Spectra of graphene nanoribbons with armchair and zigzag boundary conditions

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Introduction

Graphene

• 2D material, one layer sheet of carbon atoms

\[
\begin{array}{c}
\text{Zigzag edge} \\
\text{Armchair edge}
\end{array}
\]

• many interesting physical properties
  • both metallic and semiconducting properties
  • anomalous quantum Hall effect, nonlinear Kerr effect, spintronics, Casimir effect, \ldots \text{\textsuperscript{a}}
  • Nobel Prize in Physics for 2010 to A. Geim and K. Novoselov
• nanoribbons: strips of graphene\textsuperscript{b}
• models: tight-binding (solid state physics), further approximations for low energies and long wavelengths

\[\Rightarrow \text{Dirac operator + boundary conditions}\textsuperscript{c,d}\]

Domain $\Omega \subset \mathbb{R}^2$ and space
- finite rectangle (strip)
- arbitrary sufficiently regular
- infinite strip (waveguide)
- Hilbert space $L^2(\Omega, \mathbb{C}^k), (k = 2, 4)$

Operator

$$H := \begin{pmatrix} 0 & \tau^* & 0 & 0 \\ \tau & 0 & 0 & 0 \\ 0 & 0 & 0 & -\tau \\ 0 & 0 & -\tau^* & 0 \end{pmatrix}, \quad \tau := -i\partial_1 + \partial_2, \quad \tau^* := -i\partial_1 - \partial_2$$

Boundary conditions

**Zig-zag BC**

$$\Psi_1(x_1, -b) = 0, \quad \Psi_2(x_1, b) = 0, \quad \Psi_3(x_1, -b) = 0, \quad \Psi_4(x_1, b) = 0.$$ 

**Armchair BC**

$$\Psi_1(-a, x_2) = \Psi_3(-a, x_2), \quad \Psi_2(-a, x_2) = \Psi_4(-a, x_2), \quad \Psi_1(a, x_2) = e^{i\Theta} \Psi_3(a, x_2), \quad \Psi_2(a, x_2) = e^{i\Theta} \Psi_4(a, x_2), \quad \Theta \in \mathbb{R}. $$
Model

Boundary conditions

Zig-zag BC
\[ \Psi_1(x_1, -b) = 0, \]
\[ \Psi_2(x_1, b) = 0, \]
\[ \Psi_3(x_1, -b) = 0, \]
\[ \Psi_4(x_1, b) = 0. \]

Armchair BC
\[ \Psi_1(-a, x_2) = \Psi_3(-a, x_2), \]
\[ \Psi_2(-a, x_2) = \Psi_4(-a, x_2) \]
\[ \Psi_1(a, x_2) = e^{i\Theta} \Psi_3(a, x_2), \]
\[ \Psi_2(a, x_2) = e^{i\Theta} \Psi_4(a, x_2), \quad \Theta \in \mathbb{R}. \]

Differences in boundary conditions

- armchair BC
  - only for specific directions
  - semi-conducting and metallic properties (depend on \( \Theta \))
- zig-zag BC
  - appropriate also for non-straight boundary
  - metallic behaviour, edge states
- substantial difference from the operator point of view
- realistic physical system: combination of zig-zag and armchair or manifolds (e.g. cylinder)
Armchair Example

Operator and BC

\[
\hat{H}_{\text{ap}} := \begin{pmatrix}
0 & \tau^* & 0 & 0 \\
\tau & 0 & 0 & 0 \\
0 & 0 & 0 & -\tau \\
0 & 0 & -\tau^* & 0
\end{pmatrix}, \quad \tau := -i\partial_1 + \partial_2, \quad \tau^* := -i\partial_1 - \partial_2
\]

\[\text{Dom}(\hat{H}_{\text{ap}}) : \Psi \in C^1(\Omega, \mathbb{C}^4)\]

Armchair: \[\Psi_1(-a, x_2) = \Psi_3(-a, x_2), \quad \Psi_1(a, x_2) = e^{i\Theta} \Psi_3(a, x_2), \quad \Psi_2(-a, x_2) = \Psi_4(-a, x_2), \quad \Psi_2(a, x_2) = e^{i\Theta} \Psi_4(a, x_2), \quad \Theta \in \mathbb{R},\]

Periodic: \[\Psi_i(x_1, -b) = \Psi_i(x_1, b), \quad (i = 1, \ldots, 4).\]

