twins (F,G) verifying a given relation $W(F,G)=1$ may have a fixed *or* free tangency ratio p/q
– or again, their continuous parameter c (constructed from the leading coefficients: see (3.22)) may be fixed *or* free.

The second series of examples (3.10 through 3.20) imposes additional symmetries on twins. More precisely, for each of the 10 basic finite groups $\boldsymbol{Aut}_i$ of automorphisms of $\boldsymbol{Gr\{A,B\}}$, we construct relations $W(F,G)\equiv 1$ whose set of non-trivial solutions (F,G) is globally invariant under $\boldsymbol{Aut}_i$.

Our third series of examples (3.21 through 3.23) deals with more exceptional situations, e.g. with relations $W(F,G)=1$ whose general solution (F,G) depends on *several continuous* parameters with *fixed* or *variable* positions (i.e. parameters making their "first appearance" inside coefficients of *fixed or variable* depth t). They also illustrate related questions such as the *glueing* or *splitting* of relators.

3.4. Basic examples

Example 3.1 (Fixed ratio p/q , no continuous parameter) . — If we set

$$\begin{aligned} \boldsymbol{W} &:= \boldsymbol{U}^P\,\boldsymbol{V}^Q \quad \text{with } P,Q\in\mathbb{Z}^* \text{ and} \\ \boldsymbol{U} &= \boldsymbol{U(A,B)} := \overline{\overline{\boldsymbol{A}}}^{\,3}\,\boldsymbol{B} = \{\boldsymbol{A}\{\boldsymbol{A}\{\boldsymbol{A},\boldsymbol{B}\}\}\} && (3.23) \\ \boldsymbol{V} &= \boldsymbol{V(A,B)} := \overline{\overline{\boldsymbol{B}}}^{\,2}\,\boldsymbol{A} = \{\boldsymbol{B}\{\boldsymbol{B},\boldsymbol{A}\}\} && (3.24) \end{aligned}$$

then for each $s\in\mathbb{N}^*$, $W(F,G)=1$ has a unique (upto conjugacy) twin solution (F,G) with tangency orders $(p,q)=(s,2\,s)$ and fixed invariants c,α_*,β_*.

Proof. — Let us look for a solution (F,G) with tangency orders $p\neq q$. We may write the infinitesimal generators (F_*,G_*) in the form:

$$F_* = (1+\sum_{1\leqslant t}\alpha_t\,x^t)\,(\frac{\alpha}{p}\,x^p\,\partial) \quad \text{with } \alpha\neq 0 \qquad (3.25)$$

$$G_* = (1+\sum_{1\leqslant t}\beta_t\,x^t)\,(\frac{\beta}{q}\,x^q\,\partial) \quad \text{with } \beta\neq 0 \qquad (3.26)$$

Then if we set:

$$U_* := \log U(F,G)\ ;\ V_* := \log V(F,G)\ ; W_* := \log W(F,G) \tag{3.27}$$

we find by the Campbell-Hausdorff-like identity (1.20):

$$U_* \quad = (\sum_{t\geqslant 0} u_t\, x^t)\, (\tfrac{\alpha^3\,\beta}{p^3\,q}\, x^{1+M}\,\partial) \quad \text{with } M = 3\,p+q \tag{3.28}$$

$$V_* \quad = (\sum_{t\geqslant 0} v_t\, x^t)\, (\tfrac{\alpha\,\beta^2}{p\,q^2}\, x^{1+N}\,\partial) \quad \text{with } N = p+2\,q \tag{3.29}$$

Moreover, since U and V have as *corner components*:

$$\mathbf{u_0}(\mathbf{a},\mathbf{b}) = (\overline{\mathbf{a}})^{\mathbf{3}}\,\mathbf{b} \quad ; \quad \mathbf{v_0}(\mathbf{a},\mathbf{b}) = (\overline{\mathbf{b}})^{\mathbf{2}}\,\mathbf{a} \tag{3.30}$$

the series in U_* and V_* have initial coefficients:

$$u_0 \quad = \quad T_{q,p,p,p} \tag{3.31}$$

$$v_0 \quad = \quad T_{p,q,q} \tag{3.32}$$

and current coefficients of the form:

$$u_t \quad = \quad u_t^*\,\alpha_t + u_t^{**}\,\beta_t + \text{ earlier terms} \tag{3.33}$$

$$v_t \quad = \quad v_t^*\,\alpha_t + v_t^{**}\,\beta_t + \text{ earlier terms} \tag{3.34}$$

with

$$u_t^* \quad = \quad T_{q,p+t,p,p} + T_{q,p,p+t,p} + T_{q,p,p,p+t} \tag{3.35}$$

$$u_t^{**} \quad = \quad T_{q+t,p,p,p} \tag{3.36}$$

$$v_t^* \quad = \quad T_{p+t,q,q} \tag{3.37}$$

$$v_t^{**} \quad = \quad T_{p,q+t,q} + T_{p,q,q+t} \tag{3.38}$$

In view of (2.28); (2.29) the identity $W(F,G) = 1$ clearly implies $M = N$, which in turn implies $(p,q) = (s, 2\,s)$. It also implies the vanishing of the first coefficient in W_*:

$$P\,u_0\,\frac{\alpha^3\,\beta}{p^3\,q} + Q\,v_0\,\frac{\alpha\,\beta^2}{p\,q^2} = 0 \tag{3.39}$$

which rigidly fixes the invariant c:

$$c := \alpha^2\,\beta^{-1} = -\frac{p^2}{q}\,\frac{Q}{P}\,\frac{u_0}{v_0} = +\frac{p\,Q}{(p+q)\,P} = +\frac{Q}{3\,P} \tag{3.40}$$

As for the current coefficients (α_t, β_t), their first occurence in W_* takes place at depth t. More precisely:

$$W_* = (\sum_{1 \leqslant t} w_t\, x^t)\, (\frac{\alpha^4\, \beta^3}{p^4\, q^3}\, x^{1+5\, s}) \tag{3.41}$$

with

$$\begin{aligned} w_t &= w_t^*\, \alpha_t + w_t^{**}\, \beta_t + \text{ earlier terms} && (3.42)\\ w_t^* &= P\, \frac{\alpha^3\, \beta}{p^3\, q}\, u_t^* + Q\, \frac{\alpha\, \beta^2}{p\, q^2}\, v_t^* && (3.43)\\ w_t^{**} &= P\, \frac{\alpha^3\, \beta}{p^3\, q}\, u_t^{**} + Q\, \frac{\alpha\, \beta^2}{p\, q^2}\, v_t^{**} && (3.44) \end{aligned}$$

But due to (3.29) we may factor out the term:

$$R = -P\, u_0\, \frac{\alpha^3\, \beta}{p^3\, q} = -Q\, v_0\, \frac{\alpha\, \beta^2}{p\, q^2} \tag{3.45}$$

Eventually, after expliciting u_0,v_0 etc. and recalling that $(p, q) = (s, 2\, s)$, we find

$$\begin{aligned} w_t^* &= R\, (-\frac{u_t^*}{u_0} + \frac{v_t^*}{v_0}) = R\, \frac{1}{6}\, (6 + t + t^2)\, (t - q) && (3.46)\\ w_t^{**} &= R\, (-\frac{u_t^{**}}{u_0} + \frac{v_t^{**}}{v_0}) = \frac{1}{6}\, R\, (6 + t + t^2)\, (t - p) && (3.47) \end{aligned}$$

The presence of a common factor $(6+t+t^2)$ alongside the individual factors $(t - q)$ and $(t - p)$ is no accident, but a consequence of the identities:

$$\begin{aligned} (t - p)\, u_t^* + (t - q)\, u_t^{**} &= t - M && (3.48)\\ (t - p)\, v_t^* + (t - q)\, v_t^{**} &= t - N && (3.49) \end{aligned}$$

which are but special cases of (3.8). Thus in the end we have:

$$w_t \equiv R\, (1/6)\, (6 + t + t^2)\, [(t - q)\, \alpha_t - (t - p)\, \beta_t] + \text{ earlier terms} \tag{3.50}$$

Therefore, since solving $W(F, G) = 1$ means solving $W_*(F, G) = 0$, i.e. killing all coefficients w_t, we have one choice at each inductive step. For $t = 0$, $w_0 = 0$ is ensured by (3.39). For $t \notin \{0, p, q\}$, we may choose α_t (or β_t) as we wish, after which the other coefficient is fixed. For the exceptional values p and q, however, the choice is restricted to *one* of the coefficients. For $t = p$, β_p can be anything, but α_p is rigidly set and equal to $\alpha\, \alpha_*$ irrespective of the choice for β_p. For $t = q$, α_q can be anything, but β_q is rigidly set and equal to $\beta\,\beta_*$. This comes as no surprise, since the 'iteration residues' (α_*,β_*) are strongly invariant under conjugations. Their actual values, as given in (3.82) below, result from a general calculation that will be carried out in Example 3.4.

If we now analyse the 'earlier terms' in (3.50) with the help of formula (1.20) applied successively to $U(F,G)$, $V(F,G)$, $W(F,G)$, we find that these 'earlier terms' are generated by multiple Lie brackets involving only monomials l_{t_i} for earlier indices $t_i < t$, starting with l_p and l_q (recall that $l_n := x^{n+1}\,\partial$). We may therefore restrict ourselves to indices t in $p\,\mathbb{N}$ or $q\,\mathbb{N}$, and set all other coefficients α_t or β_t equal to 0. In particuliar, if we take advantage of the latitude afforded by the induction to normalise either F_* or G_*, then the *conormal twin* (see (3.10) (3.11)) will carry only indices of depth t in $p\,\mathbb{N}+q\,\mathbb{N}$ and so too will the connector H^{nor}. □

Example 3.2 (Fixed ratio $p/q = p_0/q_0$, one discrete parameter) . — If we set

$$\begin{aligned} \boldsymbol{W} &:= \boldsymbol{U}^P\,\boldsymbol{V}^Q \quad \text{with } P,Q\in\mathbb{Z}^* \text{ and} \\ \boldsymbol{U}=\boldsymbol{U}(\boldsymbol{A},\boldsymbol{B}) &:= (\overline{\overline{\boldsymbol{A}}})^{1+q_0}\,\boldsymbol{B} = \{\boldsymbol{A}\{\cdots\boldsymbol{A},\boldsymbol{B}\}\cdots\}\} \quad (3.51) \\ \boldsymbol{V}=\boldsymbol{V}(\boldsymbol{A},\boldsymbol{B}) &:= (\overline{\overline{\boldsymbol{B}}})^{1+p_0}\,\boldsymbol{A} = \{\boldsymbol{B}\{\cdots\boldsymbol{B},\boldsymbol{A}\}\cdots\}\} \quad (3.52) \end{aligned}$$

then $W(F,G)=1$ has a twin solution with tangency orders (p,q) iff $p/q = p_0/q_0$ $(\neq 1)$. That solution depends on a prime unit root ϵ of order s_0 (s_0 being the largest common divisor of p,q).

Proof. — Taking F_* and G_* as in (3.25), (3.26), we get this time:

$$U_* = (\sum_{t\geqslant 0} u_t\,x^t)\,(\frac{\alpha^{1+p_0}\,\beta}{p^{1+q_0}\,q}\,x^{1+M}\,\partial) \quad (3.53)$$

$$V_* = (\sum_{t\geqslant 0} v_t\,x^t)\,(\frac{\alpha\,\beta^{1+q_0}}{p\,q^{1+p_0}}\,x^{1+N}\,\partial) \quad (3.54)$$

with

$$M = p\,(1+q_0)+q \quad ; \quad N = q\,(1+p_0)+p \quad (3.55)$$

The initial coefficients are:

$$u_0 = T_{q,p^{[1+q_0]}} \quad (3.56)$$

$$v_0 = T_{p,q^{[1+p_0]}} \quad (3.57)$$

where of course $m^{[n]}$ denotes the sequence $(m,\ldots,m)$ of length n. The subsequent coefficients u_t, v_t retain their earlier expression (3.33) (3.34) but with:

$$\begin{aligned} u_t^* &= \sum T_{q,p^{[q_1]},p+t,p^{[q_2]}} \quad (\text{for } q_1,q_2 \geqslant 0;\ q_1+q_2=q_0) && (3.58)\\ u_t^{**} &= T_{q+t,p^{[1+q_0]}} && (3.59)\\ v_t^* &= T_{p+t,q^{[1+p_0]}} && (3.60)\\ v_t^{**} &= \sum T_{p,q^{[p_1]},q+t,q^{[p_2]}} \quad (\text{for } p_1,p_2 \geqslant 0;\ p_1+p_2=p_0) && (3.61) \end{aligned}$$

But the earlier identities (3.48) (3.49) remain in force (being a simple consequence of (3.8)) and lead to a drastic simplification of u_t^* and v_t^{**}:

$$\begin{aligned} u_t^* &= \frac{t-M}{t-p} - \frac{t-q}{t-p}\, T_{q+t,p^{[1+q_0]}} && (3.62)\\ v_t^{**} &= \frac{t-N}{t-q} - \frac{t-p}{t-q}\, T_{p+t,q^{[1+p_0]}} && (3.63) \end{aligned}$$

Here too, W_* may vanish only if $M=N$, which in view of (3.55) imposes $p/q = p_0/q_0$. So we may write:

$$(p,q) = (s\,p_*, s\,q_*) \quad ; \quad (p_0,q_0) = (s_0\,p_*, s_0\,q_*) \tag{3.64}$$

with s, s_0 in $\mathbf{N}^*$ and p_*, q_* coprime. The argument then proceeds as in Example 3.1, except that (3.39) now becomes:

$$P\,u_0\,(\frac{\alpha\,\beta}{p\,q})\,(\frac{\alpha}{p})^{s_0\,p_*} + Q\,v_0\,(\frac{\alpha\,\beta}{p\,q})\,(\frac{\beta}{q})^{s_0\,q_*} = 0 \tag{3.65}$$

so that the invariant $c := \alpha^{q_*}\,\beta^{-p_*}$ is now constrained only by:

$$c^{s_0} = -\frac{Q\,v_0}{P\,u_0}\,\frac{p^{s_0\,p_*}}{q^{s_0\,q_*}} \tag{3.66}$$

and so depends on a unit root ϵ of order s_0. That aside, the key identity (3.50) has its exact counterpart:

$$w_t = R\,.\,D(t)\,[(t-q)\,\alpha_t - (t-p)\,\beta_t] \ + \ \text{earlier terms} \tag{3.67}$$

with a trivial factor:

$$R = -P\,u_0\,(\frac{\alpha\,\beta}{p\,q})\,(\frac{\alpha}{p})^{s_0\,p_*} = Q\,v_0\,(\frac{\alpha\,\beta}{p\,q})\,(\frac{\beta}{q})^{s_0\,q_*} \tag{3.68}$$

and a non-trivial divisor $D(t) = T_{q,p^{[1+q_0]}}\,T_{p,q^{[1+p_0]}}\,\Delta(t)$, with

$$\Delta(t) = \Delta_{p,q;p_0,q_0}(t) = \frac{1}{(t-p)\,(t-q)}\Big(-(t-S) + (t-q)\,\frac{T_{q+t,p^{[1+q_0]}}}{T_{q,p^{[1+q_0]}}} + (t-p)\,\frac{T_{p+t,q^{[1+p_0]}}}{T_{p,q^{[1+p_0]}}}\Big) \tag{3.69}$$

and

$$S = M = N = s\,s_0\,p_*\,q_* + s\,p_* + s\,q_* \tag{3.70}$$

Here again, despite the denominator $(t-p)\,(t-q)$, $D(t)$ is necessarily a polynomial in t. Thus, if we take $s = s_0 = 1$ and therefore $(p,q) = (p_0,q_0) = (p_*,q_*)$, we find:

$$D(t) = \text{Const}\,(t^3 + 15\,t^2 + 26\,t + 120) \quad \text{for } (p_*,q_*) = (1,3)$$

$$D(t) = \text{Const}\,(2\,t^3 + 15\,t^2 + 55\,t + 210) \quad \text{for } (p_*,q_*) = (2,5)$$

$$D(t) = \text{Const}\,(t^4 + 26\,t^3 + 271\,t^2 + 606\,t + 2520) \quad \text{for } (p_*,q_*) = (1,4)$$

etc. The remarkable thing is that, although $(t-p)\,(t-q)\,D(t)$ is a real polynomial of mixed signs, $D(t)$ itself seems always to carry positive coefficients only, which automatically ensures $D(t) \neq 0$ for $t \in \mathbb{N}^*$ and thus guarantees the solvability of the relation $W(F,G) = 1$. It should be pointed out, however, that this positivity result has not been *proved* for all values $p/q = p_0/q_0$, but only checked, on a case by case basis, for all pairs (p,q), (p_0,q_0) with $p+q \leqslant 30$, $p_0+q_0 \leqslant 30$. However, reverting to the expression (3.7) of $T_\bullet$, we may rewrite $\Delta(t)$ as:

$$\Delta(t) = \Delta_{p,q;p_0,q_0}(t) = \frac{\frac{1}{2}\,(p_0\,q+p\,q_0)+(p+q)-t}{(t-p)\,(t-q)} + \frac{1}{t-q}\,\frac{\Gamma(\frac{p+t}{q}+p_0)\,\Gamma(\frac{p}{q}-1)}{\Gamma(\frac{p}{q}+p_0)\,\Gamma(\frac{p+t}{q}-1)} + \frac{1}{t-p}\,\frac{\Gamma(\frac{q+t}{p}+q_0)\,\Gamma(\frac{q}{p}-1)}{\Gamma(\frac{q}{p}+q_0)\,\Gamma(\frac{q+t}{p}-1)} \tag{3.71}$$

Then,using the asymptotic properties of the gamma function, it is a straightforward exercise to check that, for any *fixed pair* (p,q) and for $(p_0,q_0) = (\sigma\,p, \sigma\,q)$ the divisor $\Delta(t)$ is > 0 for all $t > 0$ and all σ large enough. Thus, for any given tangency orders (p,q), we can point to an explicit relation $W(F,G) = 1$ that admits twins (F,G) with that tangency order. □

Example 3.3 (Fixed ratio p/q, one continuous parameter). — If we set

$$\boldsymbol{W(A,B) := \{U(A,B), V(A,B)\}}$$

with $\boldsymbol{U}$,$\boldsymbol{V}$ as in Example 3.1, then the relation $W(F,G) = 1$ has twin solutions for the tangency ratios $p/q = 1/2$. These twins still have the same iteration residues $\alpha_* = 15/8$ and $\beta_* = -7/12$ as in Example 3.1, but they now depend on a free continuous parameter $c := \alpha^2\,\beta^{-1}$

Proof. — Starting from any pairs (p,q) and (α,β), we take (F_*,G_*) as in Example 3.1 and get the same expression for (U_*,V_*). The expression for W_*, however, does change:

$$W_* = (\sum_{0\leqslant t} w_t\,x^t)\,(\frac{\alpha\,\beta}{p\,q})(\frac{\alpha}{p})^2\,(\frac{\beta}{q})\,(x^{1+M+N}\,\partial) \tag{3.72}$$

with an initial coefficient w_0 that vanishes *iff* $M = N$. Thus $W_* = 0$ implies $p/q = 1/2$ as before but leaves (α, β) and thus $c := \alpha^2 \beta^{-1}$ completely free. As for the current coefficient w_t, it is now of the form:

$$w_t = D(t)\,((t-q)\,\alpha_t - (t-p)\,\beta_t) + \text{ earlier terms} \tag{3.73}$$

with the same divisor $D(t)$ as in Example 3.1, upto the trivial factor $\frac{u_0\, v_0}{R}$:

$$D(t) = u_0\, v_0\, t\, \frac{1}{6}\,(6 + t + t^2) \quad \text{with } u_0 = T_{q,p,p},\ v_0 = T_{p,q,q} \tag{3.74}$$

So the only change is the appearance of the continuous parameter c (invariant under general conjugacies) or the pair (α, β) (invariant under identity-tangent conjugacies) instead of the rigidity in Example 3.1. As for the values (α_*, β_*), they are special cases of (3.76), (3.77) below. □

Example 3.4 (Fixed ratio $p/q = p_0/q_0$, one continuous parameter). — If we set

$$\boldsymbol{W(A,B)} := \{\boldsymbol{U(A,B)}, \boldsymbol{V(A,B)}\}$$

with $\boldsymbol{U}$,$\boldsymbol{V}$ as in Example 3.2 (i.e. $\boldsymbol{U} := (\overline{\overline{\boldsymbol{A}}})^{1+q_0}\, \boldsymbol{B}$ and $\boldsymbol{V} := (\overline{\overline{\boldsymbol{B}}})^{1+p_0}\, \boldsymbol{A}$) then the relation $W(F,G) = 1$ has twin solutions for the tangency ratios $p/q = p_0/q_0$. These twins still possess the same iteration residues (α_*, β_*) as in Example 3.2 (see (3.81) below), but they now depend on a free parameter $c := \alpha^{q_*}\, \beta^{-p_*}$

Proof. — The transition is the same as from Example 3.1 to Example 3.3. We find this time:

$$W_* = (\sum_{0 \leqslant t} w_t\, x^t)\,(\frac{\alpha\,\beta}{p\,q})(\frac{\alpha}{p})^{q_0}\,(\frac{\beta}{q})^{p_0}\,(x^{1+M+N}\,\partial) \tag{3.75}$$

with M, N as in (3.53). The initial coefficient w_0 vanishes iff $M = N$, which implies $p/q = p_0/q_0 = p_*/q_*$ (irreducible) as in Example 3.2 but leaves (α, β) and thus $c := \alpha^{q_*}\, \beta^{-p_*}$ free. The current coefficient w_t is still of the form:

$$w_t = D(t)\,((t-q)\,\alpha_t - (t-p)\,\beta_t) + \text{ earlier terms} \tag{3.76}$$

with the same divisor $D(t)$ as in Example 2, upto the trivial factor $u_0\, v_0/R$:

$$D(t) = u_0\, v_0\, t\, \Delta(t) \quad (\text{with } \Delta(\bullet) \text{ as in (3.69)}) \tag{3.77}$$

That leaves only one point to settle, namely the invariants α_*, β_*. These are known as soon as the 'earlier terms' in (3.74) are known for $t = p$ and $t = q$. To calculate these, we must express $\boldsymbol{u(a,b)} := \log \boldsymbol{U}(e^{\boldsymbol{a}}, e^{\boldsymbol{b}})$ and

$\boldsymbol{v(a,b)} := \log \boldsymbol{V}(e^{\boldsymbol{a}}, e^{\boldsymbol{b}})$ and retain not just the terms of lowest degree (i.e. the 'corner components') but also the terms of immediately higher degree, by using formula (1.20) (with $\boldsymbol{a}_j$ either $\boldsymbol{a}$ or $\boldsymbol{b}$) and then substituting (l_p, l_q) for $(\boldsymbol{a}, \boldsymbol{b})$. Eventually, we find expressions that depend only on (p_0, q_0) but not (p, q):

$$\alpha_* \;=\; \frac{1}{2}\,\frac{(1+q_0)^2 - \dfrac{T_{p_0,q_0,p_0,q_0^{[p_0]}}}{p_0\, T_{p_0,q_0^{[1+p_0]}}}}{(q_0-p_0)\,\Delta(p_0)} \tag{3.78}$$

$$\beta_* \;=\; \frac{1}{2}\,\frac{(1+p_0)^2 - \dfrac{T_{q_0,p_0,q_0,p_0^{[q_0]}}}{q_0\, T_{q_0,p_0^{[1+q_0]}}}}{(p_0-q_0)\,\Delta(q_0)} \tag{3.79}$$

with $\Delta(\bullet) = \Delta_{p,q;p_0,q_0}(\bullet)$ as in (3.67) and (3.69). Remarkably, these formulas don't apply directly when $t = p_0$ or q_0 for then the denominators vanish. We must either calculate $\Delta(t)$ first as a polynom in t or else resort to the formulas:

$$(q_0-p_0)\,\Delta(p_0) = \frac{\sum T_{q_0,p_0^{[q_1]},2\,p_0,p_0^{[q_2]}}}{T_{q_0,p_0^{[1+q_0]}}} - \frac{T_{2\,p_0,q_0^{[1+p_0]}}}{T_{p_0,q_0^{[1+p_0]}}} \tag{3.80}$$

$$(p_0-q_0)\,\Delta(q_0) = \frac{\sum T_{p_0,q_0^{[p_1]},2\,q_0,q_0^{[p_2]}}}{T_{p_0,q_0^{[1+p_0]}}} - \frac{T_{2\,q_0,p_0^{[1+q_0]}}}{T_{q_0,p_0^{[1+q_0]}}} \tag{3.81}$$

with a summation extending to all $q_i \geqslant 0$ such that $q_1 + q_2 = q_0$ (resp. to all $p_i \geqslant 0$ such that $p_1 + p_2 = p_0$. Thus we get:

$$(\alpha_*, \beta_*) = (30/29, -7/12) \qquad \text{for } (p_0, q_0) = (1,2) \tag{3.82}$$
$$(\alpha_*, \beta_*) = (50/27, -8/45) \qquad \text{for } (p_0, q_0) = (1,3) \tag{3.83}$$
$$(\alpha_*, \beta_*) = (413/132, -625/564) \qquad \text{for } (p_0, q_0) = (2,3) \tag{3.84}$$
$$(\alpha_*, \beta_*) = (1785/856, -69/1120) \qquad \text{for } (p_0, q_0) = (1,4) \tag{3.85}$$

□

Example 3.5 (Free ratio p/q, one continuous parameter). — If we set

$$\boldsymbol{W(A,B)} \;:=\; \boldsymbol{\{U(A,B), V(A,B)\}} \quad \text{with}$$
$$\boldsymbol{U = U(A,B)} \;:=\; \overline{\overline{\boldsymbol{A}}}\,\overline{\overline{\boldsymbol{B}}}\,\overline{\overline{\boldsymbol{B}}}\,\overline{\overline{\boldsymbol{B}}}\,\boldsymbol{A} = \boldsymbol{\{A\{B\{B\{B,A\}\}\}\}} \tag{3.86}$$
$$\boldsymbol{V = V(A,B)} \;:=\; \overline{\overline{\boldsymbol{B}}}\,\overline{\overline{\boldsymbol{A}}}\,\overline{\overline{\boldsymbol{B}}}\,\overline{\overline{\boldsymbol{A}}}\,\boldsymbol{B} = \boldsymbol{\{B\{A\{B\{A,B\}\}\}\}} \tag{3.87}$$

then the relation $W(F,G)=1$ has a twin solution for each $p \neq q$ and each (α,β). That solution is unique (upto identity-tangent conjugacies), with iteration residues:

$$\alpha_* = \frac{q\,(4\,p^2+5\,p\,q+3\,q^2)}{p^2\,(p+7\,q)} \quad ; \quad \beta_* = \frac{3\,p^2\,(p+q)}{q^2\,(3\,p+q)} \tag{3.88}$$

