Bernstein-Durrmeyer operators with arbitrary weight functions

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Bernstein basis polynomials

Standard simplex in $\mathbb{R}^d$:

$$S^d := \{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d : 0 \leq x_1, \ldots, x_d \leq 1, x_1 + \cdots + x_d \leq 1 \}.$$
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Barycentric coordinates:

$$x = (x_0, x_1, \ldots, x_d), \quad x_0 := 1 - x_1 - \cdots - x_d.$$
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The $d$-variate Bernstein basis polynomials of degree $n$ are defined by

$$B_\alpha(x) := \binom{n}{\alpha} x^\alpha = \frac{n!}{\alpha_0! \alpha_1! \cdots \alpha_d!} x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_d^{\alpha_d}, $$

$$\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^{d+1} \text{ with } |\alpha| := \alpha_0 + \alpha_1 + \cdots + \alpha_d = n. $$
Bernstein basis polynomials

One-dimensional case: $n \in \mathbb{N}$ and $k = 0, 1, \ldots, n$,

$$p_{n,k}(x) := \binom{n}{k} x^k (1 - x)^{n-k}, \quad x \in [0, 1].$$
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The Bernstein operator is defined for \( f \in C(S^d) \) by

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(B_n f)(x) := \sum_{|\alpha|=n} f \left( \frac{\alpha}{n} \right) B_\alpha(x).
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This is a positive linear operator that reproduces linear functions.

Uniform convergence for every function in \( C(\mathbb{S}^d) \).
Bernstein-Durrmeyer operator

A similar construction for integrable functions?
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**Definition.** The Bernstein-Durrmeyer operator is defined for \( f \in L^q(\mathbb{S}^d) \), \( 1 \leq q < \infty \), or \( f \in C(\mathbb{S}^d) \) by

\[
(M_n f)(x) := \sum_{|\alpha|=n} \frac{\int_{\mathbb{S}^d} f(y) B_\alpha(y) \, dy}{\int_{\mathbb{S}^d} B_\alpha(y) \, dy} B_\alpha(x).
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\( M_n \) is a positive linear operator that reproduces constant functions. Convergence in \( L^q(S^d) \), \( 1 \leq q < \infty \), and in \( C(S^d) \).
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Extension to functions on the \( d \)-dimensional simplex: Derriennic (starting from 1985).
Weighted Bernstein-Durrmeyer operator

Let $\rho$ be a non-negative bounded (regular) Borel measure on $\mathbb{S}^d$ such that $\text{supp} (\rho) \setminus (\partial \mathbb{S}^d) \neq \emptyset$. 
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$L_q(\mathbb{S}^d)$, $1 \leq q < \infty$: the weighted $L^q$-space with the norm

$$\|f\|_{L_q^\rho} := \left( \int_{\mathbb{S}^d} |f(x)|^q \, d\rho(x) \right)^{1/q}.$$
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$$\| f \|_{L^q_\rho} := \left( \int_{\mathbb{S}^d} |f(x)|^q \, d\rho(x) \right)^{1/q}.$$

**Definition.** The Bernstein-Durrmeyer operator with respect to the measure $\rho$ is defined for $f \in L^q_\rho (\mathbb{S}^d), 1 \leq q < \infty$, or $f \in C(\mathbb{S}^d)$ by

$$(M_{n,\rho} f)(x) := \sum_{|\alpha| = n} \frac{\int_{\mathbb{S}^d} f(y) \, B_\alpha(y) \, d\rho(y)}{\int_{\mathbb{S}^d} B_\alpha(y) \, d\rho(y)} \, B_\alpha(x).$$
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$M_{n,\rho}$ is a positive linear operator that reproduces constant functions.
Jacobi weights

The weighted Bernstein-Durrmeyer operator $M_{n,\rho}$ is very well studied for Jacobi weights, i.e., for

$$d\rho(x) = x^{\mu} \, dx,$$

with $\mu = (\mu_0, \mu_1, \ldots, \mu_d) \in \mathbb{R}^{d+1}$, where $\mu_i > -1$, $i = 0, 1, \ldots, d$. 
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Bernstein-Durrmeyer operators with respect to Jacobi weights were introduced by Păltănea (1983), Berens and Xu (1991), in the multivariate case by Ditzian (1995).
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They were studied by many authors, e.g., Derriennic, Berens, Xu, Ditzian, Chen, Ivanov, X.-L. Zhou, Knoop, Gonska, Heilmann, Abel, Jetter, Stöckler, \ldots
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Berens and Xu noticed that the Bernstein-Durrmeyer operator with respect to Jacobi weight is a summation method for Jacobi series.
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Motivation: Learning Theory

In a joint paper with Kurt Jetter (JAT, 2010), we started to study the multivariate Bernstein-Durrmeyer operators with respect to general measure.
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Recently (2012), Bing-Zheng Li used the multivariate operators $M_{n,\rho}$ to obtain estimates for learning rates of least-square regularized regression with polynomial kernels.
Motivation: Learning Theory

Typical problems in learning theory:

- Regression. E.g., least squares.
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Convergence: examples

Numerical experiments show that convergence holds for a wide class of measures $\rho$. 
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**Example 1.** Consider the measure $d\rho(x) = w(x) \, dx$ with

\[
w(x) = \left| \left( x - \frac{1}{2} \right)^2 - \frac{1}{8} \right|
\]
Example 1

\[ d\rho(x) = \left| (x - \frac{1}{2})^2 - \frac{1}{8} \right| \, dx, \quad f(x) = x, \quad n = 5 : \]
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Convergence: examples