Spectral problem for \(\hat{H}_{\text{ap}}^2\)

\[
\hat{H}_{\text{ap}}^2 = \begin{pmatrix}
-\Delta & 0 & 0 & 0 \\
0 & -\Delta & 0 & 0 \\
0 & 0 & -\Delta & 0 \\
0 & 0 & 0 & -\Delta
\end{pmatrix}
\]
Armchair Example

Eigenvalues and eigenfunctions

- separation of variables: simple ODE problems
- eigenvalues:
  \[ \lambda_{m,n} = \sigma_m^2 + \zeta_n^2, \quad m, n \in \mathbb{Z}, \]
  \[ \sigma_m = m\pi/b, \quad \zeta_n = n\pi/(2a) - \Theta/(4a). \]
- eigenfunctions:
  \[ \Psi_{m,n} = e^{i\sigma_m x_2} \begin{pmatrix} Ae^{-i\zeta_n x_1} \\ Be^{-i\zeta_n x_1} \\ A(-1)^n e^{-i\frac{\Theta}{2}} e^{i\zeta_n x_1} \\ B(-1)^n e^{-i\frac{\Theta}{2}} e^{i\zeta_n x_1} \end{pmatrix}, \quad A, B \in \mathbb{C}. \]
- spectrum is discrete, 0 ∈ \( \sigma(\hat{H}_{ap}^2) \) for some \( \Theta \)
- eigenfunctions form orthonormal basis in \( L^2(\Omega, \mathbb{C}^4) \), \( \hat{H}_{ap}^2 \) is ess. self-adjoint
- spectrum of \( \hat{H}_{ap} \) : \( \pm \sqrt{\lambda_{m,n}} \)
- \( \text{Dom}(H_{ap}) = W^{1,2}(\Omega, \mathbb{C}^4) + \text{BC} \), graph norm \( \|\hat{H}_{ap}\Psi\|^2 + \|\Psi\|^2 = \|\Psi\|^2_{W^{1,2}} \)

Zigzag Example

Operator and BC

\[ \hat{H}_{zp} := \begin{pmatrix} 0 & \tau^* & 0 & 0 \\ \tau & 0 & 0 & 0 \\ 0 & 0 & 0 & -\tau \\ 0 & 0 & -\tau^* & 0 \end{pmatrix}, \quad \tau := -i\partial_1 + \partial_2, \quad \tau^* := -i\partial_1 - \partial_2 \]

\[ \text{Dom} (\hat{H}_{zp}) : \Psi \in C^1(\Omega, \mathbb{C}^4) \]

Zig – zag : \( \Psi_1(x_1, -b) = 0, \quad \Psi_2(x_1, b) = 0, \)

\( \Psi_3(x_1, -b) = 0, \quad \Psi_4(x_1, b) = 0, \)

Periodic : \( \Psi_i(-a, x_2) = \Psi_i(a, x_2), (i = 1, \ldots, 4). \)

Reduction

\[ \hat{H}_{zp} = \begin{pmatrix} 0 & \tau^* \\ \tau & 0 \end{pmatrix} \]

Zig – zag : \( \Psi_1(x_1, -b) = 0, \quad \Psi_2(x_1, b) = 0, \)

Periodic : \( \Psi_i(-a, x_2) = \Psi_i(a, x_2) \)
Zigzag Example

Spectrum of $\dot{H}^2_{zp}$

$$\dot{H}^2_{zp} = \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix}$$

Zig - zag : $\Psi_1(x_1, -b) = 0$, $\Psi_2(x_1, b) = 0$

$$(i\partial_1 - \partial_2)\Psi_1(x_1, b) = 0,$$  $$(i\partial_1 + \partial_2)\Psi_2(x_1, -b) = 0,$$

Periodic : $\Psi_i(-a, x_2) = \Psi_i(a, x_2)$,

$$\partial_1 \Psi_i(-a, x_2) = \partial_1 \Psi_i(a, x_2).$$

Reduction II

$$\dot{H}^2_{zp} = -\Delta$$

Zig - zag : $\psi(x_1, -b) = 0$, $$(i\partial_1 - \partial_2)\psi(x_1, b) = 0,$$

Periodic : $\psi(-a, x_2) = \psi(a, x_2)$,

$$\partial_1 \psi(-a, x_2) = \partial_1 \psi(a, x_2).$$
Zigzag Example

Spectrum of $\dot{H}_{zp}^2$

- separation of variables:
  \[ \psi(x_1, x_2) = e^{-i\sigma x_1} \xi(x_2) \]