Proof. — In the *free structures* $\boldsymbol{Gr\{A,B\}}$ and $\boldsymbol{Lie\{a,b\}}$, the elements $\boldsymbol{U}$,$\boldsymbol{V}$,$\boldsymbol{W}$ have alternance (2,3),(2,3),(4,6) and their Lie images $\boldsymbol{u}$,$\boldsymbol{v}$,$\boldsymbol{w}$ have corner components:

$$\boldsymbol{u}_0 = \overline{\boldsymbol{a}}\,\overline{\boldsymbol{b}}\,\overline{\boldsymbol{b}}\,\overline{\boldsymbol{b}}\,\boldsymbol{a} \;\;;\;\; \boldsymbol{v}_0 = \overline{\boldsymbol{b}}\,\overline{\boldsymbol{a}}\,\overline{\boldsymbol{b}}\,\overline{\boldsymbol{a}}\,\boldsymbol{b} \;\;;\;\; \boldsymbol{w}_0 = [\boldsymbol{u}_0, \boldsymbol{v}_0] \tag{3.89}$$

Going over to the *bound structures* $\tilde{\mathcal{G}}_0$ and $\tilde{\mathcal{L}}_0$ and defining U, V, W and U_*, V_*,W_* in the usual way, we find:

$$U_* = (\sum_{t\geqslant 0} u_t\,x^t)\,(\frac{\alpha}{p})^2\,(\frac{\beta}{q})^3\,(x^{1+M}\,\partial) \tag{3.90}$$

$$V_* = (\sum_{t\geqslant 0} v_t\,x^t)\,(\frac{\alpha}{p})^2\,(\frac{\beta}{q})^3\,(x^{1+M}\,\partial) \tag{3.91}$$

$$W_* = (\sum_{t\geqslant 0} w_t\,x^t)\,(\frac{\alpha}{p})^4\,(\frac{\beta}{q})^6\,(x^{1+M+N}\,\partial) \tag{3.92}$$

but with $M=N=2\,p+3\,q$ irrespective of the choice of (p,q). Furthermore, we have leading coefficients:

$$u_0 = T_{p,q,q,q,p} \quad ; \quad v_0 = T_{q,p,q,p,q} \quad ; \quad w_0 = 0 \quad (\forall p,q) \tag{3.93}$$

and the current coefficients of the form:

$$u_t = u_t^*\,\alpha_t + u_t^{**}\,\beta_t + \text{ earlier terms} \tag{3.94}$$

$$v_t = v_t^*\,\alpha_t + v_t^{**}\,\beta_t + \text{ earlier terms} \tag{3.95}$$

$$w_t = w_t^*\,\alpha_t + w_t^{**}\,\beta_t + \text{ earlier terms} \tag{3.96}$$

with (α_t,β_t) denoting the coefficients of (F_*,G_*) of depth t, as in (3.25) (3.26)

and with:

$$u_t^* = T_{p+t,q,q,q,p} + T_{p,q,q,q,p+t} \tag{3.97}$$

$$u_t^{**} = T_{p,q+t,q,q,p} + T_{p,q,q+t,q,p} + T_{p,q,q,q+t,p} \tag{3.98}$$

$$v_t^* = T_{q,p+t,q,p,q} + T_{q,p,q,p+t,q} \tag{3.99}$$

$$v_t^{**} = T_{q+t,p,q,p,q} + T_{q,p,q+t,p,q} + T_{q,p,q,p,q+t} \tag{3.100}$$

and therefore:

$$w_t^* = t \det \begin{pmatrix} u_0 & v_0 \\ u_t^* & v_t^* \end{pmatrix} \quad ; \quad w_t^{**} = t \det \begin{pmatrix} u_0 & v_0 \\ u_t^{**} & v_t^{**} \end{pmatrix} \tag{3.101}$$

But applying (3.8) to u_0 and v_0 we get:

$$(t-2\,p-3\,q)\,u_0 \quad = \quad (t-p)\,u_t^* + (t-q)\,u_t^{**} \tag{3.102}$$

$$(t-2\,p-3\,q)\,v_0 \quad = \quad (t-p)\,v_t^* + (t-q)\,v_t^{**} \tag{3.103}$$

As a consequence, $(t-p)\,w_t^* + (t-q)\,w_t^{**} = 0$ and we may write:

$$w_t = D(t)\,((t-q)\,\alpha_t - (t-p)\,\beta_t) + \text{ earlier terms} \tag{3.104}$$

with a divisor $D(t)$ given by:

$$D(t) := \frac{t}{t-q} \det \begin{pmatrix} u_0 & v_0 \\ u_t^* & v_t^* \end{pmatrix} = -\frac{t}{t-p} \det \begin{pmatrix} u_0 & v_0 \\ u_t^{**} & v_t^{**} \end{pmatrix} \tag{3.105}$$

A simple calculation then yields:

$$D(t) = p\,q\,(p-q)^2\,t^2\,(t^2 + q\,t + 6\,p\,q) \tag{3.106}$$

so that the divisor $D(t)$ is always $\neq 0$ for positive integers (p,q,t) $(p \neq q)$. The upshot is that, given initial data (p,q) and (α,β) $(p \neq q\ ;\ \alpha\,\beta \neq 0)$, the inductive resolution of $\{w_t = 0\}$ is possible, leading to 'intrinsical series' for F_*^{conor}, G_*^{conor}, H_*^{nor} that have exactly the form (3.10), (3.11), (3.18) and carry only coefficients of depth t in $p\,\mathbb{N} + q\,\mathbb{N}$. To find the exact values of the iteration residues (α_*, β_*), we must first apply (1.20) to calculate the terms of degree $(2+1,3)$ and $(2,3+1)$ in $u(\boldsymbol{a},\boldsymbol{b}) := \log U(e^{\boldsymbol{a}}, e^{\boldsymbol{b}})$ and $v(\boldsymbol{a},\boldsymbol{b}) := \log V(e^{\boldsymbol{a}}, e^{\boldsymbol{b}})$; then plug this into $w(\boldsymbol{a},\boldsymbol{b}) := \log W(e^{\boldsymbol{a}}, e^{\boldsymbol{b}})$; and lastly replace $(\boldsymbol{a},\boldsymbol{b})$ by (l_p, l_q). We find:

$$\alpha_* \quad = \quad -\frac{1}{2}\,\frac{1}{p-q}\,\frac{(u_0\,v_0' - v_0\,u_0')}{D(p)} = -\frac{1}{2\,p}\,\frac{\det \begin{pmatrix} u_0 & v_0 \\ u_0' & v_0' \end{pmatrix}}{\det \begin{pmatrix} u_0 & v_0 \\ u_p^* & v_p^* \end{pmatrix}} \tag{3.107}$$

$$\beta_* \quad = \quad -\frac{1}{2}\,\frac{1}{p-q}\,\frac{(u_0\,v_0'' - v_0\,u_0'')}{D(q)} = -\frac{1}{2\,q}\,\frac{\det \begin{pmatrix} u_0 & v_0 \\ u_0'' & v_0'' \end{pmatrix}}{\det \begin{pmatrix} u_0 & v_0 \\ u_q^{**} & v_q^{**} \end{pmatrix}} \tag{3.108}$$

with $u_0, v_0, D(t)$ as above and:

$$u_0' = T_{p,q,p,q,q,p} + T_{p,q,q,q,p,p} \tag{3.109}$$

$$u_0'' = 3\, T_{p,q,q,q,q,p} \tag{3.110}$$

$$v_0' = T_{q,p,p,q,p,q} + T_{q,p,q,p,p,q} \tag{3.111}$$

$$v_0'' = 2\, T_{q,p,q,q,p,q} + T_{q,p,q,p,q,q} \tag{3.112}$$

Which eventually leads to the values α_*, β_* mentioned in (3.85). □

Remark 3.6 . — We observe that α_*, β_*, as indeed all the secondary invariants $\alpha_{(m,n)}, \beta_{(m,n)}, \gamma_{(m,n)}$ carried by the "intrinsic series" (3.10),(3.11), (3.18), are homogeneous functions of degree 0 in (p, q).

Remark 3.7 . — For the time being, we have set aside the case of twins with identical tangency order $p = q$. But we may note that when $p/q \longrightarrow 1$, then $\alpha^{-1} F_*$ and $\beta^{-1} G_*$ in Example 3.5 tend to common, *non-trivial* series. In particuliar, α_* and β_* go to a common limit, which interestingly is $\neq 0$. As (3.85) shows, that limit is 3/2.

Remark 3.8 . — What makes Example 3.5 tick is clearly not the precise shape of $\boldsymbol{U(A,B)}$ and $\boldsymbol{V(A,B)}$, but the following three circumstances:

(i) the words $\boldsymbol{U(A,B)}$ and $\boldsymbol{V(A,B)}$ have the same alternance (d_1, d_2)

(ii) the corner components $\boldsymbol{u_0(a,b)}$ and $\boldsymbol{v_0(a,b)}$ are independent in $\boldsymbol{Lie}^{[d_1,d_2]}$

(iii) the divisor $D(t) \equiv D(p,q;t)$ is $\neq 0$ for $p \neq q$ and $t > 0$.

Example 3.5 happens to be the simplest possible instance of this type since it corresponds to $(d_1, d_2) = (2, 3)$, which is the lowest alternance for which $\boldsymbol{Lie}^{[d_1,d_2]}$ has dimension > 1. But in this case, the pair (d_1, d_2) consists of an even and an odd number, and for the sequel it will be useful to have examples corresponding to the two other combinations: odd+odd, even+even.

Example 3.9 (Free ratio p/q, one continuous parameter) . — If we set

$$\boldsymbol{W} := \{\boldsymbol{U}, \boldsymbol{V}\}$$

with distinct words $\boldsymbol{U}$, $\boldsymbol{V}$ of equal alternance (d_1, d_2), the picture is usually the same as in Example 3.5, for the condition (ii) and (iii) are generically fulfilled. In particular, for $(d_1, d_2) = (3, 3) = (odd, odd)$, we may take:

$$\boldsymbol{U} = \boldsymbol{U}(\boldsymbol{A},\boldsymbol{B}) := \overline{\overline{\boldsymbol{A}}}^{2}\,\overline{\overline{\boldsymbol{B}}}^{3}\,\boldsymbol{A} \quad ; \quad \boldsymbol{V} = \boldsymbol{V}(\boldsymbol{A},\boldsymbol{B}) := \overline{\overline{\boldsymbol{B}}}^{2}\,\overline{\overline{\boldsymbol{A}}}^{3}\,\boldsymbol{B} \tag{3.113}$$

since in this case:

$$D(t) = 3\,p\,q\,(p-q)^2\,t^2\Big(2\,t^3 + 5\,(p+q)\,t^2 + (3\,p^2 + 14\,p\,q + 3\,q^2)\,t + 24\,p\,q\,(p+q)\Big) \tag{3.114}$$

with positive coefficients only. We also find:

$$\alpha_* = \frac{q\,(p+3\,q)\,(18\,p^2 + 23\,p\,q + 6\,q^2)}{p^2\,(10\,p^2 + 43\,p\,q + 27\,q^2)} \tag{3.115}$$

$$\beta_* = \frac{p\,(q+3\,p)\,(18\,q^2 + 23\,q\,p + 6\,p^2)}{q^2\,(10\,q^2 + 43\,q\,p + 27\,p^2)} \tag{3.116}$$

Similarly, for $(d_1, d_2) = (2,4) = (even, even)$ we may take:

$$\boldsymbol{U} = \boldsymbol{U}(\boldsymbol{A},\boldsymbol{B}) := \overline{\overline{\boldsymbol{A}}}\,\overline{\overline{\boldsymbol{B}}}^{4}\,\boldsymbol{A} \quad ; \quad \boldsymbol{V} = \boldsymbol{V}(\boldsymbol{A},\boldsymbol{B}) := \overline{\overline{\boldsymbol{B}}}^{3}\,\overline{\overline{\boldsymbol{A}}}^{2}\,\boldsymbol{B} \tag{3.117}$$

since in this case:

$$D(t) = 4\,p\,q\,(p-q)^2\,(p+q)\,\Big(6\,p\,q + q\,t + t^2\Big)\,\Big(2\,q + 2\,p + t\Big) \tag{3.118}$$

again with positive coefficients only. Here again we find: $\alpha_* \neq 0$, $\beta_* \neq 0$.

We skip the proofs, since they follow axactly the same lines as in Example 3.5. Of course, the above Remarks 3.6 and 3.7 still apply.

3.5. Examples with built-in symmetries

To a given twin-generating relation $W(F,G) = 1$, one may not add any independent relation $W_1(F,G) = 1$ without forcing F and G to commute (which by our definition twins are forbidden to do). One may well, however, impose *additional symmetries.* More precisely, given any one of the ten (upto conjugacy) finite subgroups [4] of $\boldsymbol{Aut}\ \boldsymbol{Gr}\{\boldsymbol{A},\boldsymbol{B}\}$, one may look for relations $\boldsymbol{R}(\boldsymbol{A},\boldsymbol{B}) = 1$ that generate twins while being invariant under the action of the subgroup $\boldsymbol{Aut}_j$ in question:

$$\boldsymbol{R}^{\boldsymbol{T}}(\boldsymbol{A},\boldsymbol{B}) = \boldsymbol{Q}_{\boldsymbol{T}}(\boldsymbol{A},\boldsymbol{B})\,(\boldsymbol{R}(\boldsymbol{A},\boldsymbol{B}))^{\epsilon_{\boldsymbol{T}}}\,(\boldsymbol{Q}_{\boldsymbol{T}}(\boldsymbol{A},\boldsymbol{B}))^{-1} \tag{3.119}$$
$$(\,\forall \boldsymbol{T} \in \boldsymbol{Aut}_j\ ;\ \epsilon_{\boldsymbol{T}} = \pm 1\,)$$

(4) Their list was given at the end of §2, along with the notation for their elements. We recall that among these 10 finite subgroups 6 are cyclic; 2 abelian non-cyclic; and 2 non-abelian.

(Here the exponent $\epsilon_{\boldsymbol{T}}$ has to be an integer *and* a unit root. So $\epsilon_{\boldsymbol{T}} = \pm 1$). In each of these ten cases, we shall restrict ourselves to relations $R(F,G) \equiv 1$ whose general twin solution (F,G) depends only on a free ratio p/q ($\neq 1$ unless stated otherwise) and a free continuous parameter $c = \alpha^{q_*}\,\beta^{-p_*}$ (like in Examples 3.5 or 3.9). And as always in this section, we shall strive to pick those examples which are easiest to construct, rather than those with the lowest word complexity.

We begin with the six cyclic groups of automorphisms.

Example 3.10 (Invariance under $\boldsymbol{S}_1 : (\boldsymbol{A},\boldsymbol{B}) \mapsto (\boldsymbol{B},\boldsymbol{A})$) . — If we set $\boldsymbol{U} := \overline{\overline{\boldsymbol{A}}}^{\,2}\,\overline{\overline{\boldsymbol{B}}}^{\,3}\,\boldsymbol{A}$ and so $\boldsymbol{U}^{S_1} = \overline{\overline{\boldsymbol{B}}}^{\,2}\,\overline{\overline{\boldsymbol{A}}}^{\,3}\,\boldsymbol{B}$, then $R := \{\boldsymbol{U},\boldsymbol{U}^{\boldsymbol{S}_1}\}$ is a twin-generating relation invariant under $\boldsymbol{S}_1$ whose general twin solution depends only on p/q and $c = \alpha^{q_*}\,\beta^{-p_*}$.

Proof. — This is in fact nothing but Example 3.9 for $(d_1,d_2) = (3,3)$ since then we had $\boldsymbol{V} \equiv \boldsymbol{U}^{\boldsymbol{S}_1}$. So all there is to check is the $\boldsymbol{S}_1$ -invariance of the relation $\boldsymbol{R} \equiv 1$, which simply results from $\boldsymbol{R}^{\boldsymbol{S}_1} \equiv \boldsymbol{R}^{-1}$. □

Example 3.11 (Invariance under $\boldsymbol{S}_2 : (\boldsymbol{A},\boldsymbol{B}) \mapsto (\boldsymbol{B}^{-1},\boldsymbol{A}^{-1})$) . — If we take once again $\boldsymbol{U} := \overline{\overline{\boldsymbol{A}}}^{\,2}\,\overline{\overline{\boldsymbol{B}}}^{\,3}\,\boldsymbol{A}$ and so $\boldsymbol{U}^{\boldsymbol{S}_2} = \overline{\overline{(\boldsymbol{B}^{-1})}}^{\,2}\,\overline{\overline{(\boldsymbol{A}^{-1})}}^{\,3}\,\boldsymbol{B}^{-1}$, then $R := \{\boldsymbol{U},\boldsymbol{U}^{\boldsymbol{S}_2}\}$ is a twin-generating relation invariant under $\boldsymbol{S}_2$ and with a general solution depending on p/q and c.

Proof. — This new relation $\boldsymbol{R}$ not only differs from that of Example 3.10 but is not even equivalent to it. However, it has the same corner component $\boldsymbol{r}_0(\boldsymbol{a},\boldsymbol{b})$ and so the same non- vanishing divisor $D(t)$ (see (3.111)). As for the invariance, it results once again from $\boldsymbol{R}^{\boldsymbol{S}_2} \equiv \boldsymbol{R}^{-1}$. □

Example 3.12 (Invariance under $\boldsymbol{I}_0 : (\boldsymbol{A},\boldsymbol{B}) \mapsto (\boldsymbol{A}^{-1},\boldsymbol{B}^{-1})$). — Let $\boldsymbol{W} = \{\boldsymbol{U},\boldsymbol{V}\}$ be a twin-generating relation as in Example 3.5 or 3.9. Since $\boldsymbol{U}$, $\boldsymbol{V}$ necessarily share the same (global) alternance $d = d_1 + d_2$, $\boldsymbol{W}$ is always of even (global) alternance $2\,d$ and $\boldsymbol{W}_1 = \{\boldsymbol{W},\boldsymbol{A}\}$ of odd (global) alternance $2\,d+1$. If we then we set:

$$\boldsymbol{R} \;:=\; \boldsymbol{W}\,\boldsymbol{W}^{\boldsymbol{I}_0} \tag{3.120}$$

$$\boldsymbol{R}_1 \;:=\; \boldsymbol{W}_1\,(\boldsymbol{W}_1^{\boldsymbol{I}_0})^{-1} \tag{3.121}$$

we get two (non-equivalent) twin-generating and $\boldsymbol{I}_0$-invariant relations $\boldsymbol{R}$ and $\boldsymbol{R}_1$, whose general solution depends on p/q and $c = \alpha^{q_*}\,\beta^{-p_*}$, but whose iteration residues (α_*,β_*) automatically vanish.

Proof. — The $\boldsymbol{I}_0$-invariance trivially results from:

$$\boldsymbol{R}^{\boldsymbol{I}_0} := \boldsymbol{W}^{-1}\,\boldsymbol{R}\,\boldsymbol{W} \quad \text{(even alternance)} \tag{3.122}$$

$$\boldsymbol{R}_1^{\boldsymbol{I}_0} := \boldsymbol{R}_1^{-1} \quad \text{(odd alternance)} \tag{3.123}$$

The corner component and the divisor of $\boldsymbol{R}$ (resp. $\boldsymbol{R}_1$) are twice those of $\boldsymbol{W}$ (resp. $\boldsymbol{W}_1$). The divisor $D(p,q;t)$ of $\boldsymbol{W}$, in turn, being the same as in Example 3.5 or 3.9, does not vanish for $p \neq q$ and $p,q,t \in \mathbb{N}^*$. Neither does the divisor $D_1(p,q;t)$ of $\boldsymbol{W}_1$ since:

$$\boldsymbol{w}_{10}(\boldsymbol{a},\boldsymbol{b}) = [\boldsymbol{w}_0(\boldsymbol{a},\boldsymbol{b}),\, \boldsymbol{a}] \tag{3.124}$$

Hence

$$D_1(p,q;t) = ((d_1-1)\,p + d_2\,q + t)\,D(p,q;t) \tag{3.125}$$

The simpler invariance relation (3.120) (bought at the cost of a more complex $\boldsymbol{W}_1$) implies that the Lie image $\boldsymbol{r}_1(\boldsymbol{a},\boldsymbol{b})$ of $\boldsymbol{R}(\boldsymbol{A},\boldsymbol{B})$ carries only components of *odd* (global) degree in $(\boldsymbol{a},\boldsymbol{b})$. The component of lowest degree is the corner component $\boldsymbol{w}_{10}(\boldsymbol{a},\boldsymbol{b})$, with degree exactly $2\,d+1$. But the next two components (of degree $2\,d+2$) vanish, and so do the invariants α_*, β_* which stem from these components. The same conclusion also holds for the solutions of the relation $\boldsymbol{R} \equiv 1$, but the proof is slightly less direct. First, we note that (3.119) implies:

$$\boldsymbol{r}(-\boldsymbol{a},-\boldsymbol{b}) = e^{-\overline{\boldsymbol{w}(\boldsymbol{a},\boldsymbol{b})}}\,\boldsymbol{r}(\boldsymbol{a},\boldsymbol{b}) \tag{3.126}$$

So, if we set:

$$\boldsymbol{r}_2(\boldsymbol{a},\boldsymbol{b}) = \boldsymbol{r}(\boldsymbol{a},\boldsymbol{b}) + \boldsymbol{r}(-\boldsymbol{a},-\boldsymbol{b}) \tag{3.127}$$

we have:

$$\boldsymbol{r}_2(\boldsymbol{a},\boldsymbol{b}) = (1 + e^{-\overline{\boldsymbol{w}(\boldsymbol{a},\boldsymbol{b})}})\,\boldsymbol{r}(\boldsymbol{a},\boldsymbol{b}) \tag{3.128}$$

Since $1+\exp(-\overline{\boldsymbol{w}})$, as an operator on $\overline{\boldsymbol{Lie}}(\boldsymbol{a},\boldsymbol{b})$, is clearly invertible, the identity $\boldsymbol{r}(\boldsymbol{a},\boldsymbol{b}) \equiv 0$ is equivalent to $\boldsymbol{r}_2(\boldsymbol{a},\boldsymbol{b}) = 0$ (though $\boldsymbol{r}_2(\boldsymbol{a},\boldsymbol{b})$ is *not* the Lie image of any word $\boldsymbol{R}_2(\boldsymbol{A},\boldsymbol{B})$). Now, in view of its definition (3.125), $\boldsymbol{r}_2(\boldsymbol{a},\boldsymbol{b})$ carries only components of *even* (global) degree in $(\boldsymbol{a},\boldsymbol{b})$. So here again the two components of $\boldsymbol{r}_2(\boldsymbol{a},\boldsymbol{b})$ immediately superior (in degree) to the corner component necessarily vanish, so that $\alpha_* = \beta_* = 0$. □

Example 3.13 (Invariance under $\boldsymbol{I}_1 : (\boldsymbol{A},\boldsymbol{B}) \mapsto (\boldsymbol{A}^{-1},\boldsymbol{B})$). — Let $\boldsymbol{W} := \{\boldsymbol{U},\boldsymbol{V}\}$ be a twin-generating relation of alternance (d_1,d_2) in $(\boldsymbol{A},\boldsymbol{B})$, with d_1 even (as in Example 5) or odd (as in the same Example, but with $\boldsymbol{A}$ and $\boldsymbol{B}$ exchanged). Then if we set:

$$\boldsymbol{R} := \boldsymbol{W}\,\boldsymbol{W}^{\boldsymbol{I}_1} \quad (\text{for } d_1 \text{ even}) \tag{3.129}$$

$$\boldsymbol{R}_1 := \boldsymbol{W}\,(\boldsymbol{W}^{\boldsymbol{I}_1})^{-1} \quad (\text{for } d_1 \text{ odd}) \tag{3.130}$$

we get $\boldsymbol{I}_1$- invariant twin-generating relations. As usual, their general solution depends on p/q and c, but here only α_* is guaranteed to vanish.

Proof. — Same as for Example 3.12. □

Example 3.14 (Invariance under $\boldsymbol{J} : (\boldsymbol{A}, \boldsymbol{B}) \mapsto (\boldsymbol{B}^{-1}, \boldsymbol{A}\,\boldsymbol{B}^{-1})$*)* . — If we set successively:

$$\boldsymbol{C} := \{\boldsymbol{A}, \boldsymbol{B}\} \tag{3.131}$$

$$\boldsymbol{U} := \{\overline{\overline{\boldsymbol{C}}}^{\,2}\boldsymbol{A}, \overline{\overline{\boldsymbol{C}}}^{\,2}\boldsymbol{B}\} \tag{3.132}$$

$$\boldsymbol{V} := \overline{\overline{\boldsymbol{C}}}^{\,2}\{\overline{\overline{\boldsymbol{C}}}\,\boldsymbol{A}, \overline{\overline{\boldsymbol{C}}}\,\boldsymbol{B}\} \tag{3.133}$$

$$\boldsymbol{W} := \{\boldsymbol{U}, \boldsymbol{V}\} \tag{3.134}$$

$$\boldsymbol{R} := \boldsymbol{W}\,\boldsymbol{W}^{\boldsymbol{J}}\,\boldsymbol{W}^{\boldsymbol{J}^2} \tag{3.135}$$

then the $\boldsymbol{J}$-invariant relation $R(F, G) \equiv 1$ has a general twin-solution that depends on p/q and c.

Proof. — The Campbell-Hausdorff formula shows that the action of $\boldsymbol{J}$ on the images of $(\boldsymbol{A}, \boldsymbol{B})$ is:

$$(\boldsymbol{a}, \boldsymbol{b}) \mapsto (-\boldsymbol{b}, \boldsymbol{a} - \boldsymbol{b}) \mod [\boldsymbol{a}, \boldsymbol{b}] \tag{3.136}$$

So here the restriction to the terms of lowest degree is *not* a diagonal mapping. Therefore, if we want to construct an example which, like the previous ones, leads to twins with a completely free ratio p/q, special precautions have to be taken to produce words $\boldsymbol{U}$, $\boldsymbol{V}$ whose Lie images $\boldsymbol{u}$, $\boldsymbol{v}$ have corner components $\boldsymbol{u}_0, \boldsymbol{v}_0$ that are:

(i) non-proportional

(ii) separately invariant under (3.136)

(iii) and which give rise to a non-vanishing divisor.

