Bernstein-Durrmeyer operators with arbitrary weight functions

Elena E. Berdysheva

German University of Technology in Oman

Muscat, Sultanate of Oman

Bernstein basis polynomials

Standard simplex in $\mathbb{R}^d$:

$$\mathbb{S}^d := \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : \\ 0 \le x_1, \ldots, x_d \le 1,\ x_1 + \cdots + x_d \le 1\}.$$

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Barycentric coordinates:

$$\mathbf{x} = (x_0, x_1, \ldots, x_d), \quad x_0 := 1 - x_1 - \cdots - x_d.$$

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The d-variate Bernstein basis polynomials of degree n are defined by

$$B_\alpha(x) := \binom{n}{\alpha} \mathbf{x}^\alpha = \frac{n!}{\alpha_0! \alpha_1! \ldots \alpha_d!} x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_d^{\alpha_d},$$

$\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^{d+1}$ with $|\alpha| := \alpha_0 + \alpha_1 + \cdots + \alpha_d = n$.

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One-dimensional case: $n \in \mathbb{N}$ and $k = 0, 1, \ldots, n$,

$$p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}, \qquad x \in [0,1].$$

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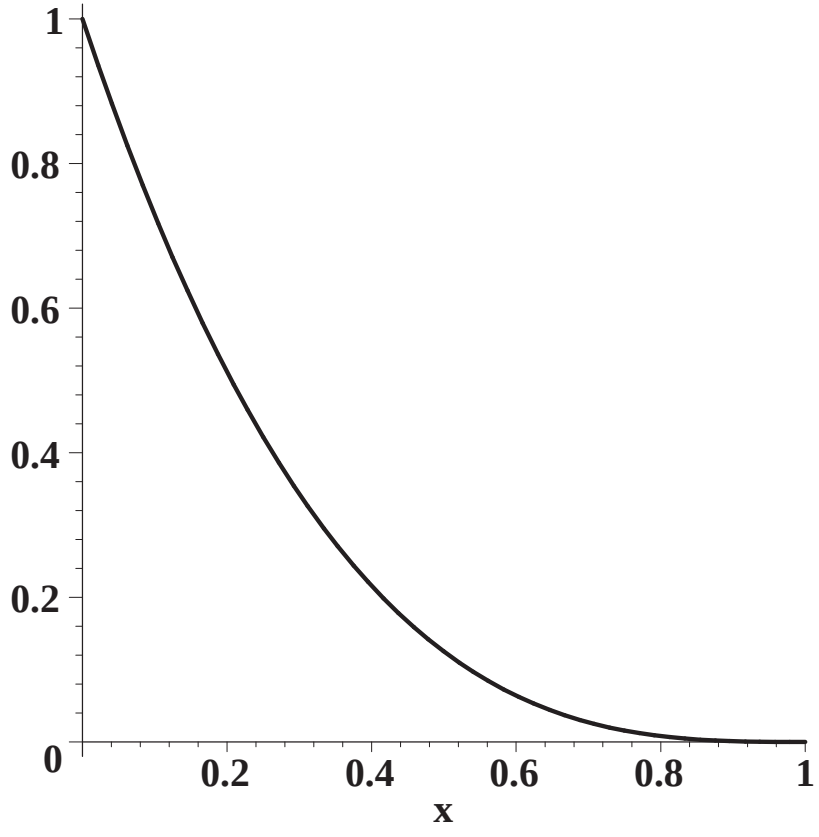

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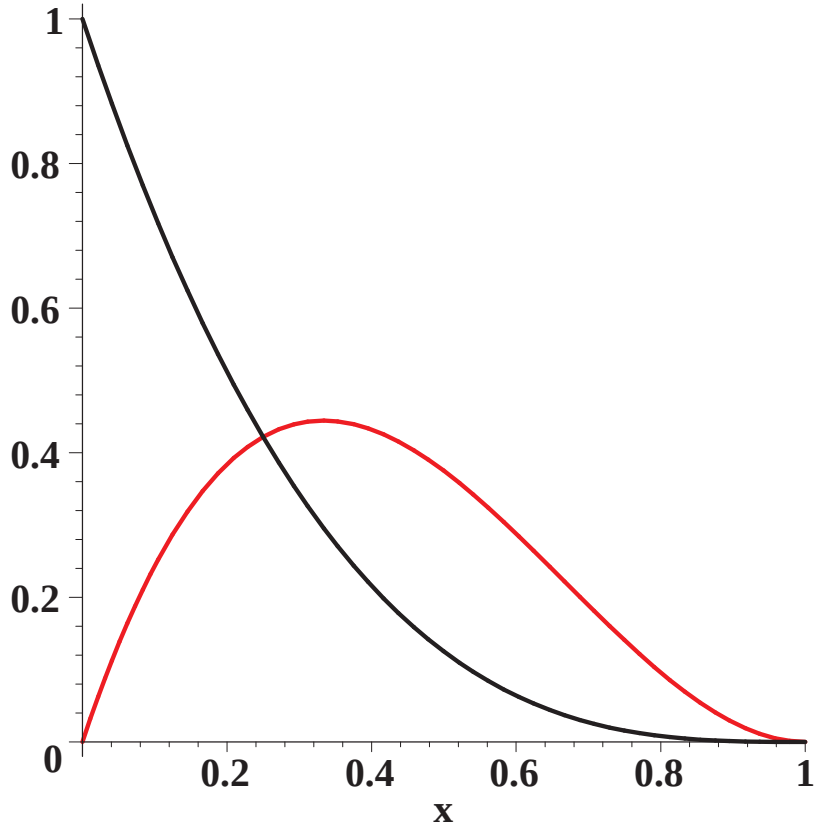

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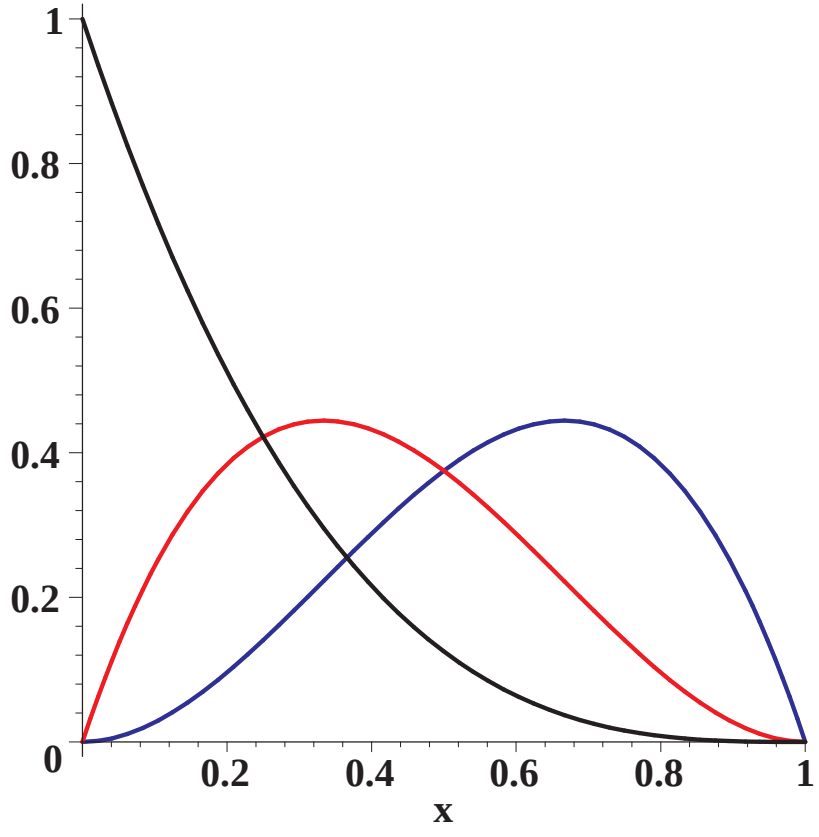