- periodic BC: $\sigma_m = m\pi/a, m \in \mathbb{Z}$

- “Cauchy-Riemann” BC:
  \[
  \begin{aligned}
  -\xi''' &= \omega^2 \xi & \text{in } (-b, b), \\
  \xi &= 0 & \text{at } -b, \\
  \xi' - \sigma_m \xi &= 0 & \text{at } b,
  \end{aligned}
  \]

- $\lambda_{m,n} = \sigma_m^2 + \omega_{m,n}^2, \ m \in \mathbb{Z}, n \in \mathbb{N}$.

- $\lambda_{m,1} \to 0$ as $m \to +\infty$, more precisely
  \[ \lambda_{m,1} = 4\sigma_m^2 e^{-4\sigma_m b} + O(\sigma_m^4 e^{-8\sigma_m b}). \]

- $0 \in \sigma_{\text{ess}}(\dot{H}_{zp}^2)$

- $\{\psi_{m,n}\}$ form orthonormal basis, $\sigma(\dot{H}_{zp}) : \pm \sqrt{\lambda_{m,n}}$

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Zigzag Example

Eigenfunctions

- “usual” states: \( \psi_{m,n} = e^{-i\sigma_m x_1} \sin(\omega_{m,n} (x_2 + b)) \)
- “edge” states: \( \psi_{m,n} = e^{-i\sigma_m x_1} \sinh(\omega_{m,1} (x_2 + b)), m > m_0 \)

- analogous effect and calculations for annulus:
General situation

Aims

- understand spectra of Dirac operator with zigzag BC
- equivalently Laplacian with “Cauchy-Riemman” BC
- essential spectrum, explain and describe the edge states
- domains of definition
- perturbations by (diagonal) potential

Available mathematical literature

- 1995 K. M. Schmidt\(^a\)
  - special case in 2D,
  - domains of definition, supersymmetry
- 1970 W. D. Evans\(^b\)\(^c\)
  - 3D problems (sphere and $\mathbb{R}^3$) with potentials
  - proofs based more on calculations (spherical symmetry, separation of variables)
  - perturbations, Glazman’s decomposition method\(^d\)

Definition of operator

Lemma (Self-adjointness):
Let $T$ be closed, densely defined operator. Then

$$ H = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} $$

is self-adjoint.

- we search for closed realization of $\tau = -i\partial_1 + \partial_2$ and its adjoint $\tau^*$

Proposition (SUSY):
Let $T$ be densely defined and closed. $T^*T, TT^*$ are associated with the closed symmetric quadratic forms $t_{T^*T}[\psi] := \|T\psi\|^2, t_{TT^*}[\psi] := \|T^*\psi\|^2$ defined on Dom ($T$), Dom ($T^*$), respectively. Moreover, $\sigma(TT^*) \cup \{0\} = \sigma(T^*T) \cup \{0\}$.

- $T^*T$ corresponds to Laplacian with “Cauchy-Riemann” BC
- Dom ($T$) is its form-domain
- proofs in 1993 Thaller$^a$

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Schmidt’s case

- arbitrary (sufficiently regular) domain $\Omega$
- $H = \begin{pmatrix} 0 & \tau^* \\ \tau & 0 \end{pmatrix}$
- different BC (not precisely zig-zag): $\psi_1 | \partial \Omega = 0$

Theorem [Schmidt’95]

Let $\tau_0 := -i\partial_1 + \partial_2$ be defined on $\text{Dom}(\tau_0) := C_0^\infty(\Omega)$. Then

\[ \tau := -i\partial_1 + \partial_2, \]
\[ \text{Dom}(\tau) := W^{1,2}_0(\Omega), \]

is the closure of $\tau_0$ and the adjoint reads

\[ \tau^* := -i\partial_1 - \partial_2, \]
\[ \text{Dom}(\tau^*) := \{ \psi \in L^2(\Omega) \cap W^{1,2}_{\text{loc}}(\Omega) : \tau^* \psi \in L^2(\Omega) \}. \]

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Schmidt’s case

Corollaries [Schmidt’95]