If furthermore we want $\boldsymbol{U}$ and $\boldsymbol{V}$ to be expressible each as *one* single multicommutator, then the "simplest" possible choice happens to be (3.132)–(3.133), yielding:

$$\boldsymbol{u}_0 = [[[\boldsymbol{a}, \boldsymbol{b}], [[\boldsymbol{a}, \boldsymbol{b}], \boldsymbol{a}]], [[\boldsymbol{a}, \boldsymbol{b}], [[\boldsymbol{a}, \boldsymbol{b}], \boldsymbol{b}]]] \tag{3.137}$$

$$\boldsymbol{v}_0 = [[\boldsymbol{a}, \boldsymbol{b}], [[\boldsymbol{a}, \boldsymbol{b}], [[[\boldsymbol{a}, \boldsymbol{b}], \boldsymbol{a}], [[\boldsymbol{a}, \boldsymbol{b}], \boldsymbol{b}]]]] \tag{3.138}$$

The non-proportionality of $\boldsymbol{u}_0, \boldsymbol{v}_0$ is easily checked by specialising $(\boldsymbol{a},\boldsymbol{b})$ to (l_p, l_q). The invariance (*mod* $[\boldsymbol{a}, \boldsymbol{b}]$) under (3.136) is immediate. As for the divisor associated with $\boldsymbol{w}_0 = [\boldsymbol{u}_0, \boldsymbol{v}_0]$, a tedious but straightforward calculation yields:

$$D(p, q; t) = 12\, p^2\, q^2\, t\, (p - q)^2\, (t + 6\, p + 6\, q)\, (t + p - q)\, (t - p + q) \quad (3.139)$$

so that $D(p, q; t) \neq 0$ for $p \neq q$ and $t \in p\,\mathbb{N} + q\,\mathbb{N}$ $(p, q, t > 0)$. The same holds for the divisor of $\boldsymbol{r}_0$, which is simply three times that of $\boldsymbol{w}_0$. That leaves only the $\boldsymbol{J}$-invariance, which as usual is the easiest part to check. Indeed, from (3.135) we get:

$$\boldsymbol{R}^{\boldsymbol{J}} = \boldsymbol{W}^{-1}\, \boldsymbol{R}\, \boldsymbol{W} \quad (3.140)$$

□

Remark. — Although we have postponed dealing with twins with equal tangency orders ($p = q$) until the "systematic" investigation of §5, it may be noted that in the above Example 3.14 (as also in Example 3.18 below) the action of $\boldsymbol{J}$ and $\boldsymbol{J}^2$ exchanges both types of twins (i.e. $p \neq q$ and $p = q$).

Example 3.15 (Invariance under $\boldsymbol{K}_1 : (\boldsymbol{A}, \boldsymbol{B}) \mapsto (\boldsymbol{B}^{-1}, \boldsymbol{A})$) . — If we set successively:

$$\boldsymbol{W} := \{\overline{\overline{\boldsymbol{B}}}^{\,2}\, \overline{\overline{\boldsymbol{A}}}^{\,3}\, \boldsymbol{B}, \overline{\overline{\boldsymbol{A}}}^{\,2}\, \overline{\overline{\boldsymbol{B}}}^{\,3}\, \boldsymbol{A}\} \quad (3.141)$$

$$\boldsymbol{Q} := \{\overline{\overline{\boldsymbol{A}}}^{\,3}\, \boldsymbol{B}, \overline{\overline{\boldsymbol{B}}}^{\,3}\, \boldsymbol{A}\} \quad (3.142)$$

$$\boldsymbol{W}_1 := \{\boldsymbol{Q}, \boldsymbol{W}\} \quad (3.143)$$

$$\boldsymbol{R} := \boldsymbol{W}_1, \boldsymbol{W}_1^{\boldsymbol{K}_1}\, \boldsymbol{W}_1^{\boldsymbol{K}_1^2}\, \boldsymbol{W}_1^{\boldsymbol{K}_1^3} \quad (3.144)$$

then the $\boldsymbol{K}_1$-invariant relation $R(F, G) \equiv 1$ has a general twin-solution that depends on p/q and c.

Proof. — Since $\boldsymbol{K}_1^4 = 1$, (3.144) gives $\boldsymbol{R}^{\boldsymbol{K}_1} \equiv \boldsymbol{W}_1^{-1}\, \boldsymbol{R} \boldsymbol{W}_1$ which takes care of the invariance. Up to the innocuous factor:

$$T_{6\,p+6\,q+t,4\,p+4\,q}\, T_{p,q,q,q}\, T_{q,p,p,p} \equiv -(2\,p+2\,q+t)\, p\, q\, (p-q)^2\, (p+q)^2 \quad (3.145)$$

the divisor associated with $\boldsymbol{W}_1$ coincides with the divisor of $\boldsymbol{W}$, which we already encountered in Example 3.9 and found (see (3.114)) to be non-vanishing.

The reason for bracketing $\boldsymbol{W}$ with $\boldsymbol{Q}$ is of course a question of parity: whereas $\boldsymbol{W}$ and $\boldsymbol{W}^{\boldsymbol{K}_1}$ have opposite corner components (since these are of even degree 6 in both $\boldsymbol{a}$ and $\boldsymbol{b}$, and so change signs under $(\boldsymbol{a},\boldsymbol{b}) \mapsto (-\boldsymbol{b},\boldsymbol{a})$), the new words $\boldsymbol{W_1}$ and $\boldsymbol{W}_1^{\boldsymbol{K}_1}$ have identical corner components:

$$\boldsymbol{W}_0^{\boldsymbol{K}_1} = -\boldsymbol{W}_0 \; ; \; \boldsymbol{q}_0^{\boldsymbol{K}_1} = -\boldsymbol{q}_0 \; ; \; \boldsymbol{W}_{10}^{\boldsymbol{K}_1} = \boldsymbol{W}_{10} \tag{3.146}$$

so that the corner components (and divisors) of all four factors in (3.144) merely add up. □

Example 3.16 (Invariance under $\boldsymbol{Aut}_1 = \{\boldsymbol{I},\boldsymbol{I}_0,\boldsymbol{I}_1,\boldsymbol{I}_2\}$) . — Take any twin-generating relation $\boldsymbol{W} = \{\boldsymbol{U},\boldsymbol{V}\}$ as in Example 3.5 or 3.9 and set $\boldsymbol{W}_1 := \{\boldsymbol{B},\{\boldsymbol{A},\boldsymbol{W}\}\}$. Then the $\boldsymbol{Aut}_1$-invariant relation:

$$\boldsymbol{R} := \boldsymbol{W}_1\,(\boldsymbol{W}_1^{\boldsymbol{I}_1})^{-1}\,\boldsymbol{W}_1^{\boldsymbol{I}_0}\,(\boldsymbol{W}_1^{\boldsymbol{I}_2})^{-1} \tag{3.147}$$

has a general twin solution that depends on p/q and c.

Proof. — The invariance follows from the relations:

$$\boldsymbol{R}^{\boldsymbol{I}_0} \equiv \boldsymbol{Q}_{\boldsymbol{I}_0}\,\boldsymbol{R}\,\boldsymbol{Q}_{\boldsymbol{I}_0}^{-1} \qquad \text{with} \qquad \boldsymbol{Q}_{\boldsymbol{I}_0} = \boldsymbol{W}_1\,(\boldsymbol{W}_1^{\boldsymbol{I}_1})^{-1} \tag{3.148}$$

$$\boldsymbol{R}^{\boldsymbol{I}_1} \equiv \boldsymbol{Q}_{\boldsymbol{I}_1}\,\boldsymbol{R}^{-1}\,\boldsymbol{Q}_{\boldsymbol{I}_1}^{-1} \quad \text{with} \quad \boldsymbol{Q}_{\boldsymbol{I}_1} = \boldsymbol{W}_1^{\boldsymbol{I}_2}\,(\boldsymbol{W}_1^{\boldsymbol{I}_0})^{-1} \tag{3.149}$$

$$\boldsymbol{R}^{\boldsymbol{I}_2} \equiv \boldsymbol{R}^{-1} \tag{3.150}$$

On the other hand, $\boldsymbol{W}$ automatically has alternance (even,even) in ($\boldsymbol{A}$,$\boldsymbol{B}$), but $\boldsymbol{W}_1$ has alternance (odd,odd), and a divisor which differs from that of $\boldsymbol{W}$ by the trivial factor $T_{2\,d\,p+2\,d\,q+t,p,q}$ (d being the global alternance of $\boldsymbol{U}$ or $\boldsymbol{V}$). As a consequence, all four factors in (3.147) contribute the same corner component and the same divisor. □

Example 3.17 (Invariance under $\boldsymbol{Aut}_2 = \{\boldsymbol{I},\boldsymbol{I}_0,\boldsymbol{S}_1,\boldsymbol{S}_2\}$) . — Take any $\boldsymbol{U} = \boldsymbol{U}(\boldsymbol{A},\boldsymbol{B})$ that has symmetrical alternance (d_1,d_2), $d_1 = d_2$, and yields a twin-generating relation $\boldsymbol{W} := \{\boldsymbol{U},\boldsymbol{U}^{\boldsymbol{S}_1}\}$. For instance, take $\boldsymbol{U} = \overline{\overline{\boldsymbol{A}}}^{2}\,\overline{\overline{\boldsymbol{B}}}^{3}\,\boldsymbol{A}$ as in (3.113). Then the $\boldsymbol{Aut}_2$-invariant relation $\boldsymbol{R} := \boldsymbol{W}^{\boldsymbol{I}_0}\,\boldsymbol{W} = 1$ has a general twin-solution that depends on p/q and c.

Proof. — The invariance follows from:

$$\boldsymbol{R}^{\boldsymbol{S}_1} = \boldsymbol{W}\,\boldsymbol{R}^{-1}\,\boldsymbol{W}^{-1} \; ; \; \boldsymbol{R}^{\boldsymbol{S}_2} = \boldsymbol{R}^{-1} \; ; \; \boldsymbol{R}^{\boldsymbol{I}_0} = \boldsymbol{W}\,\boldsymbol{R}\,\boldsymbol{W}^{-1} \tag{3.151}$$

Moreover, since $\boldsymbol{W}$ has an even (global) alternance $2\,(d_1+d_2) = 4\,d_1 = 4\,d_2$, the two factors $\boldsymbol{W}$ and $\boldsymbol{W}^{\boldsymbol{I}_0}$ which make up $\boldsymbol{R}$ contribute the same corner component and the same (non-vanishing) divisor – namely (3.114) if we take $\boldsymbol{U}$ as in (3.113). □

Example 3.18 (Invariance under $\boldsymbol{Aut}_3 = \{\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{J}^2, \boldsymbol{S}_1, \boldsymbol{S}_1\,\boldsymbol{J}, \boldsymbol{J}\,\boldsymbol{S}_1\}$). — We take the same word $\boldsymbol{W}$ of alternance $(10, 10)$ as in Example 3.14, but this time we set:

$$\boldsymbol{W}_1 = \{\{\boldsymbol{A}, \boldsymbol{B}\}, \boldsymbol{W}\} \tag{3.152}$$

$$\boldsymbol{W}_2 = \boldsymbol{W}_1\,(\boldsymbol{W_1}^{\boldsymbol{S}_1})^{-1} = \{\{\boldsymbol{A}, \boldsymbol{B}\}, \boldsymbol{W}\}\,\{\boldsymbol{W}^{\boldsymbol{S}_1}, \{\boldsymbol{B}, \boldsymbol{A}\}\} \tag{3.153}$$

$$\boldsymbol{R} = \boldsymbol{W}_2\,\boldsymbol{W}_2^{\boldsymbol{J}}\,\boldsymbol{W}_2^{\boldsymbol{J}^2} \tag{3.154}$$

Then the $\boldsymbol{Aut}_3$-invariant relation $\boldsymbol{R} = 1$ has a general twin-solution that depends on p/q and c.

Proof. — The whole point of bracketing $\boldsymbol{W}$ with $\{\boldsymbol{A}, \boldsymbol{B}\}$ is of course to get a word $\boldsymbol{W}_1$ with the same divisor as $\boldsymbol{W}$ — upto the trivial factor $T_{q,p}\,T_{10\,p+10\,q+t,p+q}$ — but such that $\boldsymbol{W}_1$ and $(\boldsymbol{W_1}^{\boldsymbol{S}_1})^{-1}$ have the same corner components (rather than opposite ones as in the case of $\boldsymbol{W}$). This yields a $\boldsymbol{W}_2$ verifying $\boldsymbol{W}_2^{\boldsymbol{S}_1} = \boldsymbol{W}_2^{-1}$ and with twice the corner component of $\boldsymbol{W}_1$. Moreover since the corner components of $\{\boldsymbol{A}, \boldsymbol{B}\}$ and $\boldsymbol{W}$ are invariant under the action (3.136) of $\boldsymbol{J}$, so too are those of $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$. The upshot is that the corner component *and* divisor of $\boldsymbol{R}$ are exactly six times those of $\boldsymbol{W}_1$, which in turn are essentially the same as those of $\boldsymbol{W}$.

As for the invariance, it is enough to check it for two generators of $\boldsymbol{Aut}_3$, e.g. $\boldsymbol{J}$ and $\boldsymbol{S}_1$. Due to $\boldsymbol{W}_2^{\boldsymbol{S}_1} = \boldsymbol{W}_2^{-1}$, the definition of $\boldsymbol{R}$ yields:

$$\boldsymbol{R}^{\boldsymbol{J}} = \boldsymbol{W}_2^{-1}\,\boldsymbol{R}\,\boldsymbol{W}_2 \quad ; \quad \boldsymbol{R}^{\boldsymbol{S}_1} = \boldsymbol{W}_2^{-1}\,\boldsymbol{R}^{-1}\,\boldsymbol{W}_2 \tag{3.155}$$

□

Example 3.19 (Invariance under $\boldsymbol{Aut}_4 = \{\boldsymbol{I}, \boldsymbol{I}_0, \boldsymbol{I}_1, \boldsymbol{I}_2, \boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{K}_1, \boldsymbol{K}_2\}$). — If we take a word $\boldsymbol{W} = \boldsymbol{W}(\boldsymbol{A}, \boldsymbol{B})$ such that:

(*) $\boldsymbol{W}$ be twin-generating

(**) $\boldsymbol{W}^{\boldsymbol{S}_1} = \boldsymbol{W}^{-1}$

(***) $\boldsymbol{W}$ be of alternance (d_1, d_2) in $(\boldsymbol{A},\boldsymbol{B})$ with $d_1 = d_2 = odd$

and if we set:

$$\boldsymbol{R} := \boldsymbol{W}\,\boldsymbol{W}^{\boldsymbol{K}_1}\,\boldsymbol{W}^{\boldsymbol{I}_0}\,\boldsymbol{W}^{\boldsymbol{K}_2} = \boldsymbol{W}\,\boldsymbol{W}^{\boldsymbol{K}_1}\,\boldsymbol{W}^{\boldsymbol{K}_1^2}\,\boldsymbol{W}^{\boldsymbol{K}_1^3} \tag{3.156}$$

then the $\boldsymbol{Aut}_4$-invariant relation $\boldsymbol{R} = 1$ has a general twin solution that depends on p/q and c. For instance, a suitable choice for $\boldsymbol{W}$ is given by the following steps:

$$\boldsymbol{W}_1 = \{\boldsymbol{U}_1, \boldsymbol{U}_1^{\boldsymbol{S}_1}\} \quad \text{and} \quad \boldsymbol{U}_1 = \overline{\overline{\boldsymbol{A}}}^2\,\overline{\overline{\boldsymbol{B}}}^3\,\boldsymbol{A} \tag{3.157}$$

$$\boldsymbol{W}_2 = \{\boldsymbol{U}_2, \boldsymbol{U}_2^{\boldsymbol{S}_1}\} \quad \text{and} \quad \boldsymbol{U}_2 = \boldsymbol{A} \tag{3.158}$$

$$\boldsymbol{W}_3 = \{\boldsymbol{U}_3, \boldsymbol{U}_3^{\boldsymbol{S}_1}\} \quad \text{and} \quad \boldsymbol{U}_3 = \overline{\overline{\boldsymbol{B}}}^3\,\boldsymbol{A} \tag{3.159}$$

$$\boldsymbol{W}_4 = \{\boldsymbol{W}_3\{\boldsymbol{W}_2, \boldsymbol{W}_1\}\} \tag{3.160}$$

$$\boldsymbol{W} = \boldsymbol{W}_4\,(\boldsymbol{W}_4^{\boldsymbol{S}_1})^{-1} \tag{3.161}$$

Proof. — As soon as $\boldsymbol{W}$ verifies (*) and (**), the relation $\boldsymbol{R}$ defined by (3.156) verifies:

$$\boldsymbol{R}^{\boldsymbol{K}_1} = \boldsymbol{W}^{-1}\,\boldsymbol{R}\,\boldsymbol{W} \quad ; \quad \boldsymbol{R}^{\boldsymbol{S}_1} = \boldsymbol{W}^{-1}\,\boldsymbol{R}^{-1}\,\boldsymbol{W} \tag{3.162}$$

which ensures the invariance, since $\boldsymbol{K}_1$ and $\boldsymbol{S}_1$ generate $\boldsymbol{Aut}_4$. Moreover, the imparity condition (***) makes sure that the corner components of the four factors in (3.156) add up rather than self-destruct.

But the existence of words $\boldsymbol{W}$ with these properties is far from obvious. So let us check the suitability of the particular $\boldsymbol{W}$ mentioned above. We observe that the divisor $D_1(p,q;t)$ of $\boldsymbol{W}_1$ is given by (3.114) and that the divisor $D(p,q;t)$ of $\boldsymbol{W}$ is essentially the same:

$$D(p,q;t) = 2\,p\,q\,(p-q)^4\,(p+q)^2\,(3\,p+3\,q+t)\,(3\,p+5\,q+t)\,D_1(p,q;t) \tag{3.163}$$

with an innocuous front factor which merely reflects the bracketing of $\boldsymbol{W}_1$ by $\boldsymbol{W}_2$ and $\boldsymbol{W}_3$, and is therefore equal to:

$$T_{6\,p+6\,q+t,p+q,4\,p+4\,q}\,T_{p,q,q,q}\,T_{q,p,p,p}\,T_{q,p}\,T_{3\,p+q,3\,q+p} \tag{3.164}$$

It is natural that $\boldsymbol{W}_2$ and $\boldsymbol{W}_3$ should contribute only trivial factors, since neither $\boldsymbol{U}_2$ nor $\boldsymbol{U}_3$ have equal alternance in $\boldsymbol{A}$ and $\boldsymbol{B}$, so that for $j = 1,2$ the corner component of $\{\boldsymbol{U}_j, \boldsymbol{U}_j^{S_1}\}$ cannot vanish identically under *all* realisations $(\boldsymbol{a},\boldsymbol{b}) \mapsto (l_p, l_q)$. $\square$

Despite its forbidding complexity, this example is the ‘simplest of its kind’, at least as long as we insist on working with multicommutators and demand a certain symmetry in $\boldsymbol{A}$ and $\boldsymbol{B}$. But if we drop these requirements, we may produce a slightly simpler example:

Example 3.20 (Invariance under $\boldsymbol{Aut}_4$: simpler example) . — If we set:

$$\boldsymbol{U} := \{(\overline{\overline{\boldsymbol{A}}}^{\,3}\,\boldsymbol{B}\,\{(\overline{\overline{\boldsymbol{A}}}\,\boldsymbol{B}), \boldsymbol{B}^{-1}\}\} \tag{3.165}$$

$$\boldsymbol{V} := \{(\overline{\overline{\boldsymbol{A}}}^{\,2}\,\boldsymbol{B}\,\{(\overline{\overline{\boldsymbol{A}}}^{\,2}\,\boldsymbol{B}), \overline{\overline{\boldsymbol{A}}}(\boldsymbol{B}^{2})\}\} \tag{3.166}$$

$$\boldsymbol{W}_1 := \boldsymbol{U}\,\boldsymbol{V} \tag{3.167}$$

$$\boldsymbol{W}_2 := \overline{\overline{\boldsymbol{B}}}^{\,2}\,\boldsymbol{W}_1 \tag{3.168}$$

$$\boldsymbol{W} := \boldsymbol{W}_2\,(\boldsymbol{W}_2^{\boldsymbol{S}_1})^{-1} \tag{3.169}$$

$$\boldsymbol{R} := \boldsymbol{W}\,\boldsymbol{W}^{\boldsymbol{K}_1}\,\boldsymbol{W}^{I_0}\,\boldsymbol{W}^{\boldsymbol{K}_2} \tag{3.170}$$

then the $\boldsymbol{Aut}_4$-invariant relation $\boldsymbol{R} = 1$ has a general twin solution that depends on p/q and c.

Proof. — It is enough to check that the new $\boldsymbol{W}$ still meets all three conditions (*) (**) (***) of Example 3.19. Clearly, if we set $\boldsymbol{b}_n := \overline{\boldsymbol{a}}^n\,\boldsymbol{b}$, we find for the corner components of $\boldsymbol{U}$, $\boldsymbol{V}$, $\boldsymbol{W_1}$, $\boldsymbol{W_2}$:

$$\boldsymbol{u}_0 = -\overline{\boldsymbol{b}}_3\,\overline{\boldsymbol{b}}_2\,\boldsymbol{b} \quad ; \quad \boldsymbol{v}_0 = 2\,\overline{\boldsymbol{b}}_2\,\overline{\boldsymbol{b}}_2\,\boldsymbol{b}_1 \tag{3.171}$$

$$\boldsymbol{w}_{10} = \boldsymbol{u_0} + \boldsymbol{v_0} \quad ; \quad \boldsymbol{w}_{20} = \overline{\boldsymbol{b}}\,\overline{\boldsymbol{b}}\,(\boldsymbol{u_0} + \boldsymbol{v_0}) \tag{3.172}$$

Next, we check that $\boldsymbol{w}_{10}$ (and so $\boldsymbol{w}_{20}$) vanish under *all* realisations $(\boldsymbol{a}, \boldsymbol{b}) \mapsto (l_p, l_q)$. Then we calculate the divisor attached to $\boldsymbol{W}_1$:

$$D_1(p,q;t) = -(p-q)^2\,q\,t^2\,(t-p+q) \tag{3.173}$$

and the one attached to $\boldsymbol{W}_2$:

$$\begin{aligned} D_2(p,q;t) &\equiv T_{3\,p+5\,q+t,q,q}\,D_1(p,q;t) \\ &\equiv (t+3\,p+4\,q)\,(t+3\,p+5\,q)\,D_1(p,q;t) \end{aligned} \tag{3.174}$$

From that we infer the divisor attached to $\boldsymbol{W} = \boldsymbol{W}_2\,(\boldsymbol{W}_2^{\boldsymbol{S}_1})^{-1}$ (beware of the + sign):

$$D(p,q;t) = D_2(p,q;t) + D_2(q,p;t) = -(p-q)^2\,t^2\,\Delta(p,q;t) \tag{3.175}$$

with

$$\begin{aligned} \Delta(p,q;t) = {}& (p+q)\,t^3 + (10\,p^2 + 10\,p\,q + 10\,q^2)\,t^2 \\ & +(29\,p^3 + 27\,p^2\,q + 27\,p\,q^2 + 29\,q^3)\,t \\ & +(p-q)^2\,(20\,p^2 + 38\,p\,q + 20\,q^2) \end{aligned} \tag{3.176}$$

Therefore neither Δ nor D vanish for $p \neq q$ and t in $p\,\mathbb{N}+q\,\mathbb{N}$ $(p,q,t>0)$. So $\boldsymbol{W}$ verifies (*) and (**). It also verifies (***), since it clearly has alternance $(5,5)$ in $(\boldsymbol{A}$,$\boldsymbol{B})$. $\square$

3.6. Further examples

Example 3.21 (Several unmovable parameters) . — Let $\boldsymbol{W}_0 := \boldsymbol{W} = \{\boldsymbol{U},\, \boldsymbol{V}\}$ be a twin-generating relation like in Examples 3.5 or 3.9, with alternance (m_0, n_0) in $(\boldsymbol{A}, \boldsymbol{B})$, and let $\boldsymbol{W}_j$, $(j = 1, \cdots, l)$ be l non-twin-generating relations with alternance (m_j, n_j), for instance:

$$\boldsymbol{W}_j := \overline{\overline{\boldsymbol{A}}}^{\,m_j-1}\ \overline{\overline{\boldsymbol{B}}}^{\,n_j}\ \ \boldsymbol{A} \tag{3.177}$$

Further, assume that m_j, n_j increase so fast that the integers μ_j, ν_j derived therefrom:

$$\begin{aligned} \mu_j \quad &:= \quad m_j - (m_0 + m_1 + \cdots m_{j-1}) \\ \nu_j \quad &:= \quad n_j - (n_0 + n_1 + \cdots n_{j-1}) \qquad (j = 1, 2, \cdots l) \end{aligned} \tag{3.178}$$

are themselves positive and strictly increasing. Then, if we set:

$$\boldsymbol{R}_0 := \boldsymbol{W}_0 \ \ ; \ \ \boldsymbol{R}_j := \{\boldsymbol{W}_j,\ \boldsymbol{R}_{j-1}\} \ \ (j = 1, \cdots, l) \tag{3.179}$$

the relation $R_l(F, G) = 1$ has a general twin solution which depends not only on the usual parameters p/q and $c = \alpha^{q_*}\, \beta^{-p_*}$, but also on l new parameters $c_1, \cdots, c_l$ which may assume any complex value, but are ‘unmovable’ in the sense that, for each of them, the place of ‘first occurence’ in F or G is within coefficients of a well-defined depth, namely $t = t_j := \mu_j\, p + \nu_j\, q$ for c_j.

Proof. — The divisor $D_l(p, q; t)$ of $\boldsymbol{R}_l$ is equal to the divisor $D(p, q; t)$ of $\boldsymbol{W}$ multiplied :

(i) by the trivial (i.e. t-independent) factors $T_{p,q^{(n_j)},p^{(m_j-1)}}$ contributed by the words $\boldsymbol{W}_j$ $(j = 1, \cdots, l)$.

(ii) by the elementary (i.e. t-affine) factors $(t - \mu_j\, p - \nu_j\, q)$ $(j = 1, \cdots, l)$ which we introduced by bracketing R_{j-1} with W_j.

Therefore $D_l(p, q; t_j) \equiv 0$ for $t_j := \mu_j\, p + \nu_j\, q$ $(j = 1, \cdots, l)$. Under normal circumstances, this should prevent the existence of twin-solutions, but here it has the opposite effect of enlarging their number, by introducing

l new free parameters c_j. Indeed, for any initial conditions (p, q) and (α, β), the relation $R_l(F, G) \equiv 1$ is equivalent to the system :

$$\log\ R_{j-1} \equiv c_j\ \log\ W_j\ \ (\text{mod. terms of depth } t_{j+1})\ \ \ (j = 1, \cdots, l) \quad (3.180)$$

which, for any given choice $(c_1, \cdots, c_l)$ in $\mathbb{C}^l$, clearly admits (upto conjugacy) a unique solution (F, G).

The coefficients of F, G with depth $t < t_1$ are calculated inductively exactly as if we were dealing with the sole relation $R_0(F, G) := W(F, G) = 1$. Then the coefficients with depth $t_1 \leqslant t \leqslant t_2$ are calculated from $\log R_0 = c_1 \log W_1$; those with depth $t_2 \leqslant t \leqslant t_3$ from $\log R_1 = c_2 \log W_2$; etc.; and lastly those of depth $t_l \leqslant t$ from $\log R_{l-1} = c_l\ \log W_l$.