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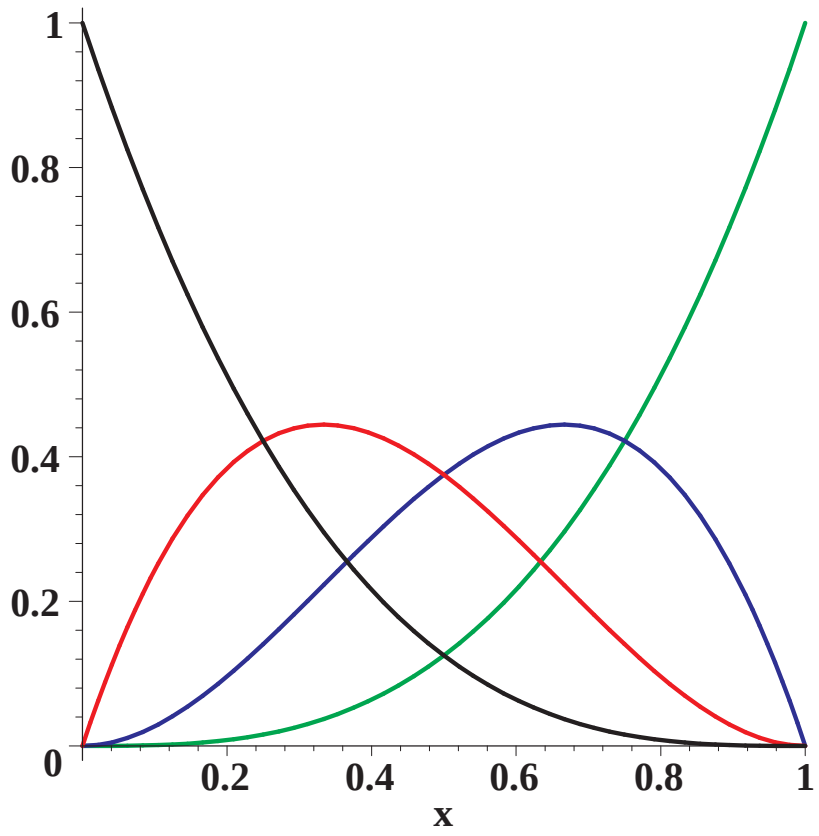

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This is a positive linear operator that reproduces linear functions.

Uniform convergence for every function in $C(\mathbb{S}^d)$.

Bernstein-Durrmeyer operator

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Definition. The Bernstein-Durrmeyer operator is defined for $f \in L^q(\mathbb{S}^d)$, $1 \le q < \infty$, or $f \in C(\mathbb{S}^d)$ by

$$\big(\mathbf{M}_n f\big)(x) := \sum_{|\alpha|=n} \frac{\int_{\mathbb{S}^d} f(y)\, B_\alpha(y)\, dy}{\int_{\mathbb{S}^d} B_\alpha(y)\, dy}\, B_\alpha(x).$$

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Extension to functions on the d-dimensional simplex: Derriennic (starting from 1985).

Weighted Bernstein-Durrmeyer operator

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Jacobi weights

The weighted Bernstein-Durrmeyer operator $\mathbf{M}_{n,\rho}$ is very well studied for Jacobi weights, i.e., for

$$d\rho(x) = \mathbf{x}^{\mu}\, dx,$$

with $\mu = (\mu_0, \mu_1, \ldots, \mu_d) \in \mathbb{R}^{d+1}$, where $\mu_i > -1$, $i = 0, 1, \ldots, d$.

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They were studied by many authors, e.g., Derriennic, Berens, Xu, Ditzian, Chen, Ivanov, X.-L. Zhou, Knoop, Gonska, Heilmann, Abel, Jetter, Stöckler, ...

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Recently (2012), Bing-Zheng Li used the multivariate operators $\mathbf{M}_{n,\rho}$ to obtain estimates for learning rates of least-square regularized regression with polynomial kernels.

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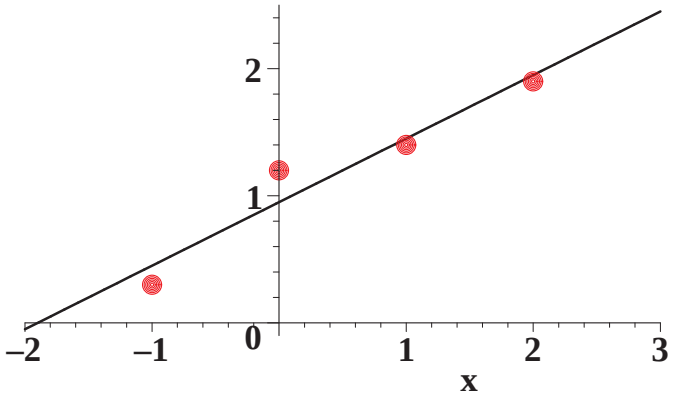
Typical problems in learning theory:

- Regression. E.g., least squares.



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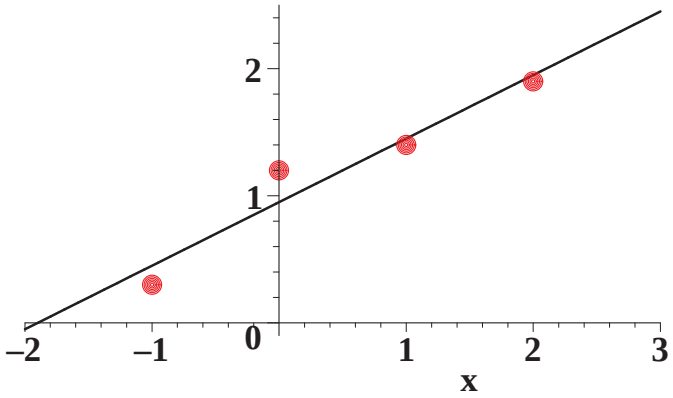
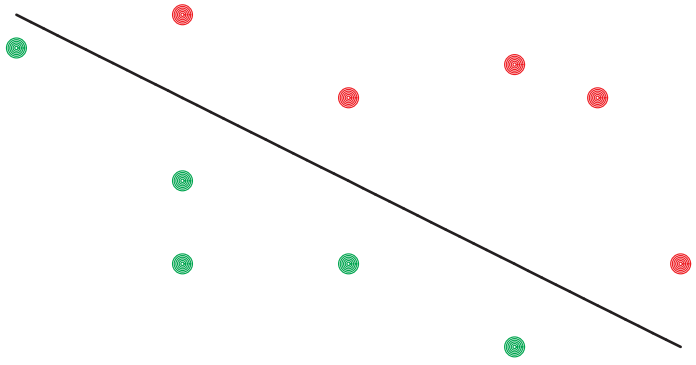


- Classification.