- $H = \begin{pmatrix} 0 & \tau^* \\ \tau & 0 \end{pmatrix}$ is self-adjoint.
- $H^2 = \begin{pmatrix} \tau^* \tau & 0 \\ 0 & \tau \tau^* \end{pmatrix}$
- $\tau^* \tau = -\Delta_D$, i.e. $\text{Dom} (\tau^* \tau) = W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega)$.
- $\sigma(\tau^* \tau) = \sigma(-\Delta_D)$, i.e. discrete spectrum
- $\text{Dom} (\tau^*)$ is larger than $W^{1,2}(\Omega)$
- $0 \in \sigma_{\text{ess}}(\tau \tau^*)$, 0 is eigenvalue of infinite multiplicity
- $\{(x_1 + i x_2)^n\} \subset \text{Ker}(\tau \tau^*)$
- $\sigma(\tau \tau^*) = \sigma(\tau^* \tau) \cup \{0\}$
More general situation

Complementary (A) and interchanging (B) BC

- in both cases: $\psi_1 \mid \partial \Omega_1 = 0$, $\psi_2 \mid \partial \Omega_2 = 0$.
- we search for closed realization of $\tau$ and its adjoint
- description of domains of definition
- analysis of spectra
More general situation

Proposition [Freitas,PS] (A)

Let $\Omega$ be a bounded connected domain with a locally Lipschitz boundary $\partial \Omega$ that is made up of two nonempty connected components $\partial \Omega_i$ ($i = 1, 2$) with $\text{dist}(\partial \Omega_1, \partial \Omega_2) > 0$.

Let $A$ be the operator acting as $A\psi := \tau \psi$ on the domain

$$\text{Dom}(A) := \{\psi \in L^2(\Omega) \cap W^{1,2}_{\text{loc}}(\Omega \cup \partial \Omega_1) : \psi \upharpoonright \partial \Omega_1 = 0, \ \tau \psi \in L^2(\Omega)\},$$

where $W^{1,2}_{\text{loc}}(\Omega \cup \partial \Omega_1)$ means the functions from $W^{1,2}(\Omega')$ for any $\Omega' \subset \Omega$, $\overline{\Omega'} \subset \Omega \cup \partial \Omega_1$.

Then $A$ is closed and $A^*$ acts as $A^*\phi := \tau^* \phi$ on the domain

$$\text{Dom}(A^*) := \{\phi \in L^2(\Omega) \cap W^{1,2}_{\text{loc}}(\Omega \cup \partial \Omega_2) : \phi \upharpoonright \partial \Omega_2 = 0, \ \tau^* \phi \in L^2(\Omega)\}.$$

Remarks

• proof is a modification of Schmidt’s case\(^a\)

• $A$ is a closure of $\tau$ defined on smooth functions satisfying BC

More general situation

Proposition [Freitas,PS] (B)

Let $\Omega$ be a bounded connected domain with locally Lipschitz and connected boundary $\partial \Omega$. Let further $p_i \in \partial \Omega$ be two distinct points such that $\partial \Omega = \partial \Omega_1 \cup \partial \Omega_2 \cup \{p_1, p_2\}$. If $B_0$ is an operator acting as $\tau$ on the domain

$$\text{Dom}(B_0) := \{\psi \in C^\infty(\Omega) : \exists \psi_0 \in C_0^\infty(\mathbb{R}^2), \psi = \psi_0 \upharpoonright \Omega, \text{supp } \psi \cap \overline{\partial \Omega_1} = \emptyset\},$$

then $B_0$ is closable and

$$\mathcal{D}_0 + \mathcal{D}_{loc} \subset \text{Dom}(B) \subset \mathcal{D},$$

where

$$\mathcal{D}_0 := \{\psi_1 \in W^{1,2}(\Omega) : \psi_1 \upharpoonright \partial \Omega_1 = 0\},$$

$$\mathcal{D}_{loc} := \{\psi_2 \in L^2(\Omega) \cap W^{1,2}_{\text{loc}}(\Omega \cup \partial \Omega_1) : \exists U_i \subset \mathbb{R}^2 \text{ open neighborhoods of } p_i, (i = 1, 2), \psi_2 \upharpoonright U_{1,2} = 0, \tau \psi_2 \in L^2(\Omega)\},$$

$$\mathcal{D} := \{\psi \in L^2(\Omega) \cap W^{1,2}_{\text{loc}}(\Omega \cup \partial \Omega_1) : \psi \upharpoonright \partial \Omega_1 = 0, \tau \psi \in L^2(\Omega)\}.$$