Actually, the conclusion would remain unchanged if the (μ_j, ν_j), instead of forming an increasing sequence, were pairwise distinct *and* comparable (for the natural order on $\mathbb{N}^2$), while of course remaining positive. For non-comparable pairs, however, there would occur some slight changes, since the order of the sequence formed by the zeros t_j of the divisor's 'elementary factors' would depend on the tangency ratio p/q. $\square$

Example 3.22 (One movable parameter) . — Let $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ be two twin-generating relations of the type encountered in Example 3.12 (see (3.120–121)), with invariance under I_0:

$$\boldsymbol{R}_j(\boldsymbol{A}^{-1},\ \boldsymbol{B}^{-1}) = (\boldsymbol{R}_j(\boldsymbol{A},\ \boldsymbol{B}))^{-1}\ \ (j = 1, 2) \quad (3.181)$$

Then if we set:

$$\boldsymbol{R}(\boldsymbol{A},\ \boldsymbol{B}) := \boldsymbol{R}_2(\boldsymbol{A},\ \boldsymbol{R}_1(\boldsymbol{AB})) \quad (3.182)$$

the relation $R(F, G) = 1$ has a general twin-solution which, on top of the usual parameters p/q and $c = \alpha^{q_*}\, \beta^{-p_*}$, depends on two new parameters: one continuous parameter c_1, which may assume any complex value, and one discrete parameter r_1, which denotes the place (or 'depth') of first occurence of c_1, and may assume any entire value.

Proof. — Fix three pairwise distinct integers p, q, r_1. Then set $p_2 := p,\ q_2 := m_1 p + n_1 q + r_1$, where (m_1, n_1) denotes the alternance of $\boldsymbol{R}_1(\boldsymbol{A}, \boldsymbol{B})$ in $(\boldsymbol{A}, \boldsymbol{B})$. Upto conjugacy, the general solution of $R_1(F, G) = 1$ may be written in normal-conormal form (see (3.9) and (3.10)) as:

$$F = F^{nor} = \alpha\, x^{p+1} \partial\ \ ;\ \ G = G^{conor} = \beta\, (x^{q+1} + \cdots)\, \partial \quad (3.183)$$

Similarly, the general solution of $R_2(F, K) = 1$ may be written as:

$$F = F^{nor} = \alpha\, x^{p_2+1} \partial\ \ ;\ \ K = K^{conor} = \gamma\, (x^{q_2+1} + \cdots)\, \partial \quad (3.184)$$

with K^{conor} depending only on $(p_2, q_2) := (p, m_1 p + n_1 q + r_1)$ and (α, β), so that F^{nor} is the same in (3.183) and (3.184). $\square$

Example 3.23 (Twin glueing) . — Let $\mathbf{R_1}(\mathbf{A}, \mathbf{B}) \ldots, \mathbf{R_r}(\mathbf{A}, \mathbf{B})$ be distinct twin-generating relations of the type encountered before, and let $\mathbf{R_0}(\mathbf{A}, \mathbf{B}) := \overline{\overline{\mathbf{A}}}^{\mathbf{n_0}} \, \overline{\overline{\mathbf{B}}}^{\mathbf{m_0}} \mathbf{A}$ be a multicommutator of very high alternance $(\mathrm{n_0}, \mathrm{m_0})$. Then for each $\mathrm{i} = 1, \ldots, \mathrm{r}$, the relation:

$$\mathbf{R}(\mathbf{A}, \mathbf{B}) := \mathbf{R_0}(\mathbf{A}, \mathbf{B}) \, \{..\{\mathbf{R_1}(\mathbf{A}, \mathbf{B}), \mathbf{R_2}(\mathbf{A}, \mathbf{B})\} \ldots \mathbf{R_r}(\mathbf{A}, \mathbf{B})\} \tag{3.185}$$

has a twin-solution which coincides upto high order (but not exactly) with the twin solution of $R_i(A, B)$.

Proof. — This is simply because for each i the corresponding *divisor* coincides, upto a non-vanishing factor, with the divisor $D_i(t)$ associated with the isolated equation $\mathbf{R_i}(\mathbf{A}, \mathbf{B})$. $\square$

The *separateness* of the i solutions (each with its own parameters) stands in sharp contrast to the *connectednes* of the solutions of what would be the differential equation analogue:

$$\underline{\mathbf{r}}(\mathbf{a}, \mathbf{b}) := \underline{\mathbf{r}}_{\mathbf{0}}(\mathbf{a}, \mathbf{b}) \; + [..[\underline{\mathbf{r}}_{\mathbf{1}}(\mathbf{a}, \mathbf{b}), \underline{\mathbf{r}}_{\mathbf{2}}(\mathbf{a}, \mathbf{b})] \ldots \underline{\mathbf{r}}_{\mathbf{r}}(\mathbf{a}, \mathbf{b})] \tag{3.186}$$

with $\underline{\mathbf{r}}_{\mathbf{i}}(\mathbf{a}, \mathbf{b})$ denoting the lowest-degree homogeneous component in the series $\mathbf{r_i}(\mathbf{a}, \mathbf{b}) := \log \mathbf{R_i}(\mathbf{e^a}, \mathbf{e^b})$ and with $\mathbf{a} = \varphi \, \partial$, $\mathbf{b} = \psi \, \partial$ for, say, φ given and ψ unknown. Here too, each 'factor' $\underline{\mathbf{r}}_{\mathbf{i}}(\mathbf{a}, \mathbf{b})$ contributes its own parameters, but the multibracket on the right-hand side of (3.186) introduces $r - 1$ additional parameters which have the effect of *connecting* (under continuous deformations) the separate solutions.

Thus, whereas the solutions of differential equations may be seamlessly *'welded'* together, the solutions of composition equations can only be *'glued'* (with only a weak interaction stemming from the perturbation $\mathbf{R_0}$). This reflects a very basic difference between the two classes of problems. Nor is it due to the fact that we are solving our (differential or composition) equations in rings of powers series: the difference persists, undiminished, when we go over to transserial solutions (see §8 *infra*).

4. Some Lie theory. Active/passive subalgebras. Divisors and universal kernels

4.1. The active/passive filtration of $\mathbf{Lie}[\mathbf{a},\mathbf{b}]$.

The twenty examples of twins reviewed in the previous section are easy to construct but somewhat atypical in so far as all of them verify relations made up of suitably arranged multi-commutators. Before turning to the description of truly 'generic' twins, we must insert a section devoted to the natural *filtrations* that arise on *free algebras* when they get *represented* as one-variable *differential algebras.*

Let us for simplicity deal with the two-generator algebra $\mathbf{Lie}[\mathbf{a},\mathbf{b}]$. It admits a natural sequence of decreasing ideals: [5]

$$\mathbf{Lie}[\mathbf{a},\mathbf{b}] = \mathbf{Uker}_0 \supset \mathbf{Uker}_1 \supset \mathbf{Uker}_2 \supset \ldots \mathbf{Uker}_\infty = \mathbf{Pass} \qquad (4.1)$$

$$[\mathbf{Uker_i},\mathbf{Uker_j}] \subset \mathbf{Uker_{i+j}} \qquad \forall i,j \quad (0 \leqslant i,j \leqslant \infty) \qquad (4.2)$$

The k-th ideal $\mathbf{Uker_k}$ is defined as consisting of all $\mathbf{w}(\mathbf{a},\mathbf{b}) \in \mathbf{Lie}[\mathbf{a},\mathbf{b}]$ which vanish up to order k:

$$\{\mathbf{w} \in \mathbf{Uker_k}\} \quad \Longleftrightarrow \quad \{\mathbf{w}(F_*,G_*) = \mathcal{O}(\epsilon^k) \quad \forall p,q,\varphi,\psi\} \qquad (4.3)$$

whenever the pair $(\mathbf{a},\mathbf{b})$ gets replaced by a pair (F_*,G_*) of the form:

$$F_* := \big(x^p + \epsilon\,\varphi(x)\big)\,x\,\partial_x \quad , \quad G_* := \big(x^q + \epsilon\,\psi(x)\big)\,x\,\partial_x \qquad (4.4)$$

or, what amounts to the same[6], of either form:

$$\begin{array}{llll} F_* := & \partial_z & , \quad G_* := & \big(z^{-q} + \epsilon\,\Psi(z)\big)\,z\,\partial_z \\ F_* := & \big(z^{-p} + \epsilon\,\Phi(z)\big)\,z\,\partial_z & , \quad G_* := & \partial_z \end{array}$$

Remark 4.1 . — The reason for limiting ourselves to pairs (F_*,G_*) consisting of perturbed *monomials* is of course that we are mostly concerned with twins in the ring of power series. But even when investigating *transserial* twins (see §8), the ordinary monomials retain their primacy, due to the fact that they alone, of all *transmonomials*, enjoy a double stability: under multiplication *and* composition.

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(5) with *Uker* standing for *universal kernel.*
(6) under a suitable variable change $x = h_\epsilon(z)$

The ideal $\mathbf{Uker_0}$ coincides with the whole algebra $\mathbf{Lie}[\mathbf{a},\mathbf{b}]$, while the ideal $\mathbf{Uker}_\infty$ consists of all elements $\mathbf{w}$ that vanish identically in all one-variable differential representations of $\mathbf{Lie}[\mathbf{a},\mathbf{b}]$. For our present purposes they are as if non-existent, and deserve to be regarded as constituting the 'passive' subalgebra **Pass**. What matters is the 'active' algebra **Act**, i.e. the quotient of $\mathbf{Lie}[\mathbf{a},\mathbf{b}]$ by **Pass**.

Actually, we shall require the full filtration:

$$\mathbf{Act} \simeq \mathbf{Act}_0 \oplus \mathbf{Act_1} \oplus \mathbf{Act}_2 \ldots, \qquad \mathbf{Act}_i = \mathbf{Uker}_i/\mathbf{Uker}_{i+1} \quad (4.5)$$

$$\mathbf{Act} \simeq \mathbf{Act}_0 \oplus \mathbf{Act}_*, \qquad \mathbf{Act}_* = \mathbf{Uker}_1/\mathbf{Uker}_\infty \quad (4.6)$$

Of special importance is the leading active algebra $\mathbf{Act_0} := \mathbf{Lie}/\mathbf{Uker_1}$. Its (homogeneous) elements $\mathbf{w}(\mathbf{a},\mathbf{b})$ vanish in all 'monomial' representations, but possess non-vanishing *divisors* $D(t) := D_\mathbf{w}(t;p,q)$. [7]

Remark 4.2 (from filtration to gradation). — In order to turn the natural *filtration* (4.1) into a *gradation* of type (4.5), one would have to actually embed each $\mathbf{Act}_i$ into $\mathbf{Lie}[\mathbf{a},\mathbf{b}]$. The most natural way of accomplishing this is by ruling that all the embedded components $\mathbf{Act}_i$ should be orthogonal to one another *and* to **Pass**, with respect to the natural scalar product on $\mathbf{Lie}[\mathbf{a},\mathbf{b}]$ [8]. This again would lead to worthwhile developments which, however, would be a distraction from our present investigation.

Next comes the question of the *dimensions*, which makes sense only inside specified homogeneous components $\mathbf{Lie}[\mathbf{a},\mathbf{b}]^{(d_1,d_2)}$.

For any given degree (d_1,d_2), the components $\mathbf{Uker}_i^{(d_1,d_2)}$ clearly become stationary after a certain critical $i := i_{crit}(d_1,d_2)$. This stationary ideal component coincides with $\mathbf{Uker}_\infty^{(d_1,d_2)}$. Of course the corresponding components $\mathbf{Act}_i^{(d_1,d_2)}$ of the active algebra turn empty.

For any given index i, the dimensions $dim(\mathbf{Uker}_i^{(d_1,d_2)})$ and also, less obviously, $dim(\mathbf{Act}_i^{(d_1,d_2)})$, are non-decreasing (but non-convex) functions of d_1 and d_2.

Remark 4.3 . — The spaces $[\mathbf{Uker}_{i'}^{(d'_1,d'_2)}, \mathbf{Uker}_{i''}^{(d''_1,d''_2)}]$ are clearly (strict) subspaces of $\mathbf{Uker}_{i'+i''}^{(d'_1+d''_1,d'_2+d''_2)}$. Taking into account the corresponding quo-

(7) We have already encountered many instances of *divisors*. For a general definition, valid even in the case of more than two generators $\mathbf{a},\mathbf{b},\mathbf{c}\ldots$, see §6.6.

(8) ie the one induced by the natural scalar product on the enveloping free associative algebra $\mathbf{Ass}(\mathbf{a},\mathbf{b})$

tients[9] $\mathbf{Act}_{(i',i'')}^{(d_1',d_1''),(d_2',d_2'')}$ would lead to a considerable refinement of our active/passive filtration, but we need not go into that here.

Before establishing the exact formulas for the main dimensions, we adduce two tables.

The first table extends to all degrees $(d_1, d_2) \leqslant (15, 10)$. In fact, in view of the interchangeability of d_1 and d_2, it covers all cases $(d_1, d_2) \leqslant (15, 15)$. The entries are three-number columns:

$$\begin{array}{lcll} top & : & \alpha & = dim(\mathbf{Act_0}^{(d_1,d_2)}) \\ middle & : & \beta & = dim(\mathbf{Act_*}^{(d_1,d_2)}) \\ bottom & : & \gamma & = dim(\mathbf{Pass_0}^{(d_1,d_2)}) \end{array}$$

When $\mathbf{Act_*}^{(d_1,d_2)}$ doesn't reduce to its first component $\mathbf{Act_1}^{(d_1,d_2)}$, the number β is entered as $^*\beta$.

The second table extends to all degrees $(4, 4) \leqslant (d_1, d_2) \leqslant (14, 14)$ and gives the exact decomposition $\beta = \beta_1 + \beta_2 + \beta_3 + \ldots$:

$$\begin{array}{lclll} top & : & \beta & = dim(\mathbf{Act_*}^{(d_1,d_2)}) & \textit{written as } {}^*\beta \textit{ when } \beta \neq \beta_1 \\ lower & : & \beta_1 & = dim(\mathbf{Act_1}^{(d_1,d_2)}) & \\ lower & : & \beta_2 & = dim(\mathbf{Act_2}^{(d_1,d_2)}) & \\ lower & : & \beta_3 & = dim(\mathbf{Act_3}^{(d_1,d_2)}) & \\ \ldots & & & & \end{array}$$

Actually, the calculations are rather costly and we did not reach the point where β_3 becomes > 0, although this certainly takes place within the limits of the table. The uncalculated numbers have been replaced by dots.[10]

(9) which may be joined to form Lie algebras.
(10) Entries for d_1 or $d_2 < 4$ were left out since for them $\beta = \beta_1$ and $0 = \beta_2 = \beta_3 \ldots$.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	1 0 0	1 0 0	1 0 0	1 0 0	1 0 0	1 0 0	1 0 0	1 0 0	1 0 0	1 0 0
2	1 0 0	1 0 0	2 0 0	2 0 0	3 0 0	3 0 0	4 0 0	4 0 0	5 0 0	5 0 0
3	1 0 0	2 0 0	3 0 0	4 1 0	4 2 1	5 3 1	6 4 2	6 6 3	7 7 4	8 9 5
4	1 0 0	2 0 0	4 1 0	5 2 1	6 5 3	6 8 6	7 12 11	8 15 17	9 21 25	9 26 35
5	1 0 0	3 0 0	4 2 1	6 5 3	7 9 9	8 15 19	8 22 36	9 *31 59	10 *40 93	11 *52 137
6	1 0 0	3 0 0	5 3 1	6 8 6	8 15 19	9 *23 43	10 *36 86	10 *49 153	11 *67 255	12 *86 399
7	1 0 0	4 0 0	6 4 2	7 12 11	8 22 36	10 *36 86	11 *52 182	12 *74 343	12 *100 603	13 *132 999
8	1 0 0	4 0 0	6 6 3	8 15 17	9 *31 59	10 *49 153	12 *74 343	13 *103 684	14 *142 1274	14 *186 2224
9	1 0 0	5 0 0	7 7 4	9 21 25	10 *40 93	11 *67 255	12 *100 603	14 *142 1274	15 *192 2493	16 *257 4589
10	1 0 0	5 0 0	8 9 5	9 26 35	11 *52 137	12 *86 399	13 *132 999	14 *186 2224	16 *257 4589	17 *339 8869
11	1 0 0	6 0 0	8 11 7	10 33 48	12 *65 196	13 *111 604	14 *168 1586	15 *243 3720	16 *332 8050	18 *445 16333
12	1 0 0	6 0 0	9 13 8	11 39 62	12 *82 270	14 *137 875	15 *214 2423	16 *306 5966	17 *426 13552	18 *568 28786
13	1 0 0	7 0 0	10 15 10	12 *48 80	13 *99 364	15 *172 1241	16 *265 3595	17 *387 9286	18 *535 22057	19 *721 49002
14	1 0 0	7 0 0	10 18 12	12 *56 100	14 *120 478	15 *208 1709	17 *328 5192	18 *477 14025	19 *668 34843	20 *897 80769
15	1 0 0	8 0 0	11 20 14	13 *67 124	15 *142 618	16 *254 2313	18 *399 7335	19 *589 20710	20 *822 53635	21 *1116 129613

	4	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
4	2 2 0	5 5 0	8 8 0	12 12 0	15 15 0	21 21 0	26 26 0	33 33 0	39 39 0	*48 47 1	*56 55 1
5	5 5 0	9 9 0	15 15 0	22 22 0	*31 29 2	*40 35 5	*52 43 9	*65 52 13	*82 63 19	*90 72 27	*120 83 37
6	8 8 0	15 20 0	*23 21 2	*36 30 6	*49 38 11	*67 47 ...	*86 56 ...	*111 66 ...	*137 76 ...	*172	*208 ... ...
7	12 12 0	22 22 0	*36 30 6	*52 37 ...	*74 48 ...	*100	*132	*168	*214	*265	*328 ... ...
8	15 15 0	*31 29 2	*49 38 11	*74 48 ...	*103	*142	*186	*243	*306	*387	*477 ... ...
9	21 21 0	*40 35 5	*67 47 ...	*100	*142	*192	*257	*332	*426	*535	*668 ... ...
10	26 26 0	*52 43 9	*86 56 ...	*132	*186	*257	*339	*445	*568	*721	*897 ... ...
11	33 33 0	*65 52 13	*111 66 ...	*168	*243	*332	*445	*579	*746	*943	*1185 ... ...
12	39 39 0	*82 63 19	*137 76 ...	*214	*306	*426	*568	*746	*956	*1219	*1525 ... ...
13	*48 47 1	*99 72 27	*172	*265	*387	*535	*721	*943	*1219	*1545	*1945 ... ...
14	*56 55 1	*120 83 37	*208	*328	*477	*668	*897	*1185	*1525	*1945	*2441 ... ...

4.2. Dimensions

Throughout this section, we shall use the following notations:

$$p(n) := \text{nb of partitions of } n \text{ with positive summands} \tag{4.7}$$

$$p^*(n) := 1 + p(1) + p(2) + \ldots + p(n) \tag{4.8}$$

$$p(n,m) := \text{nb of partitions of } n \text{ with } m \text{ non-negative summands} \tag{4.9}$$

$$:= \text{nb of partitions of } n \text{ with at most } m \text{ positive summands} \tag{4.10}$$

$$P(n,m) := \text{nb of partitions of } n \text{ with } m \text{ positive summands} \tag{4.11}$$

Clearly, $p\,(\bullet,\bullet)$ and $P(\bullet,\bullet)$ are expressible in terms of each other ;

$$p(n,m) \equiv P(n+m,m) \quad ; \quad P(n,m) \equiv p(n,m) - p(n,m-1) \tag{4.12}$$

Let us first get all the main statements and formulas out of the way (Prop. 4.4 through 4.7) and then proceed with the proofs.

PROPOSITION 4.4 (FULL ALGEBRA). — *The dimensions* $\mathrm{L}^{(\mathrm{d}_1,\mathrm{d}_2)} :=$ $\dim \mathbf{Lie}[\mathbf{a},\mathbf{b}]^{(d_1,d_2)}$ *are given by the classical formula:*

$$L^{(d_1,d_2)} = \frac{1}{d_1+d_2} \sum_{\delta|d_1\,,\,\delta|d_2} \mu(\delta)\ \frac{((d_1+d_2)/\delta)!}{(d_1/d)!\,(d_2/d)!} \tag{4.13}$$

with $\mu(.)$ as the Möbius function.

PROPOSITION 4.5 (FULL ACTIVE ALGEBRA). — *The dimensions* $\mathrm{D}^{(\mathrm{d}_1,\mathrm{d}_2)}$ $:= \dim \mathbf{Act}^{(d_1,d_2)} = \mathrm{codim}\, \mathbf{Uker}_\infty^{(\mathrm{d}_1,\mathrm{d}_2)}$ *are given by:*

$$\begin{aligned} D^{(d_1,d_2)} &= p(d_1+d_2-1,d_1)+p(d_1+d_2-1,d_2)-p(d_1{+}d_2{-}1) && (4.14)\\ &= P(2d_1+d_2-1,d_1)+P(d_1+2d_2-1,d_2)-p(d_1{+}d_2{-}1) && (4.15)\\ &= p(d_1+d_2-1,d_1)-p^*(d_1-1) \qquad (if\ \ d_2-d_1\geqslant -3) && (4.16)\\ &= p(d_1+d_2-1,d_2)-p^*(d_2-1) \qquad (if\ \ d_1-d_2\geqslant -3) && (4.17)\end{aligned}$$

PROPOSITION 4.6 (LEADING ACTIVE ALGEBRA). — *The dimensions* $\mathrm{D}_0^{(\mathrm{d}_1,\mathrm{d}_2)} := \dim \mathbf{Act}_0^{(d_1,d_2)} = \mathrm{codim}\, \mathbf{Uker}_0^{(d_1,d_2)}$ *are given by:*

$$D_0^{(d_1,d_2)} = 1 + E(\frac{(d_1-1)(d_2-1))}{d_1}) + E(\frac{(d_1-1)(d_2-1))}{d_2}) \tag{4.18}$$

where $E(x)$ denotes the entire part of the real number x.

PROPOSITION 4.7 (SECONDARY ACTIVE ALGEBRAS). — *For any fixed index* $\mathrm{i} \geqslant 0$ *and any fixed* d_1 *(resp* d_2*)* $\geqslant \sup(3,\mathrm{i}+1)$, *the dimensions* $\mathrm{D}_\mathrm{i}^{(\mathrm{d}_1,\mathrm{d}_2)} :=$ $\dim \mathbf{Act}_i^{(d_1,d_2)} = \dim \mathbf{Uker}_i^{(d_1,d_2)} - \dim \mathbf{Uker}_{i+1}^{(d_1,d_2)}$ *grow roughly like* $\mathrm{d}_2^{\mathrm{i}+1}$ *(resp* $\mathrm{d}_1^{\mathrm{i}+1}$*) times some constant.*

More precisely, though none of the generating functions

$$\textstyle\sum_{d_1,d_2}\ D_i^{(d_1,d_2)}\ x_1^{d_1}\ x_2^{d_2}$$

is rational, the partial sums $\sum_d\ D_i^{(d_0,d)}\ x^d = \sum_d\ D_i^{(d,d_0)}\ x^d$ are all rational, with denominators apparently of the form $(1-x^{i+1})(1-x^{(i+1)\,d_0})$.

4.3. Universal constraints

A non-increasing sequence $\underline{\boldsymbol{\nu}} := (n_1, n_2, \ldots, n_s)$ with $n_s \geqslant 1$ and $n = \sum n_i$ is said to be a *positive partition* of n. We write for short $\underline{\boldsymbol{\nu}} > 0$ and $\|\underline{\boldsymbol{\nu}}\| = n$. A non-increasing sequence $\overline{\boldsymbol{\mu}} := (m_1, m_2, \ldots, m_s)$ with $m_s \geqslant 0$ and $m = \sum m_i$ is said to be a *non-negative partition* of m. We write for short $\overline{\boldsymbol{\mu}} \geqslant 0$ and $\|\overline{\boldsymbol{\mu}}\| = m$.

Let ψ be any smooth one-variable function with algebraically independent derivatives $\psi, \psi', \psi'', \psi''' \ldots$
For any non-negative partition $\overline{\boldsymbol{\mu}}$ we set:

$$\psi^{\overline{\boldsymbol{\mu}}}(x) = \psi^{\overline{(m_1, \ldots, m_r)}}(x) := \psi^{(m_1)}(x) \ldots \psi^{(m_r)}(x) \tag{4.19}$$

We denote by $\mathbb{H}$ (resp $\mathbb{H}_{d_1}^{d_1+d_2-1}$) the Lie algebra over $\mathbb{C}$ spanned by all operators $\psi^{\overline{\boldsymbol{\mu}}}(x)\,\partial_x$ (resp those operators whose index $\overline{\boldsymbol{\mu}}$ is a non-negative partition of the number $d_1 + d_2 - 1$ into d_1-summands).