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Example 1. Consider the measure $d\rho(x) = w(x)\,dx$ with

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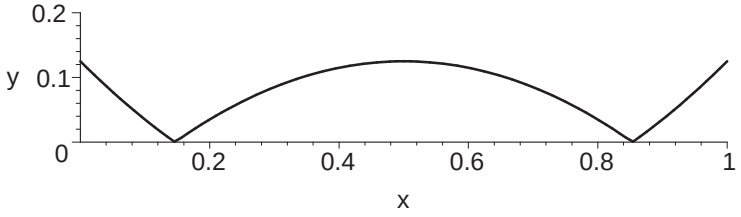

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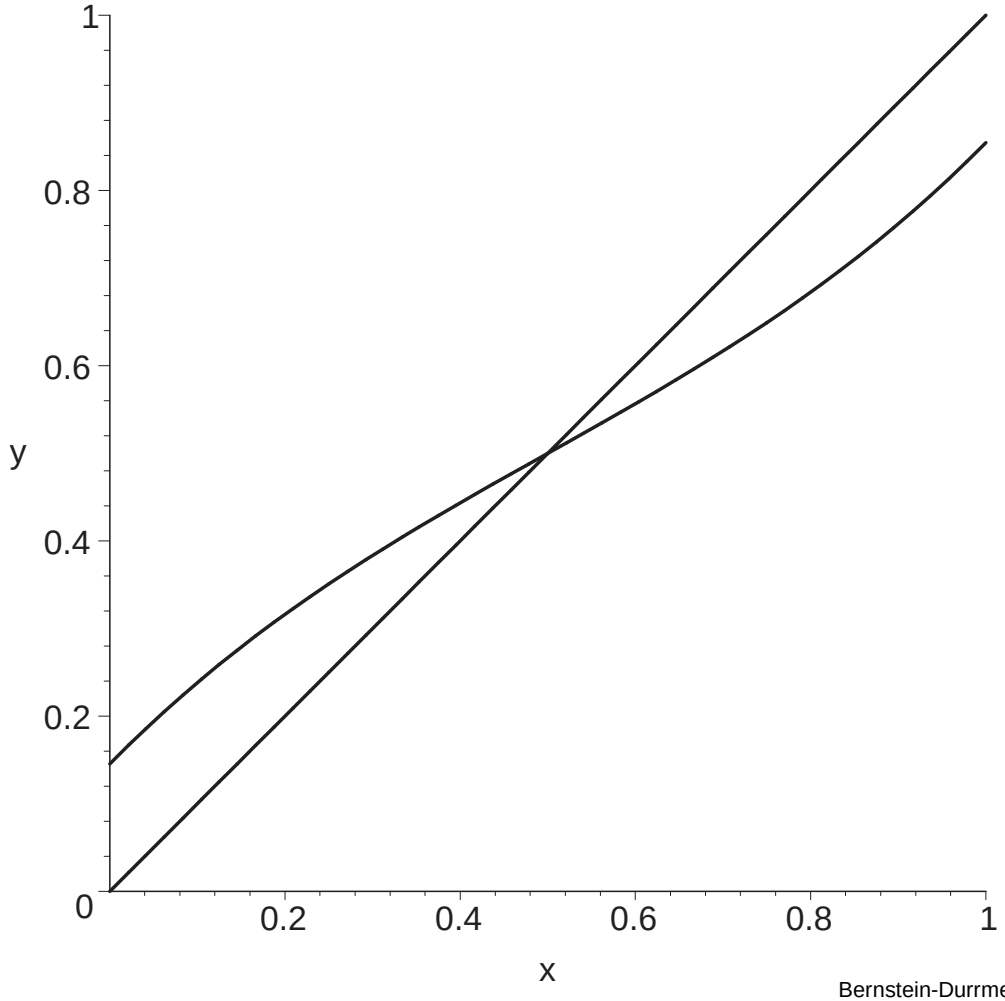
$$d\rho(x) = \left|\left(x - \tfrac{1}{2}\right)^2 - \tfrac{1}{8}\right| dx, \qquad f(x) = x, \qquad n = 5:$$



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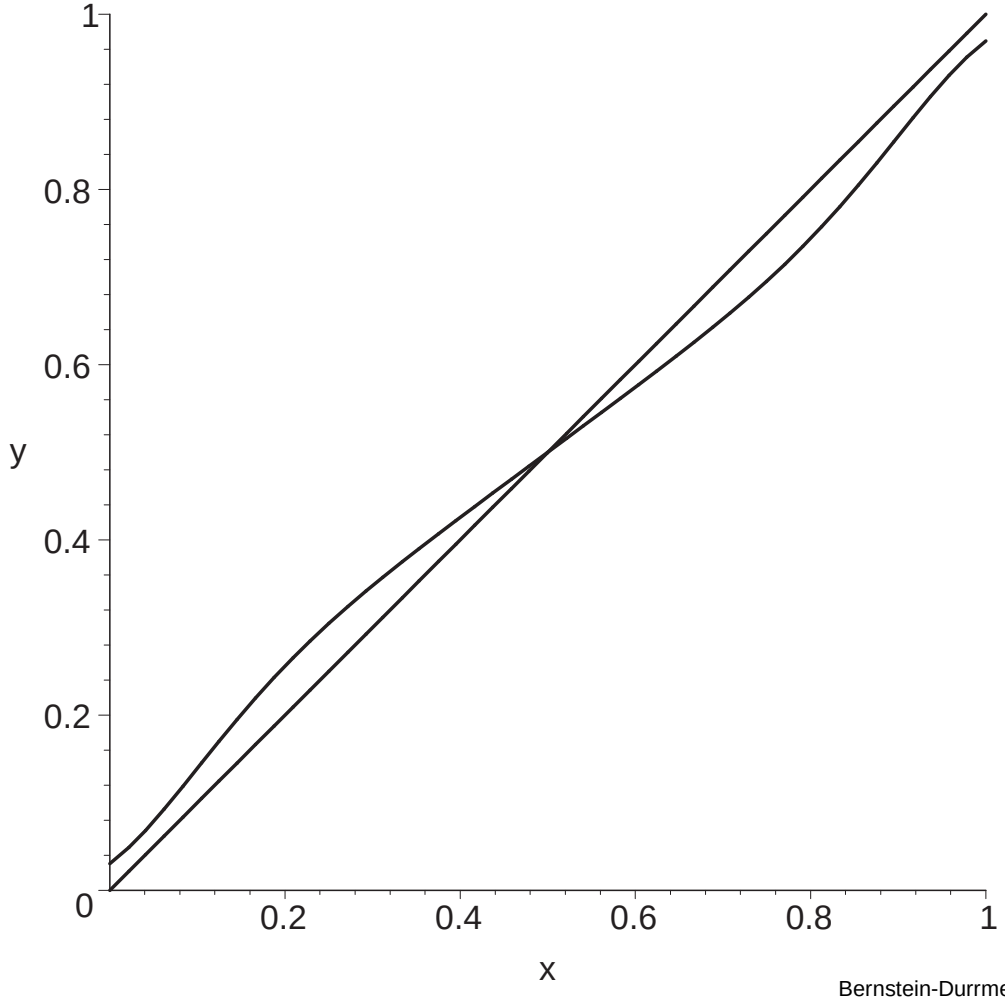