The adjoint operator $B_0^* = B^*$ reads

$$B_0^* \phi = \tau^* \phi,$$

$$\text{Dom}(B_0^*) = \{\phi \in L^2(\Omega) \cap W^{1,2}_{\text{loc}}(\Omega \cup \partial \Omega_2) : \phi \upharpoonright \partial \Omega_2 = 0, \tau^* \phi \in L^2(\Omega)\} =: \mathcal{D}^*.$$
More general situation

Remarks

- proof again a modification of the Schmidt’s case
- part with $\mathcal{D}_0$ due to 2003 Kříž\(^a\) and 2002 Dittrich & Kříž\(^b\)
  - quantum waveguides with interchanging Dirichlet and Neumann boundary conditions
  - precise description of form and operator ($-\Delta$) domains (larger than $W^{2,2}$)
- it is open if $\mathcal{D} \subset \text{Dom} (B)$

Zero is not an eigenvalue

Proposition [Freitas,PS]

Zero is not an eigenvalue of either $T^*T$ or $TT^*$, for $T = A, B$.

Proof and remarks

- eigenfunction $\psi_0$ satisfies $-\Delta \psi_0 = 0$ and $(-i \partial_1 + \partial_2)\psi_0 = 0$ in any $\Omega' \subset \Omega$, $\overline{\Omega'} \subset \Omega \cup \partial \Omega_1$
- domains of definition $(A, B)$ + embedding theorems $\Rightarrow \psi_0 \in C^\infty(\Omega')$ and $\psi_0 \in C(\overline{\Omega'})$
- $\psi_0$ is a holomorphic function in $\Omega'$ + zero on part of the boundary (with positive measure) $\Rightarrow \psi_0 = 0$
Zero is in the essential spectrum

Proposition [Freitas,PS]

Zero is in the essential spectrum of both $T^*T$ and $TT^*$, $T = A, B$.

Proof and remarks

- Dirichlet bracketing argument

\[
\begin{array}{c}
\Omega \\
\partial \Omega_1 \\
p_1 \\
\partial \Omega_2 \\
p_2 \\
\end{array}
\]

- SUSY and description of domains of definition

- particular singular sequence from “zig-zag – periodic” example
- 0 is not an eigenvalue
Local compactness

Local compactness of resolvent

- 0 is in the essential spectrum \( \Rightarrow \) the resolvent of \( H^2 \) and \( H \) is not compact
- \( \chi_K(H - z)^{-1} \) might be compact for compact \( K \subset \Omega \)
- local compactness of resolvent: typical for Schrödinger operators in \( L^2(\mathbb{R}^n) \)
  - essential spectrum \( \Leftrightarrow \) “behaviour at infinity”
  - singular sequences corresponding to \( \lambda \in \sigma_{\text{ess}} \) must vanish in limit in every compact subset of \( \mathbb{R}^n \)
- for Dirac operators in \( L^2(\mathbb{R}^n) \) valid as well

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\( ^a \) P. D. Hislop and I. M. Sigal. Springer Verlag, 1996.

Proposition [Freitas,PS]

Let \( \varphi_{1,2} \in C_0^\infty(\mathbb{R}^2) \) be such that \( \text{supp} \varphi_1 \cap \partial \Omega_2 = \emptyset \) and \( \text{supp} \varphi_2 \cap \partial \Omega_1 = \emptyset \). Then

\[
\begin{pmatrix}
\varphi_1 & 0 \\
0 & \varphi_2
\end{pmatrix}
\begin{pmatrix}
-z & A^* \\
A & -z
\end{pmatrix}^{-1}
\]

is a compact operator in \( L^2(\Omega, \mathbb{C}^2) \) for any \( z \in \mathbb{C} \setminus \mathbb{R} \).
Edge states

Proof

- description of domains of definition
- \( \| \tau \psi \|_2^2 + \| \psi \|_2^2 = \| \psi \|_{W^{1,2}}^2 \) for \( \psi \in W^{1,2}_0 \)