Proposition 4.8 (Representation of the active algebra Act). — *The component* Act^{d_1, d_2} *of the quotient*

$$\mathbf{Act} := \mathbf{Lie}[\mathbf{a}, \mathbf{b}]/\mathbf{Uker}_\infty \tag{4.20}$$

is isomorphic to the subspace of $\mathbb{H}_{d_1}^{d_1+d_2-1}$ *that is orthogonal to the 'Lagrangian constraints'* $\mathrm{Lag}_{\underline{\boldsymbol{\nu}}}$ *or to the 'power constraints'* $\mathrm{Pow}_{\underline{\boldsymbol{\nu}}}$ *with indices of the form* $\underline{\boldsymbol{\nu}} = \underline{\emptyset}$ *or of the form* $\underline{\boldsymbol{\nu}} = \underline{(n_1, \ldots, n_s)}$ *with*

$$\sum n_i \leqslant d_1 - 2 \qquad \text{and} \qquad s + \sum n_i \leqslant d_1 + d_2 - 1 \tag{4.21}$$

In symmetric fashion, it is also isomorphic to the subspace of $\mathbb{H}_{d_1}^{d_1+d_2-1}$ *that is orthogonal to the constraints* $\mathrm{Lag}_{\underline{\boldsymbol{\nu}}}$ *or* $\mathrm{Pow}_{\underline{\boldsymbol{\nu}}}$ *with indices of the form* $\underline{\boldsymbol{\nu}} = \underline{\emptyset}$ *or of the form* $\underline{\boldsymbol{\nu}} = \underline{(n_1, \ldots, n_s)}$ *with*

$$\sum n_i \leqslant d_2 - 2 \qquad \text{and} \qquad s + \sum n_i \leqslant d_1 + d_2 - 1 \tag{4.22}$$

By linearity, the constraints $Lag_{\underline{\boldsymbol{\nu}}}$ and $Pow_{\underline{\boldsymbol{\nu}}}$ are wholly determined by the 'constraint tensors' $Lag_{\underline{\boldsymbol{\nu}}}^{\overline{\boldsymbol{\mu}}}$ and $Pow_{\underline{\boldsymbol{\nu}}}^{\overline{\boldsymbol{\mu}}}$ such that:

$$< \mathrm{Lag}_{\underline{\boldsymbol{\nu}}}\, , \psi^{\overline{\boldsymbol{\mu}}} > \;=\; \mathrm{Lag}_{\underline{\boldsymbol{\nu}}}^{\overline{\boldsymbol{\mu}}} \tag{4.23}$$

$$< \mathrm{Pow}_{\underline{\boldsymbol{\nu}}}\, , \psi^{\overline{\boldsymbol{\mu}}} > \;=\; \mathrm{Pow}_{\underline{\boldsymbol{\nu}}}^{\overline{\boldsymbol{\mu}}} \tag{4.24}$$

4.4. Lagrangian constraints

Proposition 4.9 (Tensor of 'Lagrangian constraints'). — *It is explicitely given by:*

$$\mathrm{Lag}_{\underline{\boldsymbol{\nu}}}^{\overline{\boldsymbol{\mu}}} = \mathrm{Lag}_{\underline{(n_1,\ldots,n_s)}}^{\overline{(m_1,\ldots,m_r)}} \tag{4.25}$$

$$\equiv \sum_{\underline{\boldsymbol{\nu}}=\underline{\boldsymbol{\nu}_1}\oplus\ldots\oplus\underline{\boldsymbol{\nu}_r}} \mathrm{lag}_{\underline{\boldsymbol{\nu}_1}}(m_1)\ldots\mathrm{lag}_{\underline{\boldsymbol{\nu}_r}}(m_r) \tag{4.26}$$

with a sum extending to all decompositions $\underline{\boldsymbol{\nu}} = \underline{\boldsymbol{\nu}_1} \oplus \ldots \oplus \underline{\boldsymbol{\nu}_r}$ *of* $\underline{\boldsymbol{\nu}}$ *into* r *sub-partitions* $\underline{\boldsymbol{\nu}_i}$ *(some of which may be empty) and with integers* $\mathrm{lag}_{\underline{\boldsymbol{\nu}_i}}(m_i)$ *defined by:*

$$\mathrm{lag}_{\underline{\boldsymbol{\emptyset}}}(m) := 1 \qquad (\forall m \in \mathbb{N}) \tag{4.27}$$

and for non-empty partitions $\underline{\boldsymbol{\nu}}$ *by means of the identity:*

$$(-1)^m\ \psi^{(m)}(y) := \varphi^{-1}(x)\ (\varphi'(x))^m \ + \sum_{1\leqslant s\,;\,1\leqslant n_i} \mathrm{lag}_{\underline{(n_1,\ldots,n_s)}}(m)\ \varphi^{\|\mathbf{n}\|-1}\ (\varphi'(x))^{m+s-\|\mathbf{n}\|} \prod_{1\leqslant i\leqslant s} (\varphi)^{(1+n_i)}(x) \tag{4.28}$$

that connects the successive derivatives of two functions $\varphi(x)$ *and* $\psi(y)$ *linked by the reciprocity relation:*

$$\{y = \int^x \frac{dx_1}{\varphi(x_1)}\} \iff \{x = \int^y \frac{dy_1}{\psi(y_1)}\} \tag{4.29}$$

Remark 4.10 . — In practical terms, the identities (4.28) are established – and the coefficients therein can be calculated – by successive differentiations of the identities:

$$\psi \circ h\,(x) \ \equiv\ h'\,(x) \ \equiv 1/\varphi(x) \qquad \textit{with}\ \ h(x) \equiv y \tag{4.30}$$

which are reminiscent of Lagrange's inversion formula: hence the name given to the corresponding constraints.

Remark 4.11 . — An easy induction shows that for small values of m:

$$\mathrm{lag}_{\underline{\boldsymbol{\nu}}}(m) \equiv 0 \qquad \textit{if}\ \ 0 \leqslant m \leqslant s - 1 + \|\boldsymbol{\nu}\| \tag{4.31}$$

and that for larger values each $\mathrm{lag}_{\underline{\boldsymbol{\nu}}}(m)$ resolves into an exponential[11] sum:

$$\mathrm{lag}_{\underline{\boldsymbol{\nu}}}(m) \;=\; P^0_{\underline{\boldsymbol{\nu}}}(m) \;+ \sum_{1\leqslant j\leqslant \|\boldsymbol{\nu}\|} (-j)^m\; P^j_{\underline{\boldsymbol{\nu}}}(m) \tag{4.32}$$

with coefficients $P^j_{\underline{\boldsymbol{\nu}}}(m)$ that are polynomial in m, with well-defined degrees. Actually, even for m in the interval $1 \leqslant m \leqslant s-1+\|\boldsymbol{\nu}\|$, the exponential sum (4.32) still yields the right answer, i.e. 0. For $m=0$, however, it yields non-zero values, whereas the correct value is 0, as in definition (4.217).

Remark 4.12 . — Expanding (4.26) for $s=2$ and $s=3$ we find:

$$\mathrm{Lag}^{\overline{(m_1,\dots,m_r)}}_{\underline{(n_1,n_2)}} \;=\; \sum_i \mathrm{lag}_{\underline{(n_1,n_2)}}(m_i) + \sum_{i\neq j} \mathrm{lag}_{\underline{(n_1)}}(m_i)\; \mathrm{lag}_{\underline{(n_2)}}(m_j) \tag{4.33}$$

$$\begin{aligned}\mathrm{Lag}^{\overline{(m_1,\dots,m_r)}}_{\underline{(n_1,n_2,n_3)}} \;=\;& \sum_i \mathrm{lag}_{\underline{(n_1,n_2,n_3)}}(m_i) + \sum_{i\neq j} \mathrm{lag}_{\underline{(n_1)}}(m_i)\; \mathrm{lag}_{\underline{(n_2,n_3)}}(m_j)\\ &+\sum_{i\neq j} \mathrm{lag}_{\underline{(n_2)}}(m_i)\; \mathrm{lag}_{\underline{(n_1,n_3)}}(m_j)\\ &+\sum_{i\neq j} \mathrm{lag}_{\underline{(n_3)}}(m_i)\; \mathrm{lag}_{\underline{(n_1,n_2)}}(m_j)\\ &+\sum_{i\neq j,j\neq k,k\neq i} \mathrm{lag}_{\underline{(n_1)}}(m_i)\; \mathrm{lag}_{\underline{(n_2)}}(m_j)\; \mathrm{lag}_{\underline{(n_3)}}(m_k)\end{aligned} \tag{4.34}$$

Remark 4.13 . — Let us write down the exponential sums $lag_{\underline{\boldsymbol{\nu}}}(m)$ for the first seven partitions, ie for $\|\underline{\boldsymbol{\nu}}\| \leqslant 3$:

$$\begin{aligned}
\mathrm{lag}_{\underline{(\emptyset)}}(m) &= 1\\
\mathrm{lag}_{\underline{(1)}}(m) &= (1-m)\\
\mathrm{lag}_{\underline{(2)}}(m) &= (\frac{3}{4}-\frac{1}{2}m)+(-1)^m\,\frac{1}{4}\\
\mathrm{lag}_{\underline{(1,1)}}(m) &= (\frac{7}{4}-2m+\frac{1}{2}m^2)+(-1)^m\,\frac{1}{4}\\
\mathrm{lag}_{\underline{(3)}}(m) &= (\frac{11}{36}-\frac{1}{6}m)+(-1)^m\,\frac{1}{4}+(-2)^m\,(-\frac{1}{18})\\
\mathrm{lag}_{\underline{(2,1)}}(m) &= (-\frac{23}{9}+\frac{29}{12}m-\frac{1}{2}m^2)+(-1)^m\,(-1+\frac{1}{4}m)+(-2)^m\,(\frac{1}{18})\\
\mathrm{lag}_{\underline{(1,1,1)}}(m) &= (+\frac{13}{4}-\frac{49}{12}m+\frac{3}{2}m^2-\frac{1}{6}m^3)+(-1)^m\,(\frac{3}{4}-\frac{3}{4}m)
\end{aligned}$$

(11) with respect to m.

For $m = 0$ we must posit $lag_{\underline{\boldsymbol{\nu}}}(0) := 0$, but for small, positive values of m (i.e. smaller than $s - 1 + \|\boldsymbol{\nu}\|$) the above formulas, of their own, yield 0.

4.5. Power constraints

PROPOSITION 4.14 (TENSOR OF 'POWER CONSTRAINTS'). — *It is explicitely given by* $\mathrm{Pow}_{\underline{\emptyset}}^{\overline{\boldsymbol{\mu}}} = 1$ *and for non-empty partitions* $\boldsymbol{\nu}$ *by:*

$$\mathrm{Pow}_{\underline{\boldsymbol{\nu}}}^{\overline{\boldsymbol{\mu}}} = \mathrm{Pow}_{\underline{(n_1,\ldots,n_s)}}^{\overline{(m_1,\ldots,m_r)}} \tag{4.35}$$

$$(\text{if } r \leqslant s) = 0 \tag{4.36}$$

$$(\text{if } 1 \leqslant s \leqslant r) = \sum_j (1-n_1)^{m_{j(1)}} (1-n_2)^{m_{j(2)}} \ldots (1-n_s)^{m_{j(s)}} \tag{4.37}$$

with a sum ranging over all $r!/(r-s)!$ *injections* j *of the set* $\{1,\ldots,s\}$ *into the set* $\{1,\ldots,r\}$ *and with* n^m *defined in the usual way for vanishing arguments.*[12]

Proof (derivation of the power constraints) .— Let $t := (t_1,\ldots,t_r)$ be s complex variables and define linear maps:

$$\mathrm{Pow}_{[t_1,\ldots,t_s]} \; : \; \mathbb{H} \longrightarrow \mathbb{C}[t_1,\ldots,t_s] \tag{4.38}$$

by the orthogonality conditions:

$$< \mathrm{Pow}_{[\emptyset)} \, , \, \psi^{\overline{\boldsymbol{\mu}}} \partial > = 1 \tag{4.39}$$

$$< \mathrm{Pow}_{[t_1]} \, , \, \psi^{\overline{\boldsymbol{\mu}}} \partial > = \sum t_1^{m_{j(1)}} \tag{4.40}$$

$$< \mathrm{Pow}_{[t_1,t_2]} \, , \, \psi^{\overline{\boldsymbol{\mu}}} \partial > = \sum t_1^{m_{j(1)}} \, t_2^{m_{j(2)}} \tag{4.41}$$

$$\ldots \qquad \ldots$$

$$< \mathrm{Pow}_{[t_1,\ldots,t_s]} \, , \, \psi^{\overline{\boldsymbol{\mu}}} \partial > = \sum t_1^{m_{j(1)}} \, t_2^{m_{j(2)}} \ldots t_s^{m_{j(s)}} \tag{4.42}$$

for all non-negative partitions $\overline{\boldsymbol{\mu}} = (m_1,\ldots,m_s)$ and with injections j as above. Then the polynomials defined by:

$$P^{n_1,\ldots,n_r}(t_1,\ldots,t_s) := < \mathrm{Pow}_{[t_1,\ldots,t_s]} \, , \, [\psi^{(m_r)}\partial \ldots [\psi^{(m_3)}\partial, [\psi^{(m_2)}\partial, \psi^{(m_1)}\partial]..] > \tag{4.43}$$

clearly verify the following induction:

$$P^{n_1,\ldots,n_r}(\emptyset) := 0 \qquad \text{if } r \geqslant 2 \tag{4.44}$$

---

(12) ie $0^0 := 1$; $n^0 := 1$ $(\forall n \geqslant 1)$; $0^n := 0$ $(\forall n \geqslant 1)$

$$P^{n_1,\dots,n_r}(t_1) \;:=\; (t_1+r-3)\,P^{n_1,\dots,n_{r-1}}(t_1) - P^{n_1,\dots,n_r}(\emptyset)\,t_1^{n_r} \quad (4.45)$$

$$\dots \qquad\qquad \dots$$

$$P^{n_1,\dots,n_r}(t_1,\dots,t_s) \;:=\; (\|t\|+r-s-2)\,P^{n_1,\dots,n_{r-1}}(t_1,\dots,t_s) + \sum_{1\leqslant k\leqslant s}(\|t\|+r-s-2\,t_k)\,P^{n_1,\dots,n_{r-1}}(t_1,\dots,\hat{t}_k,\dots,t_s) \quad (4.46)$$

with $\|t\| := \sum t_i$ and with $\hat{}$ in $\hat{t}_k$ signalling omission.

The first two relations yield:

$$P^{n_1,\dots,n_r}(t_1) = (t_1^{n_1} - t_2^{n_2})\,(t_1-1)\,t_1\,(t_1+1)\,(t_1+2)\dots(t_1+r-3)$$

Note the absence of $t_1^{n_3}, t_1^{n_4}\dots$. Therefore:

$$P^{n_1,\dots,n_r}(t_1) = 0 \qquad for \;\; r\geqslant 2 \;\; and \;\; r\in\{1,0,-1,-2,\dots,3-r\}$$

More generally, in view of the factors $(\|t\|+r-s-2)$ and $(\|t\|+r-s-2\,t_k)$ in (4.46), it is an easy matter to check, by a double induction on r and s, that the identities:

$$P^{n_1,\dots,n_r}(t_1,\dots,t_s) = 0$$

hold true whenever:

$$r\geqslant s+2 \quad ; \quad t_k\in\{1,0,-1,-2,-3,\dots\} \quad ; \quad \sum(1-t_k)\leqslant r-2$$

which establishes the analytic expression of the 'power constraints'. □

4.6. Proofs

Derivation of the Lagrangian constraints:

The model we used for:

$$\mathbf{Act} := \mathbf{Lie}[\mathbf{a},\mathbf{b}]/\mathbf{Uker}_\infty \quad (4.47)$$

was obtained by specialising $(\mathbf{a},\mathbf{b})$ as $(\partial_y\,,\,\psi(y)\,\partial_y)$. With equal right, we might have specialised $(\mathbf{a},\mathbf{b})$ as $(\varphi(x)\,\partial_x\,,\,\partial_x)$. If we now consider the change of variable which takes us from the first to the second specialisation, we find:

$$x \;\mapsto\; y=h(x) \quad ; \quad (\partial_y\,,\,\psi(y)\,\partial_y)\mapsto(\varphi(x)\,\partial_x\,,\,\partial_x) \quad (4.48)$$

$$\varphi(x) = 1/h'(x) \quad ; \quad \psi\circ h(x)/h'(x) = 1 \quad (4.49)$$

which is readily seen to imply, for any positive m:

$$\begin{aligned}\psi^{(m)}\,\partial \quad &\mapsto \quad \frac{\varphi^{(m)}\circ h}{h'}\,\partial &(4.50)\\ &= \quad (-1)^m\,(\varphi')^m + (-1)^m \sum \mathrm{lag}_{\underline{\boldsymbol{\nu}}}(m)\,\varphi^{\|\underline{\boldsymbol{\nu}}\|}\,(\varphi')^{m-s-\|\underline{\boldsymbol{\nu}}\|}\varphi^{(1+\underline{\boldsymbol{\nu}})}\end{aligned}$$

with the same coefficients lag as in (4.28).
More generally, we have:

$$\begin{aligned}\psi^{(\overline{\boldsymbol{\mu}})}\,\partial \quad &\mapsto \quad (-1)^{\|\overline{\boldsymbol{\mu}}\|}\,(\varphi')^{\|\overline{\boldsymbol{\mu}}\|} &(4.51)\\ &+ \quad (-1)^{\|\overline{\boldsymbol{\mu}}\|}\sum \mathrm{Lag}^{\overline{\boldsymbol{\mu}}}_{\underline{\boldsymbol{\nu}}}(m)\,\varphi^{\|\underline{\boldsymbol{\nu}}\|+1-r}\,(\varphi')^{\|\overline{\boldsymbol{\mu}}\|-\|\underline{\boldsymbol{\nu}}\|-s}\varphi^{(1+\underline{\boldsymbol{\nu}})}\end{aligned}$$

with the usual notations:

$$\overline{\boldsymbol{\mu}} := \overline{(m_1,\ldots,m_r)} \quad ; \quad \psi^{(\overline{\boldsymbol{\mu}})} := \psi^{(m_1)}\ldots\psi^{(m_r)} \tag{4.52}$$
$$\underline{\boldsymbol{\nu}} := \underline{(n_1,\ldots,n_s)} \quad ; \quad \psi^{(1+\underline{\boldsymbol{\nu}})} := \psi^{(1+n_1)}\ldots\psi^{(1+n_s)} \tag{4.53}$$

and with the same tensor $Lag^{\overline{\boldsymbol{\mu}}}_{\underline{\boldsymbol{\nu}}}(m)$ as in (4.25–26).
Thus, for the generators $\psi^{(\overline{\boldsymbol{\mu}})}\,\partial$ of $\mathbb{H}$, the change of variable $x \mapsto y = h(x)$ introduces exactly $r-1$ negative powers of φ. However, in a single concatenation of Lie brackets:

$$[\psi^{(m_r)}\partial\ldots[\psi^{(m_3)}\partial,[\psi^{(m_2)}\partial,\psi^{(m_1)}\partial]..] \tag{4.54}$$

no negative powers of φ can possibly appear, since under the change $x \mapsto y = h(x)$ the elements inside the multibracket (4.54) transform according to the simple rule:

$$\psi^{(m_i)}\,\partial \quad \mapsto \quad [\varphi\,\partial\ldots[\varphi\,\partial,[\varphi\,\partial,\partial]..] \qquad (n \text{ occurences of } \varphi\,\partial) \tag{4.55}$$

There is thus a discrepancy between the behaviour of the generators of $\mathbb{H}$ and that of multibrackets, and if we write down the condition for a linear combination of generators to behave like a multibracket, we arrive precisely at the ‘Lagrangian constraints’ of Proposition 4.8 and 4.9. □

Equivalence and exhaustiveness of the Lagrangian and power constraints. Dimension of the homogeneous components of Act.

The power constraints are obviously much simpler and far easier to handle than the Lagrangian constraints. Both sets of constraints, however, are equivalent, and with the latter set, exhaustiveness is easier to prove.

The shortest way to proceed is to check, separately, the independence of the various power constraints and that of the various Lagrangian constraints. Then we provisionally *assume* that the power constraints are exhaustive and see what this says about the dimensions D^{d_1,d_2} of **Act**. It clearly implies that $D^{d_1,d_2} = D_*^{d_1,d_2} - D_{**}^{d_1,d_2}$, with $D_*^{d_1,d_2}$ being the dimension of $\mathbb{H}_{d_1}^{d_1+d_2-1}$ and $D_{**}^{d_1,d_2}$ the number of (independent !) Lagrangian constraints $Lag_{\underline{\nu}}$ subject to (4.28). Therefore:

$$D_*^{d_1,d_2} = p\,(d_1+d_2-1, d_1) = P(2\,d_1+d_2-1, d_1) \tag{4.56}$$

$$D_{**}^{d_1,d_2} = 1 + \sum_{\substack{n\leqslant d_1-2\\ n+s\leqslant d_1+d_2-1}} P(n,s) = p(d_1+d_2-1) - p(d_1+d_2-1, d_2) \tag{4.57}$$

We thus get for D^{d_1,d_2} the equivalent expression (4.14), (4.15) along with the special cases (4.16), (4.17), but *under the assumption* that the Lagrangian constraints are exhaustive. To fill this one last gap, we introduce on $\mathbb{H}$ the filtration $\mathbb{H} = \cup\,\mathbb{H}_p$, where $\mathbb{H}_p$ denotes the subspace of $\mathbb{H}$ generated by *products* of p Lie elements, and we check that, if the exhaustiveness hypothesis is valid for all components such that $d_1+d_2 \leqslant d-1$, then the next component $d_1+d_2 = d$ has codimension:

$$\operatorname{codim} \mathbb{H}_p^{d_1,d_2} = \sum_{\substack{n\leqslant d_1-2\\ n+s\leqslant d_1+d_2-p}} P(n,s) \qquad \forall p \geqslant 1 \tag{4.58}$$

But this leaves no scope for $\mathbb{H}^{d_1,d_2} (= \mathbf{Act}^{d_1,d_2})$ to have a dimension *smaller* than D^{d_1,d_2}, which by induction establishes the validity of the expression for D^{d_1,d_2}. But since the power constraints are also mutually independent and, *if exhaustive*, also lead to the same expression for D^{d_1,d_2}, it means that they, too, *must be exhaustive*. This completes the proof. □

5. Generic, low-complexity identity-tangent twins

5.1. Finite co-dimension. The general picture

The basic tools for investigating twin-begetting relations $\mathbf{W}(\mathbf{A},\mathbf{B}) = \mathbf{1}$ is not the word $\mathbf{W}(\mathbf{A},\mathbf{B})$ itself[13] but its image $\mathbf{w}(\mathbf{a},\mathbf{b})$ in the natural closure of $\mathbf{Lie}(\mathbf{a},\mathbf{b})$, defined in the usual way:

$$\mathbf{w}(\mathbf{a},\mathbf{b}) := \log \mathbf{W}(e^{\mathbf{a}}\,e^{\mathbf{b}}) = \sum_{m\geqslant 1, n\geqslant 1} \mathbf{w}_{m,n}(\mathbf{a},\mathbf{b}) \in \overline{\mathbf{Lie}}(\mathbf{a},\mathbf{b}) \tag{5.1}$$

(13) at any rate, if we are interested in power series solutions only. But when searching for general *transserial* twins (see §8), the word $\mathbf{W}(\mathbf{A},\mathbf{B})$ returns to the forefront.

We also require precise information about the nature of the homogeneous components $\mathbf{w}_{m,n}(\mathbf{a},\mathbf{b})$ and their exact position with respect to the *Uker*-filtration of $\mathbf{Lie}(\mathbf{a},\mathbf{b})$. So it is natural to set:

$$\begin{aligned}
\mathbf{Act_W} &:= \{(m,n)\in\mathbb{N}^2 \mid \mathbf{w}_{m,n}(\mathbf{a},\mathbf{b})\in\mathbf{Uker_0}\setminus\mathbf{Uker_\infty}\}\\
\mathbf{Act^i_W} &:= \{(m,n)\in\mathbb{N}^2 \mid \mathbf{w}_{m,n}(\mathbf{a},\mathbf{b})\in\mathbf{Uker_i}\setminus\mathbf{Uker_{i+1}}\} \quad (0\leqslant i)\\
\mathbf{Act_W} &= \mathbf{Act^0_W}\cup\mathbf{Act^1_W}\cup\mathbf{Act^2_W}\cup\ldots \qquad (5.2)
\end{aligned}$$

Let $\overline{\mathbf{Act}}_\mathbf{W}$, $\overline{\mathbf{Act}}^{\mathbf{0}}_\mathbf{W}$, $\overline{\mathbf{Act}}^{\mathbf{1}}_\mathbf{W}\ldots$ denote the convex hulls of these sets and let $\widehat{\mathbf{Act}}_\mathbf{W}$, $\widehat{\mathbf{Act}}^{\mathbf{0}}_\mathbf{W}$, $\widehat{\mathbf{Act}}^{\mathbf{1}}_\mathbf{W}\ldots$ be the corresponding (finite) sets of extremal points or 'summits'.

Our twenty examples of §3 were "typical" only in the sense of illustrating the main types of twins that are liable to occur, with discrete or continuous parameters, built-in symmetries or invariance properties etc. In each case, however, the words $\mathbf{W}(\mathbf{A},\mathbf{B})$ consisted of *strict multi-commutators*, with the result that the set $\mathbf{Act_W}$ was always included in an upper-right quadrant of summit $(d_1,d_2)\geqslant(2,2)$ with d_1,d_2 often quite large. As a consequence, $\mathbf{w}(\mathbf{a},\mathbf{b})$ had infinitely many vanishing components $\mathbf{w}_{m,n}(\mathbf{a},\mathbf{b})$ and the 'codimension', in the sense of §2.2, was always infinite. [14]

Put in simple terms: *our twenty 'typical' examples were fairly simple, but also embarrassingly non-generic.* So we must now tackle the generic case.

In all three tables further down this section each full dot • stands for one homogeneous component $\mathbf{Lie}^{m,n}$ and the numbers $\boldsymbol{\alpha}^{\boldsymbol{\beta}^{\boldsymbol{\gamma}}}$ next to it indicate the dimension of, respectively, $\mathbf{Act_0}^{m,n}$, $\mathbf{Act_*}^{m,n}$, $\mathbf{Pass}^{m,n}$. When $\boldsymbol{\gamma}$, or both $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$, vanish, they simply get omitted.[15]

5.2. Fixed ratio p/q and no continuous parameter

Let us first examine the *generic* counterpart of Examples 3.1 and 3.2 in §3.4.

Suppose there are in $\widehat{\mathbf{Act}}_\mathbf{W}$ two consecutive summits (m_1,n_1), (m_2,n_2) which are also in $\widehat{\mathbf{Act}}^{\mathbf{0}}_\mathbf{W}$. Suppose further that neither of the homogeneous polynomials[16] $w_{m_1,n_1}(p,q)$ and $w_{m_2,n_2}(p,q)$ has a positive rational root p/q

---

(14) Every twin-begetting word necessarily resolves into a product of *simple* commutators. But in the generic case, the 'alternance' (d_1,d_2) is $(1,1)$, which imposes only two constraints: $\mathbf{w}_{1,0}(\mathbf{a},\mathbf{b})=\mathbf{w}_{0,1}(\mathbf{a},\mathbf{b})=0$ (vanishing of the linear components).

(15) Here, the dimension $\boldsymbol{\beta}$ of $\mathbf{Act_*}^{m,n}$ matters less than the dimension $\boldsymbol{\beta}_1$ of its first subalgebra $\mathbf{Act_1}^{m,n}$, but for all points • envisaged, both actually coincide because $0=\boldsymbol{\beta}_2=\boldsymbol{\beta}_3\ldots$.