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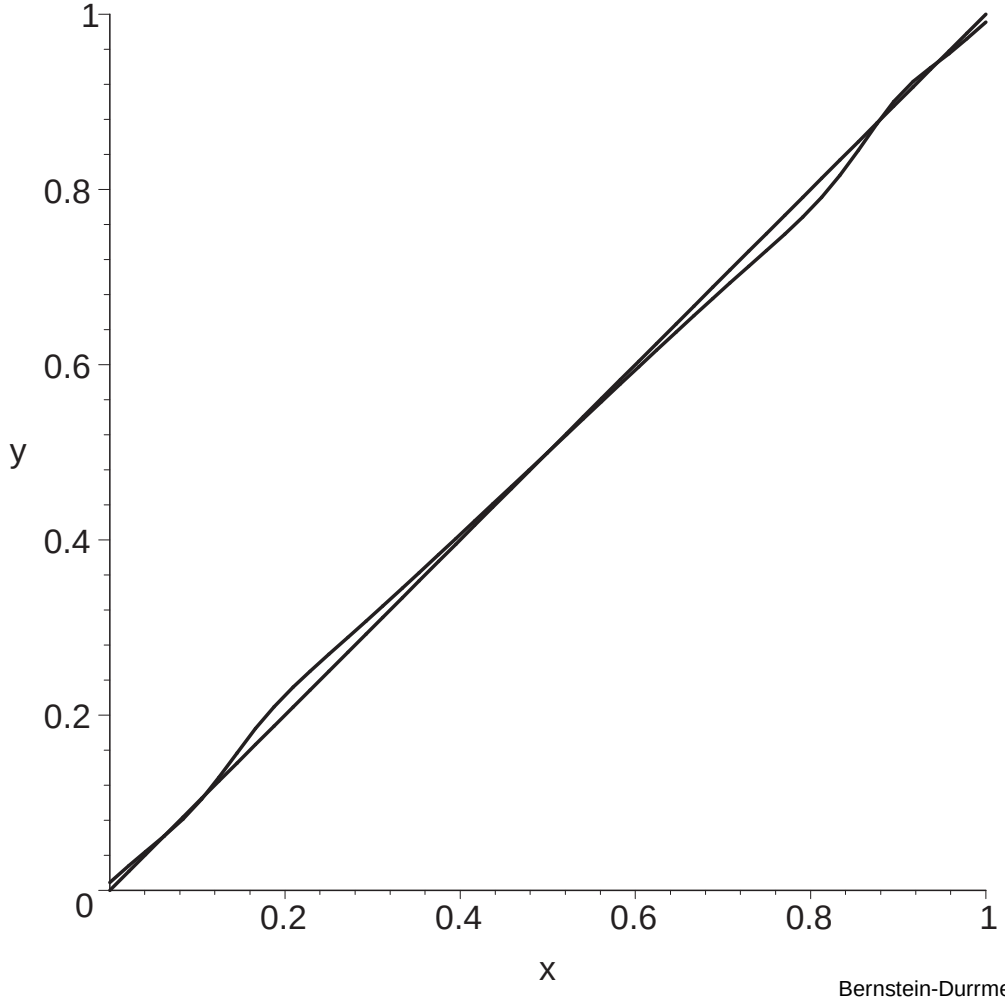

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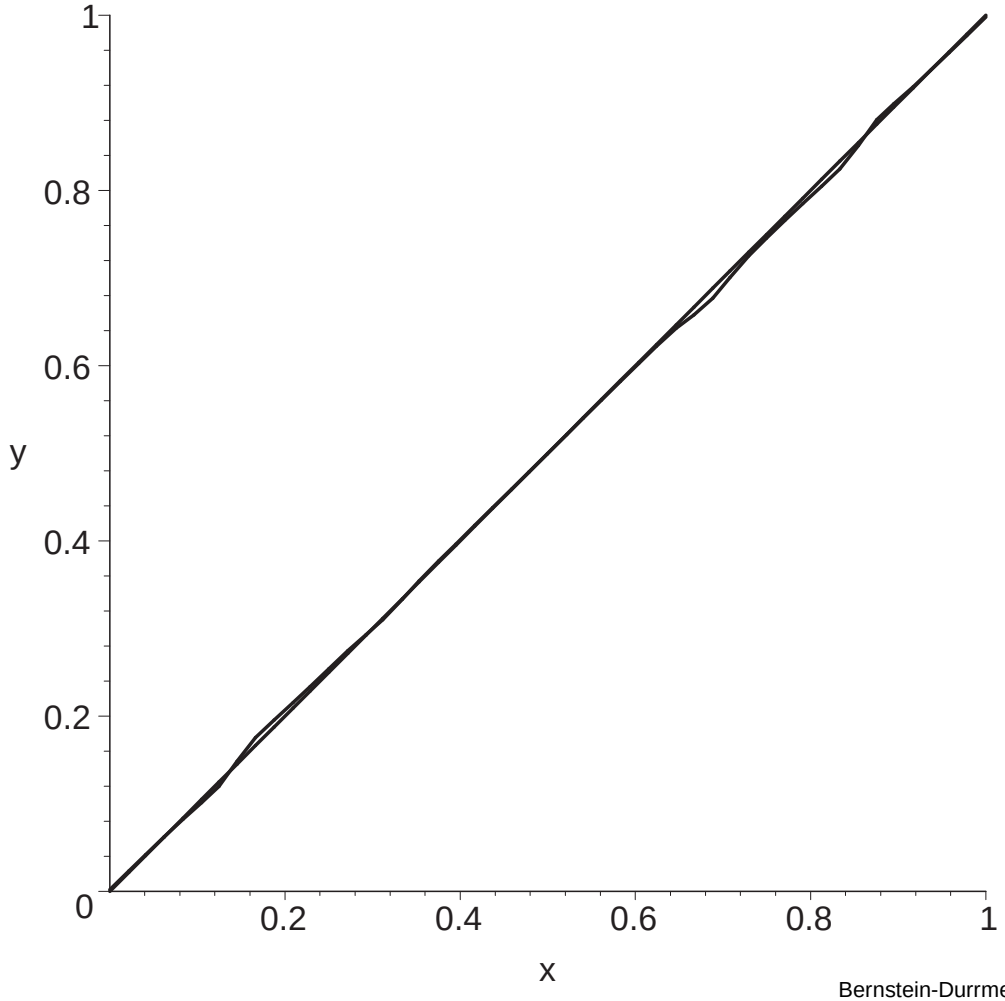