Corollaries

- 0 is the only point of essential spectrum of \( H \) in case (A)
  - singular sequences must squeeze to \( \partial \Omega_1 \) for \( (\Psi_n)_2 \) and \( \partial \Omega_2 \) for \( (\Psi_n)_1 \)
  - we can construct a singular sequence \( \{ \Phi_n \} \) for which \( (\Phi_n)_1 \) and \( (\Phi_n)_2 \) are orthogonal
  - \( \|(H - \lambda)\Phi_n\|^2 = \| \tau (\Phi_n)_1 \|^2 + \| \tau^* (\Phi_n)_2 \|^2 + |\lambda|^2 \| \Phi_n \|^2 \to 0 \)
  - then \( \Phi_n \to 0 \), but \( \| \Phi_n \| = 1 \)

- sequence of eigenfunctions associated with eigenvalues tending to zero must vanish in limit in every compact subset of \( \Omega \)
  - explanation of features of edge states
Example annulus

Edge states

\[ \lambda_{m,1} = \frac{-4m(1-m)}{r_2^2} \left( \frac{r_1}{r_2} \right)^{-2m} + \mathcal{O}\left(m^4\left(\frac{r_1}{r_2}\right)^{-4m}\right), \text{ as } m \to -\infty \]
Potential perturbations

Proposition [Freitas,PS] (A)

Let \( V \in L^\infty(\Omega) \) be a real potential. Let \( U_\delta := \{ x \in \Omega : \text{dist}(x, \partial \Omega) < \delta \} \) and let denote \( m_\delta := \text{ess inf}_{U_\delta} V \), \( M_\delta := \text{ess sup}_{U_\delta} V \). Then \( \sigma_{\text{ess}}(H_{zz}^A + V) \subset \cap_{\delta > 0}[m_\delta, M_\delta] \).

Proof and remarks

- proof: properties of singular sequences (local compactness) + simple estimates + criterion from Glazman\(^a\)
- inspired by an analogous result of Evans\(^b\) for Dirac operator in 3D ball
- if \( V \) is continuous: \( \sigma_{\text{ess}}(H + V) \subset [\min_{\partial \Omega} V, \max_{\partial \Omega} V] \)
- further amendments possible
- weaker variant in (B) case


Quantum waveguides

Dirichlet, Neumann, Robin BC

$-\Delta$

Dirichlet, Neumann, Robin BC

Motivation and literature

- straight planar strips: $\sigma_{\text{ess}}(H) = [E_0, +\infty)$
  - $E_0$ depends on BC: $E_0^{(D)} = \frac{2\pi}{d}$, $E_0^{(N)} = 0$
- what happens if the waveguide is bended?
  - bound states below essential spectrum\(^ab\)
  - similar effects for strips on manifolds\(^c\)
  - in 3D also twisting: oposite effect (Hardy inequality)\(^d\)
- graphene waveguides?

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Armchair waveguide

Armchair BC

Armchair BC

Proposition [Freitas,PS]

\[ \sigma(H_{ac}) = \sigma_{ess}(H_{ac}) = (-\infty, -E_0] \cup [E_0, +\infty), \]  where \( E_0 := \min_{n \in \mathbb{Z}} |\zeta_n| \) with 

\[ \zeta_n = n\pi/(2a) - \Theta/(4a). \]
Proposition [Freitas,PS]

Let $\Omega$ be asymptotically straight, i.e. $\gamma(s) \to 0$ as $s \to \pm \infty$, then $\sigma_{ess}(H_{zz}) = \mathbb{R}$.

Proof

- construction of singular sequences for straight waveguide
- $\sigma_{ess}(H) = \mathbb{R}$ due to edge states
- singular sequences can be modified for bent waveguide
Armchair–Zig-zag waveguide

Remarks

- difficulties with definition of operator $H$: essentially self-adjoint or more extensions?
- for $H^2$ definition via quadratic forms (Friedrichs extension)
- for $H^2$: $\sigma_{\text{ess}}(H^2) \supset \{0\} \cup [E_0^2, +\infty)$
- variational techniques in the gap (if any) of essential spectrum (much harder)
Summary and further directions

Summary

- spectra of graphene nanoribbons
- armchair and zigzag boundary conditions
- 0 in essential spectrum
- edge states and their properties
- potential perturbations
- waveguides

Further directions

- physical restrictions for appearance of edge states (necessary high-frequencies)
- optimization problems (number, highest/lowest energies) for edge states
- non-trivial geometry (manifolds), lattice defects, magnetic fields (off-diagonal perturbations)
- combined waveguides
- non-self-adjoint perturbations (gains/losses)