(16) defined as usual by monomial substitution: $\mathbf{w}_{m,n}(l_p,l_q)\equiv w_{m,n}(p,q)\,l_{mp+nq}$ with $l_n:=x^{n+1}\,\partial_x$.

other than $p/q = 1$. This is automatically the case if (m_1, n_1) and (m_2, n_2) are of the form $(1, n)$ or $(m, 1)$ or $(2, 2)$, for the corresponding components of $\mathbf{Act^0}$ are one-dimensional, but it can occur with any (m, n), for instance for $(m, n) = (3, 2)$ and

$$\mathbf{W}(\mathbf{A},\mathbf{B}) := \mathbf{A}^{\mathbf{c_1}}\,\mathbf{B}\,\mathbf{A}^{\mathbf{c_2}}\,\mathbf{B}^{-\mathbf{1}}\,\mathbf{A}^{\mathbf{c_3}}\,\mathbf{B}\,\mathbf{A}^{\mathbf{c_1}}\,\mathbf{B}^{-\mathbf{1}}\,\mathbf{A}^{\mathbf{c_2}}\,\mathbf{B}\,\mathbf{A}^{\mathbf{c_3}}\,\mathbf{B}^{-\mathbf{1}} \tag{5.3}$$

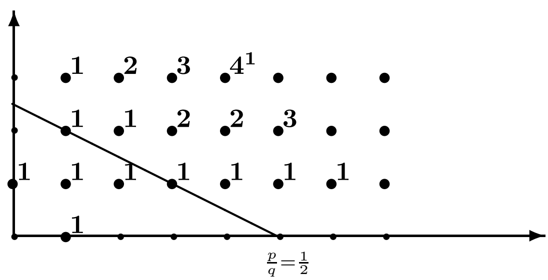
$$\mathbf{w_{2,3}}(\mathbf{a},\mathbf{b}) = \frac{1}{2}\,c_1\,c_2\,c_3\,[[\mathbf{a},\mathbf{b}],[\mathbf{a}[\mathbf{a},\mathbf{b}]]] \tag{5.4}$$

$$\text{with } c_1,\, c_2,\, c_3 \in \mathbb{Z}^*,\ c_1 + c_2 + c_3 = 0 \tag{5.5}$$

Suppose lastly that the edge linking the two summits has slope $-\frac{p_0}{q_0} := -\frac{n_1-n_2}{m_1-m_2} \neq -1$. The simplest instance of this situation is:

$$\mathbf{W}(\mathbf{A},\mathbf{B}) := \mathbf{A}\,\mathbf{B}^{\mathbf{3}}\,\mathbf{A}\,\mathbf{B}^{-\mathbf{3}}\,\mathbf{A}\,\mathbf{B}\,\mathbf{A}^{-\mathbf{3}}\,\mathbf{B}^{-\mathbf{1}} \tag{5.6}$$

$$\mathbf{w}(\mathbf{a},\mathbf{b}) = [\mathbf{a}[\mathbf{a}[\mathbf{a},\mathbf{b}]]] + 3\,[\mathbf{b}[\mathbf{b},\mathbf{a}]] + \ldots = \log \mathbf{W}(e^{\mathbf{a}}, e^{\mathbf{b}}) \tag{5.7}$$



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5.3. Fixed ratio p/q and one continuous parameter

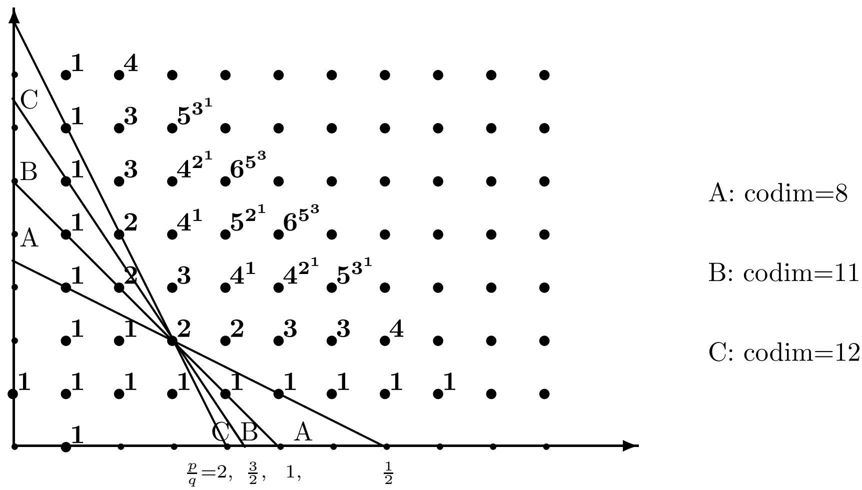
Let us now examine the *generic* counterpart of Examples 3.3 and 3.4 in §3.4.

Suppose $\widehat{\mathbf{Act}}_{\mathbf{W}}$ has a summit (m_0, n_0) which is also in $\widehat{\mathbf{Act}}^{\mathbf{0}}_{\mathbf{W}}$. Suppose further that the homogeneous polynomial $w_{m_0,n_0}(p,q)$ has one or several positive rational roots p/q (other than $p/q = 1$) such that the line D of slope $-p/q$ drawn through the summit (m_0, n_0) lies outside the convex hull $\overline{\mathbf{Act}}_{\mathbf{W}}$. This may occur with any $(m_0, n_0) \geqslant (2, 2)$ and $\neq (2, 2)$. The simplest instance of this situation corresponds to $(m_0, n_0) = (3, 2)$ and:

$$\mathbf{w}(\mathbf{a},\mathbf{b}) = +c_1\,[\mathbf{b}[\mathbf{a}[\mathbf{a}[\mathbf{a},\mathbf{b}]]]] - c_2\,[\mathbf{a}[\mathbf{a}[\mathbf{b}[\mathbf{a},\mathbf{b}]]]] + \ldots \tag{5.8}$$

$$\text{with} \quad \frac{c_1}{c_2} = \frac{2}{3}\,\frac{p+2\,q}{p+q} \tag{5.9}$$

Exercise. — Find the simplest words $\mathbf{W}(\mathbf{A},\mathbf{B})$ corresponding to the simplest values of p/q.



Drawing a line of slope p/q through the summit $(3,2)$ within one of the angular domains A, B, C and counting the dimensions $\boldsymbol{\alpha}$ of the cells $\bullet$ below that line, one finds 'codimensions' equal to, respectively, 8,11,12.

5.4. Free ratio p/q and one continuous parameter

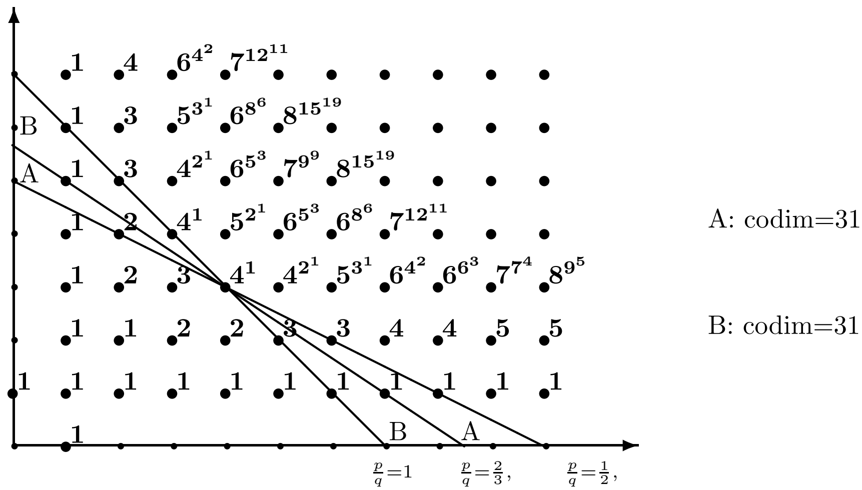
Let us now examine the *generic* counterpart of Examples 3.5 and 3.9 in §3.4.

Suppose $\widehat{\mathbf{Act}}_{\mathbf{W}}$ has a summit (m_0, n_0) which lies in $\widehat{\mathbf{Act}}^{\mathbf{1}}_{\mathbf{W}}$ rather than $\widehat{\mathbf{Act}}^{\mathbf{0}}_{\mathbf{W}}$. This may occur with any $(m_0, n_0) \geqslant (3,3)$ and $\neq (3,3)$. The polynomial $w_{m_0,n_0}(p,q)$ is then $\equiv 0$. Let p/q be any rational $\neq 1$ such that the line D of slope $-p/q$ drawn through the summit (m_0, n_0) lies outside the convex hull $\overline{\mathbf{Act}}_{\mathbf{W}}$. The simplest instance of this situation corresponds to $(m_0, n_0) = (4,3)$ with an extremal component $\mathbf{w}_{4,3}(\mathbf{a},\mathbf{b})$ in $\mathbf{Uker_1} \setminus \mathbf{Uker_2}$. That space, however, has dimension 1, so that our $\mathbf{w}_{4,3}(\mathbf{a},\mathbf{b})$ has to be proportional to:

$$[[\mathbf{b}, \overline{\mathbf{a}}^{\mathbf{3}}\, \mathbf{b}]\, \overline{\mathbf{a}}\, \mathbf{b}] + 6\, [[\mathbf{b}, \overline{\mathbf{a}}\, \mathbf{b}]\, \overline{\mathbf{a}}^{\mathbf{3}}\, \mathbf{b}] - 3\, [[\mathbf{b}, \overline{\mathbf{a}}^{\mathbf{2}}\, \mathbf{b}]\, \overline{\mathbf{a}}^{\mathbf{2}}\, \mathbf{b}] - 3\, [[\overline{\mathbf{a}}\, \mathbf{b}, \overline{\mathbf{a}}^{\mathbf{2}}\, \mathbf{b}]\, \overline{\mathbf{a}}\, \mathbf{b}]$$

or, equivalently, to:

$$\begin{aligned}&[[[\mathbf{a}, \overline{\mathbf{b}}^{\mathbf{2}}\, \mathbf{a}]\, \mathbf{a}]\, \overline{\mathbf{b}}\, \mathbf{a}] - 6\, [[[\mathbf{a}, \overline{\mathbf{b}}\, \mathbf{a}]\, \mathbf{a}]\, \overline{\mathbf{b}}^{\mathbf{2}}\, \mathbf{a}] \\ &\qquad -3\, [[\mathbf{a}, \overline{\mathbf{b}}\, \mathbf{a}]\, [\mathbf{a}\, \overline{\mathbf{b}}^{\mathbf{2}}\, \mathbf{a}]] + 4\, [[[\mathbf{a}, \overline{\mathbf{b}}\, \mathbf{a}]\, \overline{\mathbf{b}}\, \mathbf{a}]\, \overline{\mathbf{b}}\, \mathbf{a}]\end{aligned}$$



Drawing a line of slope p/q through the summit $(4,3)$ within either of the angular domains A, B and counting the dimensions $\boldsymbol{\alpha}$ of the cells $\bullet$ below that line, then adding 4, which is the dimension of $\mathbf{Act_0}^{4,3}$, one finds the same 'codimension' with A and B, namely 31.

Exercise. — Find the simplest words $\mathbf{W}(\mathbf{A},\mathbf{B})$ that induce an extremal component $\mathbf{w}_{4,3}(\mathbf{a},\mathbf{b})$ as above and for which all components lying below the bissectrix of A (or B) vanish.

Remark 5.1 (additional symmetries). — The above constructions in §5.2, §5.3, §5.4 provide *generic* analogues to the *typical* situations exemplified in §3, in Ex 1-2, 3-4, 5-9 respectively. As for the other situations reviewed in §3 (Ex 10 through 23) and which mostly involve twins with extra symmetries or invariance properties, they too have their generic counterparts. More precisely, the reader may satisfy himself by going *seriatim* through these 16 examples that *each achievable set of symmetries can also be achieved in finite 'codimension'.*

Remark 5.2 (precautions with the divisors). — The constructions of this section, as indeed the earlier constructions of §3, do work provided the corresponding divisor $D(t)$ doesn't vanish on $\mathbb{N}$. Simple calculations confirm that this is indeed the case with all three lowest-complexity examples produced in §5.2, §5.3, §5.4. There exist, however, exceptional cases in which the polynomial $D(t)$ may have k roots on $\mathbb{N}$. Even in such cases, there always exist *transserial* twins[17], but for there to be true *power series* twins, k additional conditions have to be fulfilled, bearing on the relevant homogeneous components of $\mathbf{w}(\mathbf{a},\mathbf{b})$ [18]. When these conditions are fulfilled, the twins automatically inherit k new continuous parameters.

6. Non identity-tangent twins

6.1. Twins of tangency $(0,q)$

Let us now deal with words $\mathbf{W}(\mathbf{A},\mathbf{B})$ that are no longer products of commutators (multiple or even simple). As a result, their Lie image $\mathbf{w}(\mathbf{a},\mathbf{b})$ is supported by a quadrant of apex (m_0,n_0) which is no longer $\geqslant (1,1)$ but either $(0,1)$ or $(1,0)$. Let us settle for the first choice:

$$\mathbf{W}(\mathbf{A},\mathbf{B}) : \;=\; \mathbf{A}^{n_1}\,\mathbf{B}^{m_1}\dots\mathbf{A}^{n_r}\,\mathbf{B}^{m_r} \tag{6.1}$$

$$\text{with}\quad n_j\in\mathbb{Z}^*,\ m_j\in\mathbb{Z}^*\quad,\quad \sum n_j=0\quad\text{but}\quad \sum m_j\neq 0 \tag{6.2}$$

$$F^{n_1^*}\,G^{m_1}\,F^{-n_1^*}\dots F^{n_r^*}\,G^{m_r}\,F^{-n_r^*}=1\quad\text{with}\quad n_j^*-n_{j-1}^*=n_j \tag{6.3}$$

with relative integers n_j^* defined upto some arbitrary additive constant n_0^*. We adjust n_0^* so as to have $\inf(n_j^*)=0$ and form the polynomial:

$$\Delta_W(\theta) := \sum_j\; m_j\,\theta^{n_j^*}\qquad \big(\Delta_{\mathbf{W}}(0)\neq 0\ ,\ \Delta_W(1)\neq 0\big) \tag{6.4}$$

These new *divisors* $\Delta_W(\theta)$, though rather different in form from the familiar $D_{\mathbf{w}}(t)$, play much the same role, since the inductive calculation of the n-th coefficient b_n of g leads to relations of the form $\Delta_W(a^n)\,b_n$ = *earlier terms.* As a consequence, everything hinges on the “resonance” of $\Delta_W(\theta)$, which is the largest integer s for which $\Delta_W(\theta)$ admits s roots of the form $\theta_i = a^i\,, i = 1..s$, with s distinct integers q_i and a not a unit root. To normalise the situation and fix a, we may, and always shall, assume that the $\{q_i\}$ are co-prime.

Then, *depending on whether the resonance* k *is 1,2,3…, our equation (6.256), which may be rewritten as:*

$$G^{m_1}_{n_1^*}\,\dots\,G^{m_r}_{n_r^*}=1\quad\textit{with}\quad G^m_n := F^n\,G^m\,F^{-n} \tag{6.5}$$

---

(17) See §8. Here, the relevant transseries closely resemble power series. In fact, they are usually power series of x and some block $x\,(\log x)^s$.

(18) ie on the $\mathbf{w}_{m,n}(\mathbf{a},\mathbf{b})$ for which $t=(m-m_0)\,p+(n-n_0)\,q$. Of course, that keeps us in “finite codimension”.

admits a general 1,2,3...-parameter twin solution of the form:

$$G = G^{<b_1>} \quad := \quad H^{b_1}_{q_1} \qquad (b_1 \in \mathbb{C}\ ,\ q_1 \in \mathbb{N}) \quad (6.6)$$
$$G = G^{<b_1,b_2>} \quad := \quad H^{b_1}_{q_1}\, H^{b_2}_{q_2}\, K^{<b_1,b_2>} \qquad (b_i \in \mathbb{C}\ ,\ q_i \in \mathbb{N}) \quad (6.7)$$
$$G = G^{<b_1,b_2,b_3>} \quad := \quad H^{b_1}_{q_1}\, H^{b_2}_{q_2}\, H^{b_3}_{q_3}\, K^{<b_1,b_2,b_3>} \qquad (b_i \in \mathbb{C}\ ,\ q_i \in \mathbb{N}) \quad (6.8)$$

etc.

with operators H^b_q and K corresponding to post-composition by mappings of the form:

$$h^{\circ b_i}_{q_i} \quad : \quad x \mapsto x\,(1 + b_i\, x^{q_i})^{-1/q_i} \qquad (b_i \in \mathbb{C}\ ,\ q_i \in \mathbb{N}) \quad (6.9)$$
$$k^{<b_1,b_2>} \quad : \quad x \mapsto x\,(1 + \mathcal{O}(x^{q_1+q_2})) \quad (6.10)$$
$$k^{<b_1,b_2,b_3>} \quad : \quad x \mapsto x\,(1 + \mathcal{O}(x^{q_1+q_2})) \qquad \textit{if } q_1 < q_2 < q_3 \quad (6.11)$$

etc.

provided (for $\mathrm{k} \geqslant 3$*) that the* $\mathrm{q_i}$*'s verify no dependence relation of the form:*

$$q_i = \sum_{j \neq i} m_j\, q_j \qquad \textit{with } m_j \in \mathbb{N} \textit{ and } \textstyle\sum_{m_j > 0} 1 \geqslant 2 \quad (6.12)$$

For $k = 1$ the solution (6.6) is of the type which we have dismissed as 'elementary' in §1.1. For $k \geqslant 2$, however, the solutions (6.7), (6.8) etc are genuine twins as soon as two of the free complex parameters b_i are chosen $\neq 0$.

Proof. — Straightforward: apply Campbell-Hausdorff to rephrase equation (6.5) in terms of the infinitesimal generators G^m_{n*} of the factors G^m_n and observe that, whenever all parameters b_i but one vanish, the 'corrective factor' $H^{<b_1,b_2>}, H^{<b_1,b_2,b_3>} \ldots$ reduces to the identity operator. □

6.2. Simple examples

The simplest possible 'resonant' divisor is:

$$\Delta_{\mathbf{W}}(\theta) := (\theta - a^{q_1})(\theta - a^{q_2}) = m_0 + m_1\,\theta + m_2\,\theta^2 \quad (6.13)$$
$$a \in \mathbb{N}^*\ ,\ 1 \leqslant q_1 < q_2\ ,\ m_0 = a^{q_1+q_2}\ ,\ m_1 = -a^{q_1} - a^{q_2}\ ,\ m_2 = 1 \quad (6.14)$$
$$\text{e.g.}\quad a = 2\ ,\ q_1 = 1\ ,\ q_2 = 2\ ,\ m_0 = 8\ ,\ m_1 = -6\ ,\ m_2 = 1 \quad (6.15)$$

and may be obtained from these words:

$$
\begin{aligned}
W_1(F,G) &:= G_0^8\, G_1^{-6}\, G_2 \qquad (\text{with } G_n := F^n\, G\, F^{-n}) && (6.16)\\
W_2(F,G) &:= G_1^{-6}\, G_0^8\, G_2 && (6.17)\\
W_3(F,G) &:= G_0^4\, G_1^{-6}\, G_0^4\, G_2 && (6.18)\\
W_4(F,G) &:= G_1^{-3}\, G_0^8\, G_1^{-3}\, G_2 && (6.19)
\end{aligned}
$$

The last two words have the advantadge of forcing a symmetry within the general twin solution:

$$
\begin{aligned}
\{W_i(F,G)=1\} &\iff \{W_i(F,G^{-1})=1\} \quad (i=3,4) && (6.20)\\
(F,G^{<b_1,b_2>}) \text{ twin solution} &\iff (F,G^{<-b_1,-b_2>}) \text{ twin solution} && (6.21)
\end{aligned}
$$

and of halving the quantity of unknown coefficients to compute.

For the sequel we also require examples with a on the unit circle but not a unit root. The simplest cluster of examples corresponds to divisors of the form:

$$
\begin{aligned}
\Delta_{\mathbf{W}}(\theta) &:= \Delta_1(\theta)\,\Delta_2(\theta) = \sum_{0\leqslant i\leqslant 4} m_i\,\theta^i \quad \text{with} && (6.22)\\
\Delta_1(\theta) &:= l_0\,(\theta-a)(\theta-\bar a) = l_0 + l_1\,\theta + l_0\,\theta^2 && (6.23)\\
\Delta_2(\theta) &:= l_0^2\,(\theta-a^2)(\theta-\bar a^2) = l_0^2 + (2\,l_0^2 - l_1^2)\,\theta + l_0^2\,\theta^2 && (6.24)\\
\text{for} \quad & l_0\in\mathbb{Z}^*,\ l_1\in\mathbb{Z}^*,\ 1\neq |l_1/l_0| < 2 && (6.25)\\
\text{and so} \quad & |a|=1,\ a^n\neq 1\ (\forall n) && (6.26)\\
& m_0 = m_4 = l_0^3 && (6.27)\\
& m_1 = m_3 = 2\,l_0^3 + l_0^2\,l_1 - l_0\,l_1^2 && (6.28)\\
& m_2 = 2\,l_0^3 + 2\,l_0^2\,l_1 - l_1^3 && (6.29)
\end{aligned}
$$

Here are some of the simplest choices:

$l_0=2$	$l_1=-1$	$m_0=8$	$m_1=10$	$m_2=9$	$m_3=10$	$m_4=8$
$l_0=2$	$l_1=1$	$m_0=8$	$m_1=18$	$m_2=23$	$m_3=18$	$m_4=8$
$l_0=2$	$l_1=3$	$m_0=8$	$m_1=10$	$m_2=13$	$m_3=10$	$m_4=8$
$l_0=3$	$l_1=-2$	$m_0=27$	$m_1=24$	$m_2=26$	$m_3=24$	$m_4=27$

The first choices may be realised by either of the words:

$$
\begin{aligned}
W_5(F,G) &:= G_0^8\, G_1^{10}\, G_2^9\, G_3^{10}\, G_4^8 \quad \text{with } G_n := F^n\, G\, F^{-n} && (6.30)\\
W_6(F,G) &:= G_0^4\, G_4^4\, G_1^5\, G_3^5\, G_2^9\, G_3^5\, G_1^5\, G_4^4\, G_0^4 && (6.31)
\end{aligned}
$$

Here again, $\mathbf{W}_6$, unlike $\mathbf{W}_5$, forces a reflection-symmetry of type (6.21).

From the formal viewpoint, the value of a doesn't matter, as long a is not a unit root, but from the viewpoint of analysis, as we shall see in a moment, the picture changes completely depending on whether a lies on, or outside, the unit circle.

6.3. Twins of tangency $(0,0)$

Pick any word $\mathbf{W}(\mathbf{A},\mathbf{B})$ leading to twins of tangency order $(0,q)$. Then the word:

$$\mathbf{W}_*(\mathbf{A},\mathbf{B}) := \mathbf{W}(\mathbf{A},\{\mathbf{A},\mathbf{B}\}) \tag{6.32}$$

admits a general twin solution of tangency order $(0,0)$.

To see this, normalise F by setting $f(x) := a\,x$ for the right value of a.[19] Next, solve $W(F,G_0)=1$ with respect to G_0 as above. Then end by solving $G_0=\{F,G\}$ with respect to G, which is always (uniquely) possible for any choice of $b := g'(0)$, provided b is not a unit root.

6.4. Simple examples

With the word $\mathbf{W}_*$ as in (6.32) and any one of the words $\mathbf{W}$ in §6.2.

6.5. Pre-identity-tangent 'twins': type $(0^*,0^*)$

Let p be prime $\geqslant 5$ and let e_1,e_2 be two distinct unit roots:

$$e_1^p = e_2^p = 1\ ,\ e_1\neq 1, e_2\neq 1\ ,\ e_1\,e_2\neq 1$$

Then the system

$$f \ := \ x\mapsto e_1\,x+\mathcal{O}(x^2) \tag{6.33}$$

$$g \ := \ x\mapsto e_2\,x+\mathcal{O}(x^2) \tag{6.34}$$

$$F^p \ = \ G^p \ = \ (F\,G)^p \ = \ 1 \tag{6.35}$$

is easily seen to possess a general non-elementary[20] solution that inherently[21] depends on infinitely many parameters. Similar results hold for non-prime powers $p\geqslant 8$, also for a great many systems analogous to (6.35). But here the dichotomy holds throughout: *either* there exist no non-elementary

(19) ie for a 'resonant' root of the divisor $\Delta_{\mathbf{W}}(\theta)$.
(20) in the usual sense of not being reducible, even under a joint ramified change of coordinate, to a pair of homographies.
(21) ie after normalising F,G under a joint change of coordinate.

solutions, *or* there exist an infinity of them, with an *infinite* degree of freedom. This is a sharp departure from the familiar situation, when twins depend only on finitely many parameters. Actually, since (6.35) consists of three equations with just two unknowns, the corresponding pairs (F, G) should not be regarded as proper 'twins'. As we shall see (§7 *infra*), their analytic properties also set them apart from genuine twins.

6.6. Simple examples

With p prime: $p = 5\ ,\ e_1 = \exp(2\pi i/5)\ ,\ e_2 = \exp(4\pi i/5)$

With p non-prime: $p = 8\ ,\ e_1 = i\ ,\ e_2 = -1$

6.7. Siblings

Let us revert to the case of identity-tangent mappings. Proper *twins*, as we saw, are non-elementary pairs of mappings bound by one relation. The natural generalisation is that of *'siblings'*, or non-elementary [22] systems of r mappings bound by $r-1$ relations.

$$1 = W_1(F) = \ldots = W_{r-1}(F) \tag{6.36}$$

$$\text{with } F := (F_1, \ldots, F_r)$$

A special but important sub-case arises when we impose the pair-wise commutation of r words $V_i(F)$. Indeed, for identity-tangent mappings (or again, for one-variable, formally real mappings), this requirement amounts to just $r-1$ independent relations rather than $r(r-1)/2$, as would be the case with higher-dimensional mappings: see §6.8 infra.

Let $\mathbf{Lie}(\mathbf{a}) = \mathbf{Lie}(\mathbf{a_1}, \ldots, \mathbf{a_r})$ be the free Lie algebra generated by the $\mathbf{a_i}$'s. To each multi-integer $\mathbf{p} := (p_1, \ldots, p_r)$ we associate the projectors $\mathbf{p}^*$ and $\mathbf{p}^i$ defined by specialising as usual the free generators $\mathbf{a_i}$'s to ordinary differential operators $l_t := x^{1+t}\partial_x$:

$$\mathbf{p}^* := \mathbf{Lie}(\mathbf{a}) \to \mathbb{Z} \quad , \quad \mathbf{w} \mapsto \mathbf{p}^* . \mathbf{w} \tag{6.37}$$

$$\mathbf{p^i} := \mathbf{Lie}(\mathbf{a}) \to \mathbb{Z}[t] \quad , \quad \mathbf{w} \mapsto \mathbf{p^i} . \mathbf{w} \qquad (1 \leqslant i \leqslant r) \tag{6.38}$$

---

(22) in the customary sense of not being conjugate to a system of homographies – with allowance, as usual, for ramified changes of coordinate.

with

$$\mathbf{p}^* \,.\, \mathbf{w} \quad := \quad \frac{w(l_{p_1}, \ldots, l_{p_r}).x}{x^{1+p_1+\ldots+p_r}} \tag{6.39}$$

$$\mathbf{p}^{\mathbf{i}} \,.\, \mathbf{w} \quad := \quad \frac{\partial_\epsilon\, w(l_{p_1}, \ldots, l_{p_i} + \epsilon\, l_{t+p_i} \ldots, l_{p_r}).x\,||_{\epsilon=0}}{x^{1+t+p_1+\ldots+p_r}} \tag{6.40}$$

If $\mathbf{w} \in \mathbf{Lie}(\mathbf{a})$ is homogeneous of degree d_1 in $\mathbf{a_1}$, d_2 in $\mathbf{a_2}, \ldots$, then its $\mathbf{p}$-degree is simply defined as $\mathbf{p}\text{-}deg(\mathbf{w}) := \sum p_i\, d_i$.