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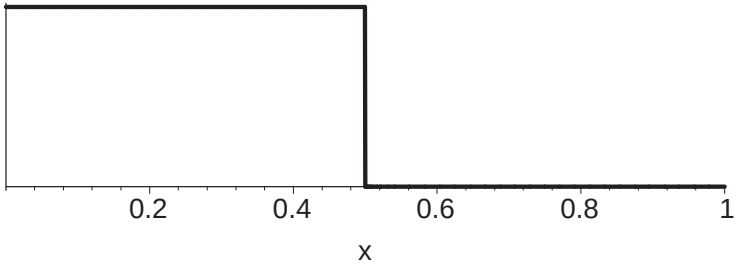

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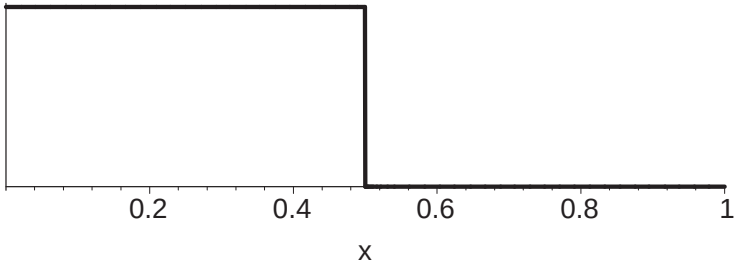

The Bernstein-Durrmeyer operator has the form

$$(\mathbf{M}_{n,\rho}\, f)(x) = \sum_{k=0}^{n} \frac{\int_0^{\frac{1}{2}} f(y)\, y^k\, (1-y)^{n-k}\, dy}{\int_0^{\frac{1}{2}} y^k\, (1-y)^{n-k}\, dy} \binom{n}{k} x^k\, (1-x)^{n-k}.$$

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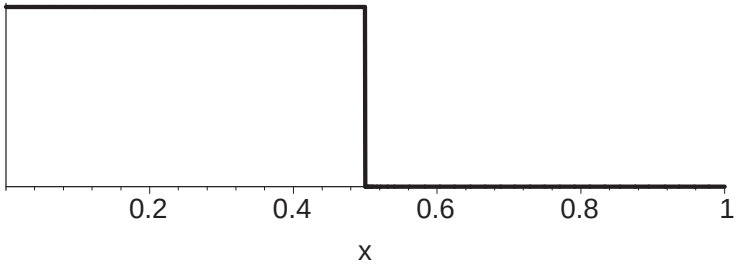

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Then for $f(x) = x$ we have $(\mathbf{M}_{n,\rho}\, f)(x) \le \frac{1}{2}$, $x \in [0,1]$.

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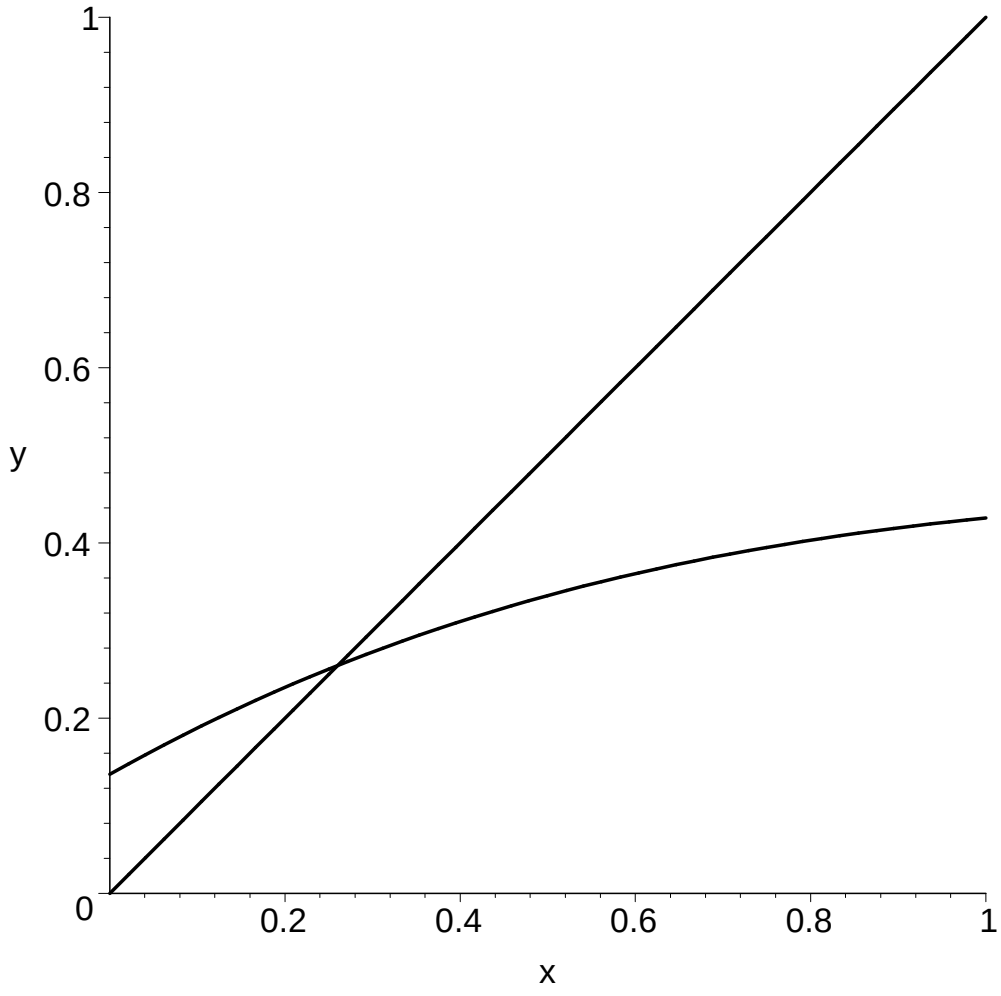

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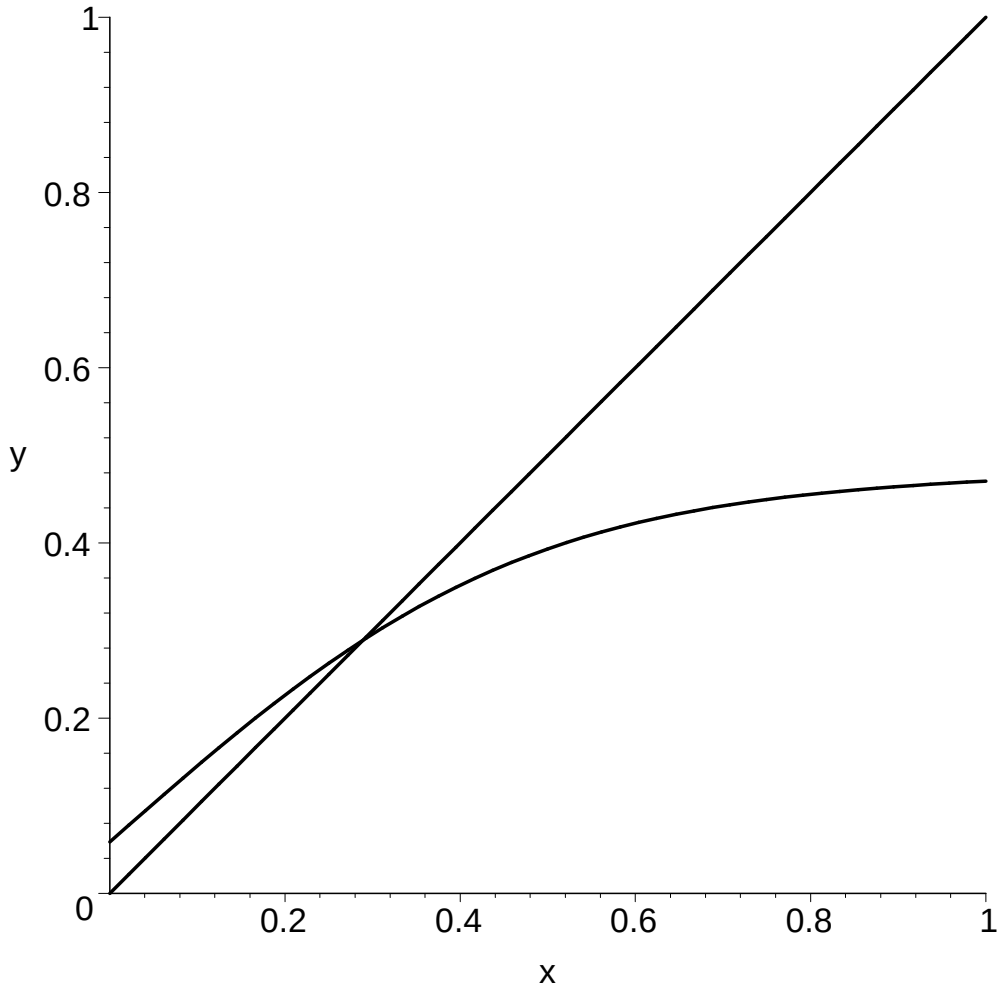

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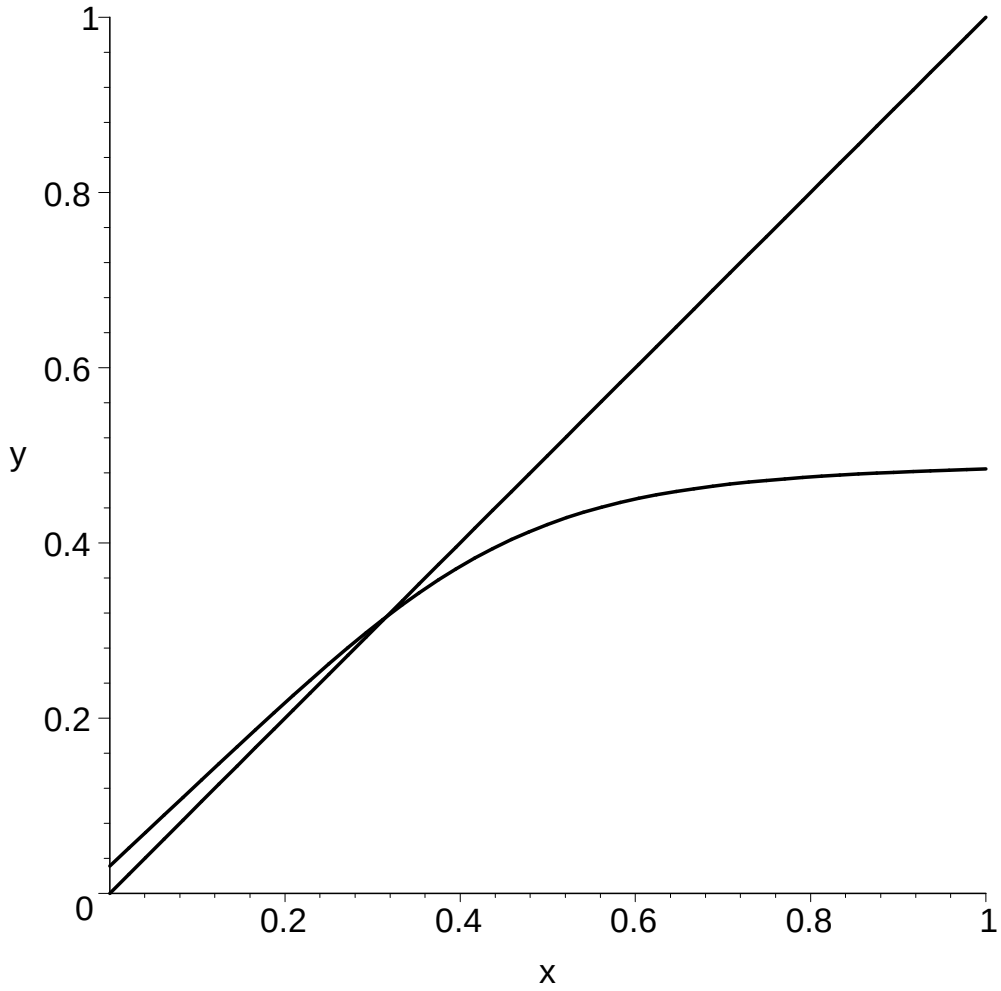
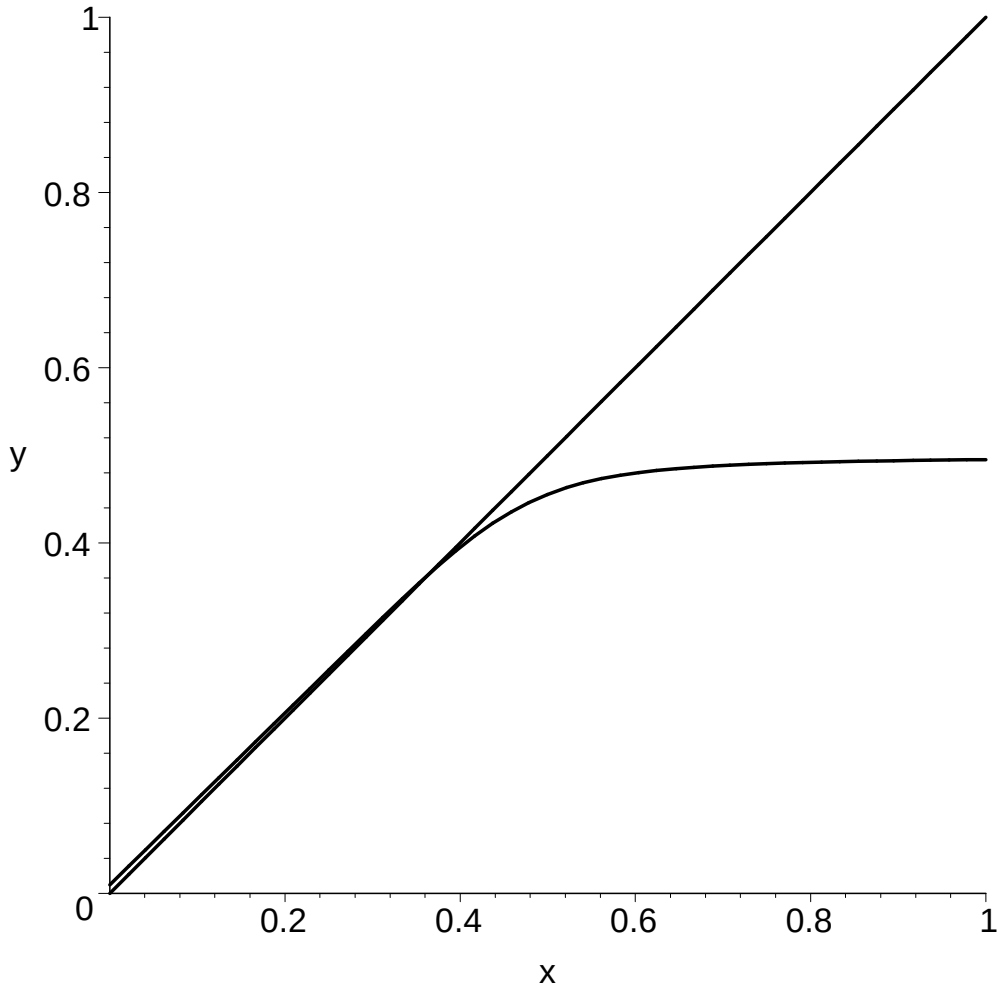