Clearly, if $\mathbf{w}$ is of $\mathbf{p}$-degree P, these two identities hold:

$$(t-P)\,\mathbf{p}^*.\mathbf{w} \quad = \quad \sum_1^r (t-p_i)\,\mathbf{p}^{\mathbf{i}}.\mathbf{w} \tag{6.41}$$

$$\mathbf{p}^{\mathbf{i}}.\mathbf{w} \quad = \quad d_i\, \mathbf{p}^*.\mathbf{w} \,+\, \mathcal{O}(t) \quad as \quad t \to 0 \tag{6.42}$$

Now, fix a multi-integer $\mathbf{p} := (p_1, \ldots, p_r)$ and a shorter (one element less!) sequence $\mathbf{w} := (\mathbf{w_1}, \ldots, \mathbf{w}_{r-1})$ of homogeneous elements of $\mathbf{Lie}(\mathbf{a})$, of various or identical $\mathbf{p}$-degrees, it doesn'st matter, but all subject to the orthogonality condition:

$$\mathbf{p}^* \,.\, \mathbf{w_1} \;=\; \mathbf{p}^* \,.\, \mathbf{w_2} = \ldots = \; \mathbf{p}^* \,.\, \mathbf{w_{r-1}} \,=\, 0 \tag{6.43}$$

In view of (6.41), (6.42) the relation:

$$D(t) := \frac{(-1)^i}{(t-p_i)} \;\det[\mathbf{p^1}.\mathbf{w}, \ldots, \widehat{\mathbf{p^i}.\mathbf{w}}, \ldots, \mathbf{p^r}.\mathbf{w}] \tag{6.44}$$

defines a function $D(t) = D_{\mathbf{w}}(t, \mathbf{p})$ that is not only independent of i [23] but also polynomial in t. This function is the exact generalisation for siblings of the 'divisors' which we came up against when discussing twins.

Proposition 6.1 (**Siblings**). — *Consider a system of type (6.36) and fix a multi-integer* $\mathbf{p}$. *Assume that each Lie element* $\mathbf{w_i}$ *has exactly one lowest-degree homogeneous component* $\underline{\mathbf{w}}_i$. *In other words:*

$$\mathbf{w_i}(\mathbf{a}) := \log \mathbf{W_i}(e^{\mathbf{a}}) = \underline{\mathbf{w}}_{\mathbf{i}}(\mathbf{a}) + \textit{terms of higher } \mathbf{p}\textit{-degree} \quad (i = 1 \ldots r) \tag{6.45}$$

(23) The term $\widehat{\mathbf{p^i}.\mathbf{w}}$ is omitted in (6.44) and of course $\mathbf{p^j}.\mathbf{w}$ stands for

$$[\mathbf{p^j}.\mathbf{w_1}, \ldots, \mathbf{p^j}.\mathbf{w_{r-1}}]$$

Then, for the system (6.36) to admit a non-elementary 'sibling' solution of the form

$$f_i(x) = x\,\big(1 + a_i\,x^{p_i} + \sum_{t\geqslant 1} a_i(t)\,x^{t+p_i}\big) \quad (i = 1\ldots r) \tag{6.46}$$

with t *in* $\mathbb{N}$ *or, what here amounts to the same, in* $\mathbf{p}.\mathbb{N} := p_1\,\mathbb{N} + \ldots + p_r\,\mathbb{N}$, *it is* necessary *that the orthogonality conditions* $\mathbf{p}\,.\,\underline{\mathbf{w}}_{\mathbf{i}} = 0$ *be fulfilled for each* i *and, when they are, it is* sufficient *that the corresponding divisor* $D(t) = D_{\underline{\mathbf{w}}}(t, \mathbf{p})$ *should be* $\not\equiv 0$ *for* $t \in \mathbf{p}.\mathbb{N}$.

We skip the proof, as it hardly differs from the one for twins ($r = 2$).

6.8. Simple examples

Since we haven't developped the *Uker*-graduation of algebras **Lie** $(\mathbf{a_1}, \ldots, \mathbf{a_r})$ for $r \geqslant 3$, we shall have to be content with a *typical* rather than a *generic* example, i.e. one relying on highly multiple commutators and, consequently, of infinite 'codimension'. Moreover, to bring home the point that, with siblings as with twins, one can achieve invariance under any given *periodic automorphism*[24] of the r-generator free group, we shall choose $r = 3$ and pick the simplest example with free orders of tangency (p_1, p_2, p_3) and invariance under the circular permutation $(F_1, F_2, F_3) \mapsto (F_2, F_3, F_1)$. First we set:

$$\mathbf{V}(\mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3}) \;:=\; \{\mathbf{U}(\mathbf{A_3}, \mathbf{A_1}), \mathbf{U}(\mathbf{A_2}, \mathbf{A_1})\} \tag{6.47}$$

$$\text{with} \quad \mathbf{U}(\mathbf{B}, \mathbf{A}) \;:=\; \mathbf{A}\{\mathbf{B^{-1}}\{\mathbf{B}, \mathbf{A^2}\}\}\mathbf{A^{-1}} \tag{6.48}$$

$$= \mathbf{A\,B\,A^{-2}\,B^{-1}\,A^2\,B^{-1}\,A^{-2}\,B\,A} \tag{6.49}$$

$$= (\mathbf{U}(\mathbf{B^{-1}}, \mathbf{A})^{-\mathbf{1}})^{-1} \tag{6.50}$$

Now, in a general free group setting, imposing the pair-wise commutation of the three words:

$$\mathbf{V}(\mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3}) \;,\; \mathbf{V}(\mathbf{A_2}, \mathbf{A_3}, \mathbf{A_1}) \;,\; \mathbf{V}(\mathbf{A_3}, \mathbf{A_1}, \mathbf{A_2}) \tag{6.51}$$

would amounts to imposing *three* independent relations. However, when dealing with identity-tangent mappings F_i, imposing the pair-wise commutation of the three words:

$$V_1 := V(F_1, F_2, F_3)\;,\;\; V_2 := V(F_2, F_3, F_1)\;,\;\; V_3 := V(F_3, F_1, F_2) \tag{6.52}$$

---

(24) or any given group of periodic automorphisms

amounts to imposing only *two* independent relations, e.g.:

$$W_{12}(F_1,F_2,F_3) := \{V_1,V_2\} = 1 \qquad (6.53)$$
$$W_{23}(F_1,F_2,F_3) := \{V_2,V_3\} = 1 \qquad (6.54)$$

This 'transitiveness' property of commutation for identity-tangent mappings is simply due to the fact that F, G do commute if and only if their logarithms $F_* := \log(F) = f_*\partial_x$, $G_* = \log(G) = g_*\partial_x$ coincide, as operators, upto multiplication by a constant.

Thus we have our system of 2 relations (6.53), (6.54) with three unknowns F_i, and we can easily see than it admits a general sibling solution with free tangency orders (p_1,p_2,p_3) by observing that:

$$\begin{aligned}\log V(F_1,F_2,F_3) \quad &= \quad -2\,p_1\,p_1\,(p_1-p_2)\,(p_2-p_3)\,(p_3-p_1)\;x^{1+2p_1+2p_2+2p_3}\,\partial_x \\ &\quad\; + \text{ terms of higher degree}\end{aligned}$$

and then reasoning as in §3.4, Example 3.5 or 3.9.

7. Analytic nature of twins

7.1. Multiplier $|a| \neq 1$: convergence

Let us start with the case which, from the analytic viewpoint, is simplest: that of a multiplier $|a| \neq 1$. We first require an auxiliary lemma.

The 'sandwich equation':
Consider the equation:

$$H^{m_1}\,K_1\,H^{m_2}\,K_2\ldots H^{m_{r-1}}\,K_{r-1}\,H^{m_r}\,K_r \;=\; 1 \qquad (7.1)$$
$$h(x) \;\text{ unknown } \;\sim a\,x \qquad (7.2)$$
$$k_i(x) \;\text{ given } \;\sim b_i\,x \text{ and analytic} \qquad (7.3)$$
$$m_1+\ldots+m_r \neq 0 \qquad m_i \in \mathbb{Z} \qquad (7.4)$$
$$a^{m_1+\ldots+m_r}\,b_1\ldots b_r = 1 \qquad (7.5)$$

which – conditions (7.2-5) aside – is the most general type of equation that can arise in a non-abelian group. It goes by the name of 'sandwich equation' (see [F]) because the iterates of the unknown mapping h alternate with given factors k_i. However, the assumption $\sum m_i \neq 0$ is really one of non-alternance. It says that the right-hand (7.1) doesn't resolve into a product of commutators, so that the sandwich equation is truly the 'opposite' of the highly alternate equations which we have been reviewing in our search for identity-tangent twins. It is, however, quite relevant for the non identity-tangent case, in particular for proving analyticity.

Changing both data and unknown $H \mapsto \underline{H}\, H$, $K_i \mapsto \underline{K}_i\, K_i$, with suitable analytic $\underline{H}, \underline{K}_i$ we get an equivalent equation:

$$H_{(0)}^{\epsilon_0}\; B_0\; H_{(1)}^{\epsilon_1}\; B_1 \ldots H_{(l-1)}^{\epsilon_{l-1}}\; B_{l-1}\; H_{(l)}^{\epsilon_l}\; B_l \;=\; 1 \quad \text{with} \tag{7.6}$$

$$H_{(i)} := C_i\, H\, C_i^{-1} \tag{7.7}$$

$$h(x) \;\text{ new unknown } \;\; \sim x \tag{7.8}$$

$$b_i(x) \text{ new analytic data } \;\sim\; x \tag{7.9}$$

$$c_i(x) \text{ new analytic data } \;\sim \gamma_i\, x \tag{7.10}$$

$$\epsilon_1 + \ldots + \epsilon_l \neq 0 \qquad \epsilon_i \in \{1, -1\} \tag{7.11}$$

Up to reindexation and multiplication by a common constant, the c_i depend only on the m_i and a_i. If $|\gamma_i|$ attains its maximum for exactly *one* index i (let us call this the "peak condition"), we can show that *the sandwich equation admits a (clearly unique) analytic solution.*

Proof. — We may always arrange for the peak value $|\gamma_{i_0}|$ to be 1 and re-index so as to have $i_0 = 0$. We may also arrange for $C_0 = 1$. Eventually, our H is the unique solution of the fixed point problem

$$H = \mathcal{P}\,(H) \;\text{ with }\; \mathcal{P}\,(H) := (B_0\; H_{(1)}^{\epsilon_1}\; B_1 \ldots H_{(l-1)}^{\epsilon_{l-1}}\; B_{l-1}\; H_{(l)}^{\epsilon_l}\; B_l)^{-1} \tag{7.12}$$

The number of factors in $\mathcal{P}^n\,(1)$ is exactly $\sigma(l, n) := \frac{l+1}{l-1}\,(l^n - 1)$. Moreover, as soon as n exceeds n_0, the factors of the form

$$B_{i(j_1, j_2 \ldots, j_{n_0})}^{\pm 1} := C_{j_{n_0}} \ldots C_{j_2}\, C_{j_1}\; B_i^{\pm 1}\; C_{j_1}^{-1} C_{j_2}^{-1} \ldots C_{j_{n_0}}^{-1} \tag{7.13}$$

present inside $\mathcal{P}^n\,(1)$ do not change, although their exact location does. Their total number is clearly $(l+1)\, l^{n_0}$. Their total contribution, therefore, will be small of order $(l+1)\, l^{n_0}\; \gamma_*^{n_0\, k_*}$ with $\gamma_* := inf|\gamma_i| < 1$ for $i = 1..l$ and with $k_* > 1$ denoting the lowest tangency order for the new unknown h and the new data b_i. But if we choose $\underline{H}$ analytic and close enough (i.e. tangent to a high enough order) to a *formal solution* of H_0, we can make k as large as we wish and in particular ensure that $|l\, \gamma_*^k| < 1$. Fixing such a k_* , we see that the analytic operators $\mathcal{P}^n(1)$ converge *normally* to an analytic operator $\mathcal{P}^\infty(1)$ which necessarily coincides with H_0. Which means that the formal H_0 was analytic in the first place. $\square$

Application to twins. — The scheme clearly applies to all four examples $W_1(A,B)\dots W_4(A,B)$ above since $m_2 = 1$. It would apply equally well to any example constructed from a *monic polynomial*[25] $\Delta_{\mathbf{W}}(\theta)$ with at least one 'resonant' root $|a| > 1$. It applies, in fact, with minor adaptations, to most situations involving multipliers $|a| \neq 1$.

An interesting aside is this: once f has been normalised to $f(x) = a\,x$, what can be said of g and its natural Riemann surface? The latter appears to possess, for almost all values of the parameters b_i, a highly fractal boundary.

7.2. Multiplier $|a| = 1$: divergence

When a is on the unit circle but not a unit root, we should expect generic divergence of the formal twins since, in a suitable z-chart ($z \sim \infty$), solving the twin equation reduces to solving an infinite sequence of affine equations:

$$P\,\varphi_n = \psi_n \quad \text{with} \quad P\varphi(z) := \sum c_k\,\varphi(a^k\,z + b_k) \quad \text{and} \quad z \sim \infty \qquad (7.14)$$

where, at the n-th inductive step, ψ_n is known and φ_n unknown [26]. Now, even for an analytic[27] input ψ_n, the solution φ_n is generically divergent and non-summable. Of course, compensation within the series $\sum \varphi_n$ cannot be ruled out off hand, but when the solution depends on a continuous parameter, a simple argument shows that we must have divergence for almost all values of that parameter.

7.3. Pre-identity-tangent 'twins': convergence

As already pointed out, these are not genuine twins, since they are constrained by a system S of more than one relation and depend on countably many parameters. However, that very circumstance *makes it possible to construct analytic solutions.* The proof, which we skip, relies on the fact:
– that S resolves itself into an inductive sequence of affine systems $S_1, S_2 \dots$, each of which admits analytic solutions
– that suitable a priori bounds can be imposed on the solution of S_n to ensure convergence in the resulting series.

Needless to say, there doesn't seem to exist any 'privileged' choice among these analytic solutions.

(25) ie a polynomial with integer coefficients and a leading term with unit coefficient.
(26) The sum $\sum$ in the definition of P is finite but it never reduces to a single term.
(27) at infinity.

7.4. Identity-tangent twins: the general picture

This is by far the most interesting, but also the most tricky case. The broad picture with identity-tangent twins of tangency order (p,q), i.e. of the form:

$$f(x) = x\,(1 + \alpha\, x^p + \ldots), \quad g(x) = x\,(1 + \beta\, x^q + \ldots) \qquad (\alpha \neq 0, \beta \neq 0)$$

with, say, $p < q$, is as follows:

P1: Due to twins (f, g) being defined only upto a joint conjugation, the question that makes sense is: how simple (convergent, Gevrey, resurgent, etc) can f and g be rendered *simultaneously*? Another way of approaching the problem is to *normalise* f to f^{nor} and then look at the nature of the corresponding *conormal* form g^{conor} of g. Or *vice versa*. Or again, more intrinsically, one may investigate the *connector* h^{nor} defined in (3.14). As it happens, *it is enough to understand any one of these functions to understand the rest* [28] and so we shall refer to them collectively as *"the intrinsic twin functions"*.

P2: *The intrinsic twin functions are doubly resurgent, with respect to the slower critical time* $z_1 := x^{-p}$ *and to the faster critical time* $z_2 := x^{-q}$.

P3: *The intrinsic twin functions are accelero-summable. Moreover, the axis* $\mathbb{R}^+$ *is free of singularities in either of the Borel planes* $\{\zeta_1\}$ *and* $\{\zeta_2\}$ *conjugate to the multiplicative plane* $\{z_1\}$ *and* $\{z_2\}$. *As a consequence there exists a privileged real accelero-summation.*[29]

P4: *The privileged sum thus obtained coincides with the one produced by the 'geometric' method sketched in §8.4.*

P5: *The invariants associated with the* z_1*- and* z_2*-resurgence regroup naturally to form a geometric object, the "shadow twins", consisting of* two *formal series*[30] *also connected by* one *relation. These "shadow twins" carry all the obstructions to the analyticity of the original twins* (f, g).

---

(28) This is by no means obvious, since the normalisation of an analytic identity tangent f to f^{nor} usually involves a divergent-resurgent change of coordinate h: see §1.3. In the present instance, however, the divergence-resurgence of h cannot 'add' to the divergence-resurgence already present in g.

(29) consisting in calculating on $\mathbb{R}^+$ both the acceleration integral in the ζ_1-plane and the Laplace integral in the ζ_2-plane.

(30) They are not power series, though, and, unlike the original twins, they carry transcendental rather than rational coefficients

P6: *Whenever twins depend on one or several continuous parameters, the intrinsic twin functions are guaranteed to be non-analytic (i.e. strictly resurgent) except (at most) for a discrete set of parameter values.*

P7: *It would seem reasonable to conjecture the non-analyticity of all genuine*[31] *identity-tangent twins.*

7.5. Identity-tangent twins: autonomous differential-difference equations and double resurgence

The phenomenon of double resurgence in the 'intrinsic twin functions' is simplest to understand in cases with a free ratio p/q and a free continuous parameter $\gamma = \alpha/\beta$, like in Ex 3.5 and Ex 3.6 of §3.4. To further simplify, we impose symmetries that make the iterative residues α_*, β_* of F, G vanish [32] so that F and G may be *separately* normalised to:

$$F^{nor} = \exp(-\alpha x^{p+1}\,\partial_x) \qquad G^{nor} = \exp(-\beta x^{q+1}\,\partial_x) \quad (\gamma := \alpha/\beta) \tag{7.15}$$

Here is one such example:

$$\begin{aligned}
\mathbf{W}(\mathbf{A},\mathbf{B}) &:= \{\mathbf{U}(\mathbf{A},\mathbf{B}),\mathbf{V}(\mathbf{A},\mathbf{B})\} \;=\; 1 \quad \text{with}\\
\mathbf{U} &:= \mathbf{U}^{++}\,\mathbf{U}^{+-}\,\mathbf{U}^{-+}\,\mathbf{U}^{--}\\
\mathbf{V} &:= \mathbf{V}^{++}\,\mathbf{V}^{+-}\,\mathbf{V}^{-+}\,\mathbf{V}^{--}\\
\mathbf{U}^{++}(\mathbf{A},\mathbf{B}) &:= \mathbf{B}\,\{\{\mathbf{B}\{\mathbf{B},\mathbf{A}\}\},\{\mathbf{B},\mathbf{A}\}\}\,\mathbf{B}^{-1}\\
\mathbf{V}^{++}(\mathbf{A},\mathbf{B}) &:= \mathbf{B}\,\{\mathbf{B}\{\mathbf{A}\{\mathbf{B}\{\mathbf{B},\mathbf{A}\}\}\}\}\,\mathbf{B}^{-1}\\
\mathbf{U}^{-+}(\mathbf{A},\mathbf{B}) &:= \mathbf{U}^{++}(\mathbf{A}^{-1},\mathbf{B})\\
\mathbf{U}^{+-}(\mathbf{A},\mathbf{B}) &:= \mathbf{U}^{++}(\mathbf{A},\mathbf{B}^{-1})^{-1}\\
\mathbf{U}^{--}(\mathbf{A},\mathbf{B}) &:= \mathbf{U}^{++}(\mathbf{A}^{-1},\mathbf{B}^{-1})^{-1}\\
\mathbf{V}^{-+}(\mathbf{A},\mathbf{B}) &:= \mathbf{V}^{++}(\mathbf{A}^{-1},\mathbf{B})\\
\mathbf{V}^{+-}(\mathbf{A},\mathbf{B}) &:= \mathbf{V}^{++}(\mathbf{A},\mathbf{B}^{-1})^{-1}\\
\mathbf{V}^{--}(\mathbf{A},\mathbf{B}) &:= \mathbf{V}^{++}(\mathbf{A}^{-1},\mathbf{B}^{-1})^{-1}
\end{aligned}$$

Going over to the Lie algebra, we find:

$$\mathbf{u}(\mathbf{a},\mathbf{b}) \;:=\; \log\mathbf{U}(\mathbf{e^b}\mathbf{e^b}) = 4\,[\overline{\mathbf{b}}\,\overline{\mathbf{b}}\mathbf{a},\overline{\mathbf{b}}\mathbf{a}] + \text{higher degree} \geqslant 7 \tag{7.16}$$

$$\mathbf{v}(\mathbf{a},\mathbf{b}) \;:=\; \log\mathbf{V}(\mathbf{e^b}\mathbf{e^b}) = 4\,[\overline{\mathbf{b}}\,\overline{\mathbf{a}}\,\overline{\mathbf{b}}\,\overline{\mathbf{b}}\mathbf{a}] + \text{higher degree} \geqslant 7 \tag{7.17}$$

---

(31) ie non-elementary, in the sense of §1.1
(32) See (1.19).

However, whereas in the formal investigation it was permissible to linearise with respect to **A** and **B** simultaneously, the analytic investigation must take global effects into account. More precisely, in order to understand the double resurgence, with must linearise with respect to **A** alone, then **B** alone, and then, in order to establish accelerablity, we must see how these two resurgences interact.

First linearisation. — We first linearise in **A**, i.e. in F. We find in the groups of diffeos:

$$\begin{aligned} U^{++}(F,G) &= F_1^{-1}\,F_0\,F_{-1}^{-1}\,F_0\,F_1^{-1}\,F_{-1}\,F_0^{-1}\,F_1\,F_0^{-1}\,F_1 \\ V^{++}(F,G) &= F_0^{-1}\,F_{-1}\,F_{-2}^{-1}\,F_{-1}\,F_0^{-1}\,F_{-1}^{-1} \\ &\qquad F_{-2}\,F_{-1}^{-1}\,F_0^{2}\,F_1^{-2}\,F_0\,F_{-1}^{-1}\,F_0\,F_1\,F_0^{-1}\,F_{-1}\,F_0^{-1}\,F_1 \\ \text{with } F_n &:= G^n\,F\,G^{-n} \end{aligned}$$

and then in the Lie algebra:

$$u(F,G).x = \log(U(F,G)).x \;= 4\,\Phi_{-1,0} - 4\,\Phi_{-1,1} + 4\,\Phi_{0,1} + \deg \geqslant 4 \quad (7.18)$$
$$v(F,G).x = \log(V(F,G)).x \;= -2\,\Phi_{-1,0} +'\;\Phi_{-1,1} - 2\,\Phi_{0,1} + \deg \geqslant 4 \quad (7.19)$$

with

$$\begin{aligned} F_* &:= \log F = \Phi(x)\,\partial_x \\ F_{*\,n} &:= \log F_n = \Phi_n(x)\,\partial_x = \frac{\Phi\circ g^{\circ n}(x)}{\partial_x\,g^{\circ n}(x)}\,\partial_x \\ \Phi_{m,n} &:= [\Phi_m,\Phi_n] := \Phi_m\,\Phi_n' - \Phi_m'\,\Phi_n \end{aligned}$$

Switching over to the 'critical variable' $z_2 := x^{-q}$ that normalises G we get:

$$\begin{aligned} G_* &:= \log G = \partial_{z_2} \qquad\Rightarrow\qquad g(z_2) := z_2 + 1 \\ F_* &:= \log F = \varphi(z_2)\,\partial_{z_2} \\ F_{*\,n} &:= \log F_n = \varphi_n(z_2)\,\partial_{z_2} = \varphi(z_2+n)\,\partial_{z_2} \\ \varphi_{m,n} &:= [\varphi_m,\varphi_n] = \varphi(z_2+m)\,\varphi'(z_2+n) - \varphi'(z_2+m)\,\varphi(z_2+n) \end{aligned}$$

Freezing to $\beta = 1$ the parameter associated with G and expanding everything in powers of the parameter α associated with F, we find:

$$\varphi(z_2) = \alpha\,\varphi^{<1>}(z_2) + \alpha^2\,\varphi^{<2>}(z_2) + \ldots \quad \text{with}$$
$$P_2\,\varphi^{<1>} = 0 \quad (7.20)$$
$$P_2\,\varphi^{<k>} = \text{earlier terms} \quad (k = 2,3\ldots) \quad (7.21)$$

with a bilinear difference-differential operator P_2 of the form:

$$P_2\,\varphi(z_2) := \sum_{m<n} c_{m,n}\,[\varphi(z_2+m),\varphi(z_2+n)]$$

and coefficients $c_{m,n}$ affine in $\sigma{:=}p/q$ and easily calculable from the linearisations (7.18), (7.19):

$$\begin{aligned}
c_{-2,0} &= c_{0,2} = -\frac{1}{2}(\sigma-1)\\
c_{-1,0} &= c_{0,1} = \ \ 2\,(2\sigma+1)\\
c_{-1,1} &= \qquad\quad -3\,(\sigma+1)
\end{aligned}$$

Second linearisation. — If we now linearise in $\mathbf{B}$ i.e. in G, the picture remains much the same, with $(F,\Phi,\varphi,\alpha,z_2,\sigma,P_2)$ and $(G,\Psi,\psi,\beta,z_1,\sigma^{-1},P_1)$ exchanging places, but with a trilinear operator P_1 instead of the bilinear P_2. Let us, for definiteness, go through this second list of parallel equations. In the group we get:

$$\begin{aligned}
U^{++}(F,G) &= G_0\,G_{-1}^{-1}\,G_0^{-1}\,G_{-1}\,G_0\,G_{-1}^{-2}\,G_0\,G_{-1}\,G_0^{-1}\,G_{-1}\,G_0^{-1}\\
V^{++}(F,G) &= G_{-1}^{-1}\,G_0^{-1}\,G_{-1}\,G_0\,G_{-1}^{-1}\,G_{-2}^{-1}\,G_{-1}\,G_{-2}\,G_0\\
&\qquad G_{-2}^{-1}\,G_{-1}^{-1}\,G_{-2}\,G_{-1}\,G_{-1}^{-1}\,G_{-1}^{-1}\,G_0\,G_{-1}\,G_0^{-1}\\
\text{with }\ G_n &:= F^n\,G\,F^{-n}
\end{aligned}$$

Going over to the Lie algebra, these relations become:

$$u(F,G).x \;=\; \log(U(F,G)).x \;=\sum_{n_1,n_2,n_3} c^{*}_{n_1,n_2,n_3}\;\Psi_{n_1,n_2,n_3}\;+\;\deg\geqslant 6 \quad (7.22)$$

$$v(F,G).x \;=\; \log(V(F,G)).x \;=\sum_{n_1,n_2,n_3} c^{**}_{n_1,n_2,n_3}\;\Psi_{n_1,n_2,n_3}\;+\;\deg\geqslant 6 \quad (7.23)$$

when expressed in the given chart x:

$$\begin{aligned}
G_* &:= \log G = \Psi(x)\;\partial_x\\
G_{*\,n} &:= \log G_n = \Psi_n(x)\;\partial_x = \frac{\Psi\circ f^{\circ n}(x)}{\partial_x\,f^{\circ n}(x)}\;\partial_x\\
\Psi_{n_1,n_2,n_3} &:= [[\Psi_{n_1},\Psi_{n_2}]\Psi_{n_3}]
\end{aligned}$$

Both sums $\sum$ in (7.22), (7.23) are finite, and their coefficients $c^{*}_{n_1,n_2,n_3}$, $c^{**}_{n_1,n_2,n_3}$ are easily calculable from the above factorisations of U^{++}, V^{++}.