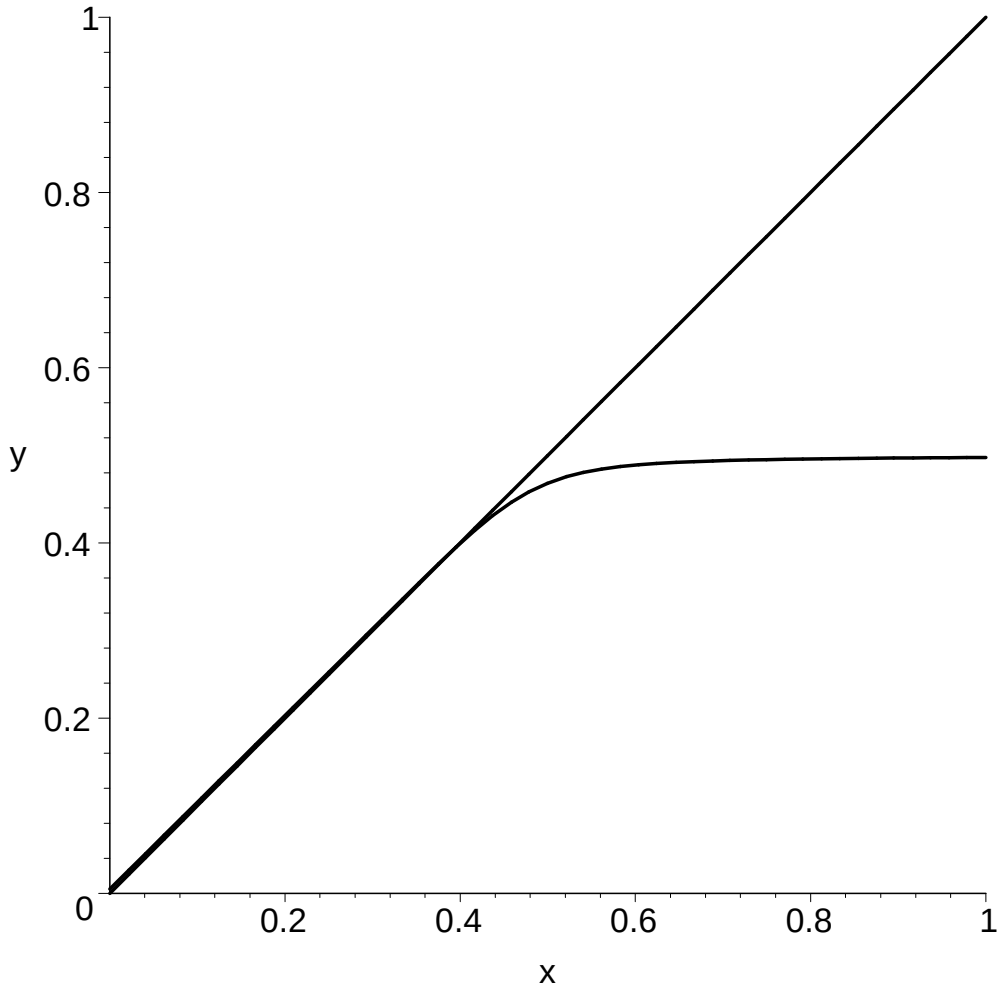
Example 2

$$d\rho = \chi_{[0,\frac{1}{2}]}\, dx, \qquad f(x) = x, \qquad n = 100:$$



Example 2

$$d\rho = \chi_{\left[0,\frac{1}{2}\right]}\, dx, \qquad f(x) = x, \qquad n = 200:$$



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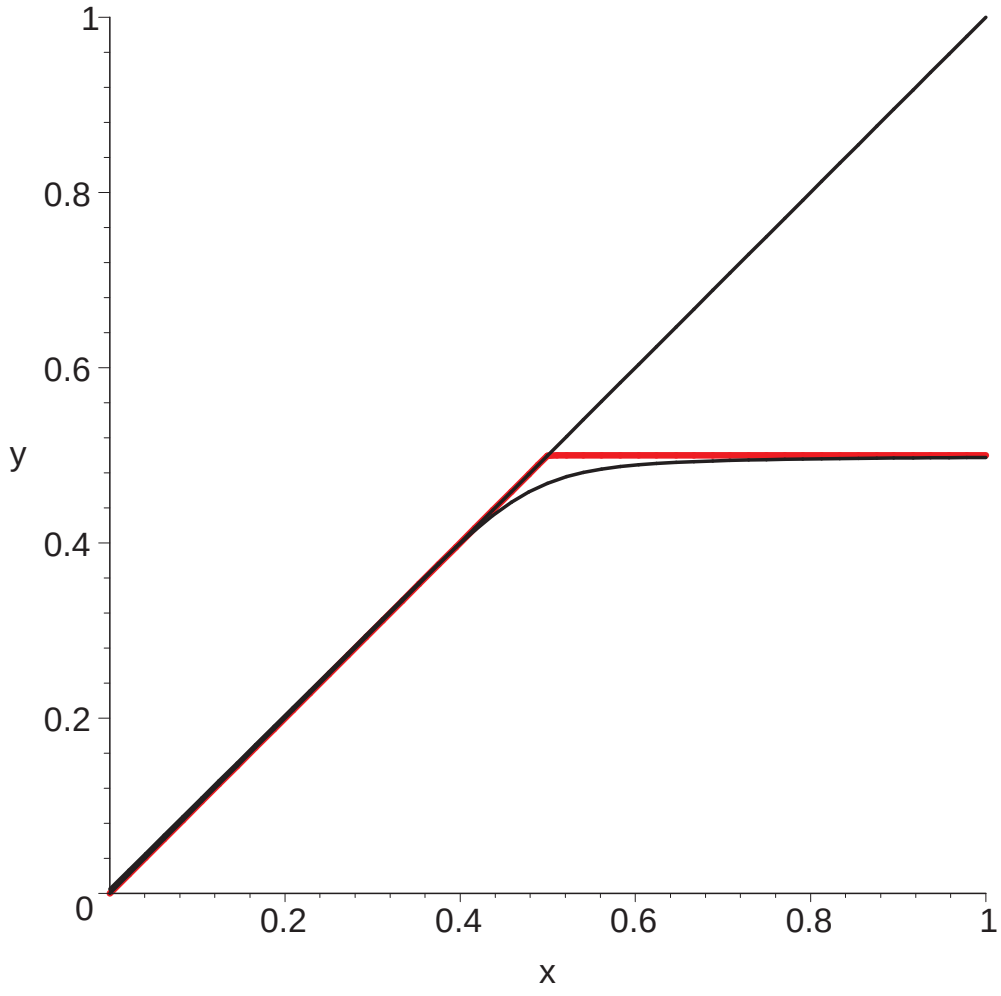

Convergence in $C(\mathbb{S}^d)$

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This is equivalent to the fact that $\operatorname{supp}(\rho) = \mathbb{S}^d$.

Theorem. (EB, JMAA, 2012) We have

$$\lim_{n\to\infty} \|f - \mathbf{M}_{n,\rho} f\|_C = 0 \qquad \text{for all} \qquad f \in C(\mathbb{S}^d)$$

if and only if ρ is strictly positive on $\mathbb{S}^d$.

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Denote $\varphi_{\mathbf{e}_i}(x) = x_i$, $\varphi_{2\mathbf{e}_i}(x) = x_i^2$.

Convergence in $C(\mathbb{S}^d)$

Theorem. (Kurt Jetter-EB, JAT, 2010) Let ρ be a Jacobi-like measure with $|\nu| - |\mu| < 1$. Then

$$\|\varphi_{\mathbf{e}_i} - \mathbf{M}_{n,\rho}(\varphi_{\mathbf{e}_i})\|_C \le C\, n^{-\frac{1-(|\nu|-|\mu|)}{2}}$$

and

$$\|\varphi_{2\mathbf{e}_i} - \mathbf{M}_{n,\rho}(\varphi_{2\mathbf{e}_i})\|_C \le C\, n^{-\frac{1-(|\nu|-|\mu|)}{2}}.$$

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Corollary. Let ρ be a Jacobi-like measure with $|\nu| - |\mu| < 1$. Let $f \in C(\mathbb{S}^d)$. Then

$$\|f - \mathbf{M}_{n,\rho}\, f\|_C \le C\, \omega\left(f, n^{-\frac{1-(|\nu|-|\mu|)}{4}}\right),$$

where $\omega(f,\delta) = \sup\{|f(x) - f(t)| : \|t - x\|_2 < \delta\}$ denote the modulus of continuity of f.

Convergence on $\operatorname{supp} \rho$

Theorem. (EB, 2012) Let $x \in (\operatorname{supp} \rho)^\circ$. Let f be bounded on $\operatorname{supp} \rho$ and continuous at x. Then

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Theorem. (EB, 2012) Let A be a compact set, $A \subset (\operatorname{supp}\rho)^\circ$. Let f be bounded on $\operatorname{supp}\rho$ and continuous on A. Then

$$\lim_{n\to\infty} \|f - \mathbf{M}_{n,\rho} f\|_{C(A)} = 0.$$

Convergence in $L^q_\rho(\mathbb{S}^d)$

Theorem. (Bing-Zheng Li, 2012) Let ρ be a non-negative bounded (regular) Borel measure on $\mathbb{S}^d$ such that $\operatorname{supp}(\rho) \setminus (\partial\mathbb{S}^d) \neq \emptyset$. Let $1 \leq q < \infty$. Then

$$\lim_{n\to\infty} \|f - \mathbf{M}_{n,\rho} f\|_{L^q_\rho} = 0$$

for every $f \in L^q_\rho(\mathbb{S}^d)$.

Convergence in $L^q_\rho(\mathbb{S}^d)$

Consider the K-functional

$$\mathcal{K}_q(f,t) = \inf\{\|f-g\|_{L^q_\rho} + t \max_{i=1,\dots,d} \|\partial_i g\|_C : g \in C^1(\mathbb{S}^d)\}.$$

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Theorem. (Bing-Zheng Li-EB, 2012) Let ρ be a non-negative bounded Borel measure on $\mathbb{S}^d$ such that $\operatorname{supp}\rho \setminus \partial\mathbb{S}^d \neq \emptyset$, and let $f \in L^q_\rho(\mathbb{S}^d)$, $1 \le q < \infty$. Then

$$\|f - \mathbf{M}_{n,\rho} f\|_{L^q_\rho} \le 2\,\mathcal{K}_q\left(f, \frac{C_q}{\sqrt{n}}\, d\left[\rho(\mathbb{S}^d)\right]^{\frac{1}{q}}\right), \quad 1 \le q < \infty,$$

where C_q is a constant that depends only on q. Moreover, $C_q = 1$ for $1 \le q \le 2$.

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Remark: the cases $q = 1$, $q = 2$ are due to Bing-Zheng Li.

Thank you for your attention!