Turning to the critical chart $z_1 := x^{-p}$ that normalises F, we find:

$$\begin{array}{rcll} F_* & := & \log F = \partial_{z_1} & \Rightarrow \qquad f(z_1) := z_1 + 1 \\ G_* & := & \log G = \psi(z_1)\, \partial_{z_1} & \\ G_{*\,n} & := & \log G_n = \psi_n(z_1)\, \partial_{z_1} = \psi(z_1+n)\, \partial_{z_1} & \end{array}$$

the components of the β-expansion of ψ are recursively given by the system:

$$\psi(z_1) = \beta\, \psi^{<1>}(z_1) + \beta^2\, \psi^{<2>}(z_1) + \ldots$$

$$P_1\, \psi^{<1>} = 0 \tag{7.24}$$

$$P_1\, \psi^{<k>} = \text{earlier terms} \qquad (k = 2, 3 \ldots) \tag{7.25}$$

with a difference-differential operator P_2 of the form (finite sum):

$$P_1\, \psi(z_1) := \sum_{n_1, n_2, n_3} c_{n_1,n_2,n_3}\, [\psi(z_1+n_1)[\psi(z_1+n_2), \psi(z_1+n_3)]]$$

whose coefficients c_{n_1,n_2,n_3} are affine in $\sigma{:=}p/q$ and easily deducible from the linearisations (7.22), (7.23):

$$c_{n_1,n_2,n_3} \equiv 2\,(2\,\sigma+1)\; c^{*}_{n_1,n_2,n_3} \;+\; (\sigma-1)\; c^{**}_{n_1,n_2,n_3}$$

Singular autonomous difference-differential operators P. — The homogeneous equations (7.20) or (7.24) which start the induction can easily be shown to admit a unique formal solution of the form:

$$\varphi^{<1>}(z_2) = z_2^{1-\sigma}\, (1 + \sum \gamma_{2n}(\sigma)\, z_2^{-2n}) \qquad (\sigma := p/q) \tag{7.26}$$

$$\psi^{<1>}(z_1) = z_1^{1-\frac{1}{\sigma}}\, (1 + \sum \delta_{2n}(\sigma)\, z_1^{-2n}) \qquad (\sigma^{-1} := q/p) \tag{7.27}$$

that is divergent-resurgent in its single "critical variable", z_2 or z_1, and *always* possesses a countable infinity of non-zero *alien derivatives.*

The same holds for the non-homogeneous equations (7.21) or (7.25) that continue the induction, as long as their right-hand sides are themselves resurgent.

Here, the 'invariants' or 'resurgence coefficients' or 'Stokes constants' which enter the resurgence equations as their only transcendental ingredient, are particularly interesting entire functions of σ, of so-called *autarkic* type[33].

(33) ie with an asymptotics completely defined by a finite set of – equally autarkic – entire functions. Actually, the class of autonomous difference-differential equations and their invariants would warrant a special investigation, but there is no room for that here.

7.6. Identity-tangent twins: canonical accelero-summability

Thus, things are fairly unproblematic as long as we expand our 'intrinsic twin-related functions' in powers of the free parameter and consider each contribution $\varphi^{<k>}$ or $\psi^{<k>}$ in isolation. But the moment we attempt to sum all these contributions, a sharp dissymmetry makes itself felt between the two critical variables. Consider for definiteness the choice $p < q$ [34] so that here z_1 is the 'slower' and z_2 the 'faster' variable: $z_1 \ll z_2$.

Summing $\varphi(z_2)$ in α is no problem *formally*, since only a finite number of terms $\varphi^{<k>}(z_2)$ contribute to any given (negative) power of z_2, but Borel transforming $\varphi(z_2)$ to $\hat{\varphi}(\zeta_2)$ leads to a divergent power series, which is no use at all.

On the other hand, summing $\psi(z_1)$ in β and Borel transforming it to $\hat{\psi}(\zeta_1)$ yields a convergent germ at $\zeta = 0$, with a sum that can be analytically continued along the whole of $\mathbb{R}^+$ with a faster-than-exponential growth pattern:

$$|\hat{\psi}(\zeta_1)| < C_0 \ \exp(C_1 \, |\zeta_1|^{\frac{q}{q-p}}) \qquad (0 < C_i < +\infty) \tag{7.28}$$

This is not good enough to apply the Laplace transform[35] but just about right to apply the *acceleration integral* [36] from ζ_1 to ζ_2 and get a new function that now has exponential growth in ζ_2 and so can be subjected to Laplace integration.

Accelero-summation here is straightforward, as there are no singularities over the axis $\arg(\zeta_1) = 0$ or $\arg(\zeta_2) = 0$ in either of the Borel planes. All told, we get a canonical sum, resulting in 'germinal' twins (f, g) that are defined and regular in a sectorial neighbourhood of $\mathbb{R}^+$ in the original geometric plane x.

For an alternative method – which in this case leads to the same result, and is more geometric, but not nearly as constructive – see §8.4 *infra*.

7.7. Identity-tangent twins: 'shadow twins' and obstructions to analyticity

There is a standard way of attaching to any resurgent or multi-resurgent function f two richer objects, *display*(f) and *restrict*(f), which automat-

(34) Remember that here the ratio p/q is *free*.
(35) which reverses Borel
(36) See [E6], Lecture 3, p114-115.

ically satisfy the same relations[37] as f, while "displaying" in user-friendly manner *all* the resurgence properties of f and *all* its transcendental 'invariants' or 'Stokes constants'. The *display* carries *all* alien derivations of f, of *all* orders, and combines the original, resurgence-bearing variable z with a countable infinity of so-called *pseudo-variables* $Z^{\omega_1,\dots,\omega_r}$. The *restriction* is obtained therefrom by jettisoning the true variable and retaining only the pseudo-variables. For details, see for instance [E9], §2.4.

Applying this to a pair of twins (f,g) [38] linked by a relation (7.29), we get first a mixed object (7.30), and then a 'pure' one, the *'shadow twins'*, linked by a relation (7.31) formally identical to (7.29), and which concentrates all the obstructions to the analyticity of (f,g).

$$\textit{twins:} \qquad 1 = W \quad (f,g) \tag{7.29}$$

$$\Downarrow$$

$$1 = W \quad (\mathrm{display}(f), \mathrm{display}(g)) \tag{7.30}$$

$$\Downarrow$$

$$\textit{shadow twins:} \qquad 1 = W \quad (\mathrm{restrict}(f), \mathrm{restrict}(g)) \tag{7.31}$$

8. Transserial twins

8.1. Reminders about transseries

This section is something of an aside, and we shall be extremely sketchy.

Let $z \sim +\infty$. The formal trigebra $\mathbb{T}$ or $\mathbb{R}[[[z]]]$ consists, very roughly, of the natural completion of $\mathbb{R}[[z^{-1}]]$ under the basic operations $\{+, \times, \circ, \partial\}$ and their inverses. Its elements, the so-called *transseries*, may involve extremely intricate concatenations of exponentials and logarithms, but always admit a (unique) distinguished or 'canonical' representation obtained by

– expelling all infinitesimals from inside the exponentials

– expelling all sums from inside the logarithms[39]

There is a natural, total order on the trigebra of transseries. In particular, in the distinguished representation, each transseries appears in the

(37) whatever their form or number, provided they make use "only" of the operations $+, \times, \circ, \partial$ and their inverses.

(38) expressed in any chart that makes them simultaneously resurgent.

(39) using in both cases the functional equations and Taylor expansions of exp and log.

guise of a transfinite, well-ordered sum of *transmonomials* which, despite their 'atomicity', may carry infinitely many coefficients, with a complex arborescent structure on them.

So much for the formal transseries. The geometric counterpart is the trigebra $\mathbb{R}\{\{\{z\}\}\}$ of so-called *analysable germs* at $+\infty$. These are in one-to-one correspondance with a small (or huge, depending on how you look at it) subalgebra of $\mathbb{R}[[[z]]]$, consisting either of 'convergent' or divergent but 'accelero-summable' transseries.

The correspondance, needless to say, commutes with the full array of basic operations $\{+, \times, \circ, \partial\}$ and respects the natural order. Both trigebras enjoy a long list of auspicious properties: stability, closure, algorithmic solvability of many types of differential or functional equations, etc.

A number of applications, though, call for even larger extenstions, resulting from the introduction of extremely fast-growing functions $\mathcal{E}_n$, the so-called *transexponentials* of strength n, or the even faster growing *ultra-exponential* $\mathcal{E}_\infty$.

Before defining the latter, let us recall the usual abbreviations for commutators and conjugations:

$$\overline{\overline{\mathbf{A}}}.\mathbf{B} \;:=\; \{\mathbf{A},\mathbf{B}\} \;:=\; \mathbf{A}^{-1}\,\mathbf{B}^{-1}\,\mathbf{A}\,\mathbf{B} \tag{8.1}$$

$$\mathbf{A}:\mathbf{B} \;:=\; \mathbf{B}\,\mathbf{A}\,\mathbf{B}^{-1} \tag{8.2}$$

$$\mathbf{A}:\mathbf{B}:\mathbf{C} \;:=\; \mathbf{A}:(\mathbf{B}:\mathbf{C}) \tag{8.3}$$

$$\neq \quad (\mathbf{A}:\mathbf{B}):\mathbf{C} \;=\; \mathbf{A}:(\mathbf{C}\,\mathbf{B})$$

Let T be the unit shift and E the exponential mapping. The transexponentials $\mathcal{E}_n$ are 'characterised' by "smoothness" and the conjugation equations:

$$\begin{aligned} T:\mathcal{E}_1 &\equiv E \\ T:\mathcal{E}_2 &\equiv \mathcal{E}_1 \\ T:\mathcal{E}_3 &\equiv \mathcal{E}_2 \\ &\dots \end{aligned}$$

The ultraexponential $\mathcal{E}_\infty$ is also 'characterised' by "smoothness" and a growth pattern faster than that of any $\mathcal{E}_n$ of finite strength n.

Actually, no matter what amount of "smoothness" we choose to impose, there is a huge indeterminacy inherent in this construction [40], which cannot

(40) roughly, one smooth periodic conjugation per step, and an even more formidable indeterminacy for $\mathcal{E}_\infty$.

be removed by purely asymptotic criteria at $+\infty$. This is the bad news, or the 'negative side' of the so-called *indiscernibility theorem.* However, by the same token, no matter what choice we strike, the resulting trans- or ultra-exponential extensions of our trigebra of transseries are going to be essentially *one*, i.e. isomorphic under the full struxture $\{+, \times, \circ, \partial, <\}$. This is the good news, or the 'positive side' of the *indiscernibility theorem.*

The trigebras of transseries and analysable functions were introduced in the late 90s to solve the so-called Dulac problem about the finiteness of limit-cycles. See [E5], [E6], [E7], [E8]. The emphasis there was squarely on the analytic side, and developments about the formal construction – ie the trigebra of transseries – were kept to a minimum. A far more sophisticated theory of formal transseries was subsequently developed by van der Hoeven (see [H1], [H2], [H3]), with special attention to the algorithmic resolution of differential, functional etc equations. The present section §8, incidentally, benefited from exchanges we had with van der Hoeven. Lastly, for complements about the indiscernibility theorem, the fast/slow functions (such as the trans- or ultra-exponentials and their reciprocals) and the whole subject of *'universal fast/slow asymptotics'*, we refer to [E5], chap. 7-10.

8.2. Transserial twins of exponential or transexponential type

Switching from power series to transseries, especially of the transexponential sort, brings significant changes to the *typology* of twins. There appears a whole new class of twin-begetting relations which may be be called *'regular'* or *'orderly'* and which had no true equivalent in the more restrictive setting of power series.

Consider for instance the following series of (F, G)-relations, where F has been normalised to the unit shift T. They consist of a principal, highly-alternate, $(1+k)$-shrinking[41] factor $\{T\,,\, T^{n_1} : \ldots : T^{n_k} : G\}$ and a perturbative, even more alternate, $(2+k)$-shrinking factor. $W_k(T, G)$.

twin relation	*nature of G*	*par. nb.*
$1 = W_0(T,G)\{T\,,\, G\}$	translations	1 param.
$1 = W_1(T,G)\{T\,,\, T^{n_1} : G\}$	dilatations	2 param.
$1 = W_2(T,G)\{T\,,\, T^{n_1} : T^{n_2} : G\}$	exponentials	3 param.
$1 = W_3(T,G)\{T\,,\, T^{n_1} : T^{n_2} : T^{n_3} : G\}$	transexp. of str. 1	4 param.
... .	....	......
$1 = W_k(T,G)\{T\,,\, T^{n_1} : \ldots : T^{n_k} : G\}$	transexp. of str. $k-2$	k param.

(41) An operator $G \mapsto \mathcal{P}(G)$ is *k-shrinking* if, $\forall m > 0$, it 'shrinks' any transexponential of strength $m+k$ to something that lies in the transexponential growth range of strength m.

The k-th equation of this list has a neat, predictable set of twin solutions, which constitute a k-dimensional real variety.

Then, alongside these *'regular'* or *'orderly'* twin-begetting relations, we have also, like with power series but in an even more bewildering variety of shapes, a collection of *'exceptional'* or *'sporadic'* relations, with twin solutions whose existence, number, nature, connectedness/disconnectedness etc ... depend on a 'fluke of algebra' – like $\log \mathbf{W}(e^{\mathbf{a}}, e^{\mathbf{b}})$ accidentally falling into this or that cell of the $Uker$-filtration – that is not at all apparent on the word $\mathbf{W}(\mathbf{A}, \mathbf{B})$ itself.

8.3. Van der Hoeven's intermediate value theorem

Let P be a differential polynomial with transserial coefficients. Van der Hoeven has shown in [H2] that, *given any pair of transseries* g_1, g_2 *such that* $P(g_1) < 0 < P(g_2)$ *there exists at least one solution* $P(g_0) = 0$ *with* $g_0 \in]g_1, g_2[$. He even supplies an algorithm for constructing g_0. He has also extended this *intermediate value theorem* to an even broader class of functional equations – not broad enough, however, to cover all composition equations $W(f,g) = id$ [42]. Nevertheless, it seems highly likely that the theorem holds in that context also.

8.4. Canonical resummation of transserial twins

Fundamental squares. — Let $W(f,g) = id$ be a twin-begetting relation and $(\tilde{f}_0, \tilde{g}_0)$ a formal transserial solution, normalised to $\tilde{f}_0 = T =$ *unit shift*. We wish to "sum" $(\tilde{f}_0, \tilde{g}_0)$ to a geometric twin (f_0, g_0) consisting of true germs at $+\infty$.

Here, it is convenient to write W in this way:

$$\begin{aligned} W(f,g) \quad &:= \quad k_{\eta_r}^{\epsilon_r} \circ k_{\eta_{r-1}}^{\epsilon_{r-1}} \circ \ldots k_{\eta_2}^{\epsilon_2} \circ k_{\eta_1}^{\epsilon_1} \\ \epsilon_i, \eta_i \quad &\in \quad \{+,-\} \;\; , \;\; k_+ := f \;\; , \;\; k_- := g \;\; , \;\; \{\eta_i = \eta_{i+1}\} \Rightarrow \{\epsilon_i = \epsilon_{i+1}\} \end{aligned} \tag{8.4}$$

A "fundamental square" in $\mathbb{R}^+ \times \mathbb{R}^+$ is a square $[a,b] \times [a,b]$ that contains a "tentative portion" of the trivial graphs $[z, f_0(z)], [f_0(z), z]$ and of the non-trivial graphs $[z, g_0(z)], [g_0(z), z]$, with a full "cycle" of points Z_i duly

(42) with, say, f known and g unknown.

located on these graphs and appearing in the *proper succession.*[43]

$$\begin{aligned} h_i &:= k^{\epsilon_i}_{\eta_i} \circ \ldots \circ k^{\epsilon_1}_{\eta_1} \qquad (8.5) \\ Z_1 &:= (h_0(z)\,,\, h_1(z)) = (z\,,\, h_1(z)) \\ Z_2 &:= (h_1(z)\,,\, h_2(z)) \\ \ldots & \\ Z_{r-1} &:= (h_{r-2}(z)\,,\, h_{r-1}(z)) \\ Z_r &:= (h_{r-1}(z)\,,\, h_r(z)) = (h_{r-1}(z)\,,\, z) \end{aligned}$$

Functional continuation and periodic adjustment. — For any $(\tilde{f}_0, \tilde{g}_0)$ we may always construct such a square with an approximate $(\overline{f}_0 := T, \overline{g}_0)$, with $\overline{f}_0 := T$ and $\overline{g}_0$ smooth, even analytic, except possibly at the extremal points $z = a$ and $z = b$. Provided $\overline{g}_0$ is close enough to a leading section of $\tilde{g}_0$ (itself taken large enough to determine *all* the finitely many, discrete or continuous parameters on which the general twin solution (f, g) depends) there clearly exists a *unique* functional continuation of $\overline{g}_0$ (that of $\overline{f}_0$ is trivial) over the whole interval $[b, +\infty[$, with $\overline{g}_0$ everywhere smooth (or analytic) except possibly at a sequences of 'images' $z_3, z_4, z_5 \ldots$ of z_1, z_2 and with the consecutive images of the r points Z_i retaining their proper order inside the succesive images of the 'fundamental square'.

An asymptotic analysis of $\overline{g}_0$ at $+\infty$ will reveal an oscillatory part interfering with the proper transserial part. But these parasitical oscillations may *always* be *uniquely* corrected by conjugation $\overline{g}_0 \mapsto g_0 = h^{-1} \circ \overline{g}_0 \circ h$ with a suitable 1-periodic mapping h, i.e. one that commutes with the unit shift T. And not only does this unique h-conjugation spirit away the parasitical oscillations; it also has the automatic effect of restoring smoothness at the points $z_1, z_2, z_3 \ldots$. The pair (f_0, g_0) thus obtained may be regarded as the exact geometric counterpart, or 'sum', of the formal pair $(\tilde{f}_0, \tilde{g}_0)$.

There exits, however, a significant difference depending on whether the formal pair $(\overline{f}_0, \overline{g}_0)$ does or does not involve transexponentials $\mathcal{E}_n$.

If *it doesn't*, then (f_0, g_0) is determined *absolutely* and *real-analytic* on $[\ldots, +\infty[$, though of course usually not at $+\infty$.

If *it does*, then (f_0, g_0) is determined *relatively* to a choice of $\mathcal{E}_n$ (and here the 'indiscernibility principle' comes into play) and *probably* (though this hasn't been proved) *real-analytic* on $[\ldots, +\infty[$ for a real-analytic choice

(43) That order is always unambiguously determined by the formal twin $(\tilde{f}_0, \tilde{g}_0)$, due to the full order that exists on transseries.

of $\mathcal{E}_n$. This is by no means a foregone conclusion, for with respect to general transserial functional equations, an analytic choice of $\mathcal{E}_n$ doesn't guarantee the solution's analyticity, but only its *cohesiveness* – which is a strong form of smoothness and quasi-analyticity: see [E5], [E6], [E7]. But the fact is that twin equations are a very special sub-class of composition equations, and enjoy a whole range of specific properties. So the jury is still out on that one.

8.5. The ordering of free or bound groups. Natural and exotic orders. Ultraexponential arbitration

Natural and exotic orders on the composition group of transseries. — Let $\mathbb{IT}$ denote the composition group of all (formal) identity-tangent transserial mappings $f : z \mapsto z + \mathcal{O}(1)$ $(z \sim +\infty)$. Alongside its natural order $\prec$, the group $\mathbb{IT}$ admits a non-countable infinity of exotic orders $\prec^{\mathcal{I},\epsilon}$.

Here is how to construct them. A transmonomial interval $I := [A_1, A_2]$ or $]A_1, A_2]$ or $[A_1, A_2[$ or $]A_1, A_2[$ bounded by two infinitesimal transmonomials $A_1 \succ A_2$ is declared shift-invariant if for any transmonomial $A \in I$ and any real shift t, the transseries $A_t(z) := A(z+t)$ has as its leading term a transmonomial $leadA_t$ that also lies in I. Not all intervals, of course, are shift-invariants, but those bounded by ordinary (negative) powers, for instance, are. Now, choose any partition $\mathcal{I} = I_1 \cup I_2 \ldots$, finite or not, of the fully ordered set of all infinitesimal transmonomials (for the natural order) into shift-invariant intervals I_i and assign a sign $\epsilon_i \in \{+, -\}$ to each of these.

Then define on $\mathbb{IT}$ a binary relation $\prec^{\mathcal{I},\epsilon}$ as follows:

$$\{g \prec^{\mathcal{I},\epsilon} f\} \iff \{0 \prec \epsilon_i\,(f \circ g^{-1} - id) \text{ if } \operatorname{lead}(f \circ g^{-1} - id) \in I_i\} \quad (8.6)$$

The relation $\prec^{\mathcal{I},\epsilon}$ is clearly transitive and trivially stable under right composition by $h \in \mathbb{IT}$. It is also stable under left compostion since, due to the shift-invariance of all I_i, the leading terms of $f \circ g^{-1} - id$ and $h \circ f \circ g^{-1} \circ h^{-1} - id$ always fall within the same interval of the partition $\mathcal{I}$. Each relation $\prec^{\mathcal{I},\epsilon}$ therefore defines an exotic order on $\mathbb{IT}$ that is compatible with the group structure as well as the natural topology.

Embedding-induced orders on free groups. — Any isomorphism $\mathbf{A}_i \mapsto f_i$ of the r-generator free group $\mathbf{Gr}(\mathbf{A})$ into a subgroup of the group $\mathbb{IT}$ endowed with its natural order $\prec$ or some exotic $\prec^{\mathcal{I},\epsilon}$, clearly induces a total group order (with left- and right-stability) on $\mathbf{Gr}(\mathbf{A})$.

Similarly, for any system $\{f_i\}$ of transserial twins or siblings , such an imbedding induces a total order on a suitable quotient $\mathbf{Gr}(\mathbf{A})/\{\mathbf{W}(\mathbf{A})\}$.

While there probably exist, on the free $\mathbf{Gr}(\mathbf{A})$ or its 'bound' quotients, other group orders than those obtainable in the above manner, this huge class of *embedding-induced* group orders is nonetheless quite interesting, if only because it points to three rather strange dichotomies:
– first, the dichotomy between *natural/exotic* orders, depending on whether the underlying embedding is into $\{\mathbb{IT}, \prec\}$ or $\{\mathbb{IT}, \prec^{\mathcal{I},\epsilon}\}$.
– second, the dichotomy between *convergent/divergent* orders, depending on whether all $\{f_i\}$ can/cannot be chosen simultaneously convergent.
– third, the dichotomy between *summable/non-summable* orders, depending on whether all $\{f_i\}$ can/cannot be chosen simultaneously summable[44].

Ultraexponential arbitration. — In the case of twins or siblings $\{f_i\}$, the skilful addition to them of infinitesimal terms of type $1/\mathcal{E}_\infty$, i.e. small beyond all transexponential orders, can always restore independence, and induce on $\mathbf{Gr}(\mathbf{A})$ group orders which sometimes can be obtained in no other way.

Thus, the order on the two-generator group $\mathbf{Gr}(\mathbf{A_1}, \mathbf{A_2})$ corresponding to the independent pair $\{f_1(z) := \mathcal{E}_\infty(z)\,,\ f_2(z) := z+1\}$ or, what amounts to the same[45], to the pair $\{\bar{f}_1(z) := z+1\,,\ \bar{f}_2(z) := z+1/\mathcal{E}_\infty(z)\}$, is specific to that (very) particular embedding, and capable of a remarkable combinatorial interpretation.

9. Conclusion: settled points, open questions

Synoptic table. — With the usual abbreviations:

$$\{F, G\} := FGF^{-1}G^{-1} \quad , \quad F_n := G^n F G^{-n} \quad , \quad G_n := F^n G F^{-n}$$

(44) This third dichotomy is quite distinct from the second one!
(45) Indeed, both pairs $\{f_1, f_2\}$ and $\{\bar{f}_1, \bar{f}_2\}$ are conjugate under a "superfast" $\bar{\mathcal{E}}_\infty$. Note, however, that replacing $\mathcal{E}_\infty$ by $\bar{\mathcal{E}}_\infty$ in the above embedding would make no difference to the order induced on $\mathbf{Gr}(\mathbf{A_1}, \mathbf{A_2})$. Indeed, although $\mathcal{E}_\infty$ and $\bar{\mathcal{E}}_\infty$ are "distinguishable" when used jointly in the same relation – the latter is much faster growing than the former and all its finite iterates – they are nonetheless "undistinguishable" when occuring in isolation!

tangency order	*defining relation* $W(F,G)=1$	*word complexity*	*freedom degree*	*analytic nature*
....	....	....	....	....
(p,q)	$W_1(F,G)=\{U_1,V_1\}$ *with* $U_1=\{F\{F\{F,G\}\}\}$ $V_1=\{G\{G,F\}\}$	*not optim.*	$\mathbb{C}$	*divergent biresurgent resummable*
(p,q)	$W_2(F,G)=\{U_1,V_1\}$ *with* $U_2=\{F\{G\{G\{G,F\}\}\}\}$ $V_2=\{G\{F\{G\{F,G\}\}\}\}$	*not optim.*	$\mathbb{C}\times\mathbb{Q}$	*divergent biresurgent resummable*
$(0,q)$	$W_3(F,G)=G_1^{-3}G_0^4G_2^1G_0^4G_1^{-3}$	$(6,15)$	$\mathbb{C}$	*analytic*
$(0,q)$	$W_4(F,G)=G_0^8G_1^{-10}G_2^9G_3^{-10}G_4^8$	$(8,45)$	$\mathbb{C}$	*divergent*
$(0,0)$	$W_5(F,G)=W_3(F,\{F,G\})$	$(6,30)$	$\mathbb{C}^2$	*analytic*
$(0,0)$	$W_6(F,G)=W_4(F,\{F,G\})$	$(10,30)$	$\mathbb{C}^2$	*divergent*
$(0^*,0^*)$	$F^7=G^7=(FG)^7=1$		$\mathbb{C}^\infty$	*analytic*

Main settled points. —

- Existence of numerous but sporadic types of formal twins.
- Generic convergence for non identity-tangent twins.
- Generic divergence-cum-resurgence for identity-tangent twins.
- Extension to 'siblings': $r{+}1$ mappings constrained by r relations.

Main open questions. —

- Are identity-tangent twins *always* divergent?
- What is the arithmetic nature [46] of their *resurgence* invariants?
- What *autarky* relations do these verify [47]?
- Do all transserial formal twins possess a *real-analytic* (as opposed to merely *cohesive*) geometric realisation?

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(46) presumably transcendental
(47) when there are free continuous parameters.

Some avenues for exploration. —

- Extend the *Uker*-filtration to the case of r-generator free algebras and higher dimensional differential representations.
- Calculate the dimensions that go with these filtrations.
- Investigate the higher-dimensional analogue of twins and siblings.
- Explore the potential[48] of representations of free or 'nearly free' groups into groups of (one or many dimensional) germ mappings.

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