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**Example 2.** $d\rho(x) = w(x) \, dx$ with

$$w(x) = \begin{cases} 1, & 0 \leq x \leq \frac{1}{2}, \\ 0, & \frac{1}{2} < x \leq 1. \end{cases}$$
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The Bernstein-Durrmeyer operator has the form

$$(M_{n, \rho} f)(x) = \sum_{k=0}^{n} \frac{\int_{0}^{\frac{1}{2}} f(y) y^k (1 - y)^{n-k} \, dy}{\int_{0}^{\frac{1}{2}} y^k (1 - y)^{n-k} \, dy} \binom{n}{k} x^k (1 - x)^{n-k}.$$
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Then for $f(x) = x$ we have $(M_{n,\rho} f)(x) \leq \frac{1}{2}$, $x \in [0,1]$. 
Example 2

\[ d\rho = \chi_{[0, \frac{1}{2}]} \, dx, \quad f(x) = x, \quad n = 5 : \]
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Convergence in $C(S^d)$

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Recall that a measure $\rho$ on $\mathbb{S}^d$ is called strictly positive, if $\rho(A \cap \mathbb{S}^d) > 0$ for every open set $A \subset \mathbb{R}^d$ with $A \cap \mathbb{S}^d \neq \emptyset$. 
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This is equivalent to the fact that $\text{supp} (\rho) = \mathbb{S}^d$. 
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This is equivalent to the fact that $\text{supp}(\rho) = \mathbb{S}^d$.

**Theorem.** (EB, JMAA, 2012) We have
\[ \lim_{n \to \infty} \| f - M_{n, \rho} f \|_C = 0 \]
for all $f \in C(\mathbb{S}^d)$ if and only if $\rho$ is strictly positive on $\mathbb{S}^d$. 
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and there are two exponents $\nu \geq \mu > -e$ and two constants $0 < a, A < \infty$ such that

$$a \, x^\nu \leq w(x) \leq A \, x^\mu, \quad x \in \mathbb{S}^d.$$
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Denote $\varphi_{e_i}(x) = x_i$, $\varphi_{2e_i}(x) = x_i^2$. 
Theorem. (Kurt Jetter-EB, JAT, 2010) Let $\rho$ be a Jacobi-like measure with $|\nu| - |\mu| < 1$. Then

$$\| \varphi_{e_i} - M_{n, \rho}(\varphi_{e_i}) \|_{C} \leq C n^{-\frac{1-(|\nu| - |\mu|)}{2}}$$

and

$$\| \varphi_{2e_i} - M_{n, \rho}(\varphi_{2e_i}) \|_{C} \leq C n^{-\frac{1-(|\nu| - |\mu|)}{2}}.$$
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**Corollary.** Let $\rho$ be a Jacobi-like measure with $|\nu| - |\mu| < 1$. Let $f \in C(S^d)$. Then

$$\| f - M_{n,\rho} f \|_C \leq C \omega \left( f, n^{-\frac{1 - (|\nu| - |\mu|)}{4}} \right),$$

where $\omega(f, \delta) = \sup \{|f(x) - f(t)| : \|t - x\|_2 < \delta\}$ denote the modulus of continuity of $f$. 

Bernstein-Durrmeyer operators with arbitrary weight functions – p. 28/32
Convergence on $\text{supp } \rho$

**Theorem.** (EB, 2012) Let $x \in (\text{supp } \rho)^\circ$. Let $f$ be bounded on $\text{supp } \rho$ and continuous at $x$. Then

$$\lim_{n \to \infty} |f(x) - M_{n, \rho} f(x)| = 0.$$
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**Theorem.** (EB, 2012) Let $A$ be a compact set, $A \subset (\text{supp }\rho)^\circ$. Let $f$ be bounded on $\text{supp }\rho$ and continuous on $A$. Then

$$\lim_{n \to \infty} \|f - M_{n,\rho} f\|_{C(A)} = 0.$$
Convergence in $L^q_{\rho}(S^d)$

**Theorem.** (Bing-Zheng Li, 2012) Let $\rho$ be a non-negative bounded (regular) Borel measure on $S^d$ such that $\text{supp}(\rho) \setminus (\partial S^d) \neq \emptyset$. Let $1 \leq q < \infty$. Then

$$\lim_{n \to \infty} \| f - M_{n,\rho} f \|_{L^q_{\rho}} = 0$$

for every $f \in L^q_{\rho}(S^d)$. 
Consider the K-functional

$$K_q(f, t) = \inf \{ \| f - g \|_{L^q_\rho} + t \max_{i=1, \ldots, d} \| \partial_i g \|_C : g \in C^1(S^d) \}.$$
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**Theorem.** (Bing-Zheng Li-EB, 2012) Let $\rho$ be a non-negative bounded Borel measure on $S^d$ such that $\text{supp} \rho \setminus \partial S^d \neq \emptyset$, and let $f \in L^q_\rho(S^d)$, $1 \leq q < \infty$. Then

$$
\| f - M_{n,\rho} f \|_{L^q_\rho} \leq 2K_q \left( f, \frac{C_q}{\sqrt{n}} d \left[ \rho(S^d) \right]^{\frac{1}{q}} \right), \quad 1 \leq q < \infty,
$$

where $C_q$ is a constant that depends only on $q$. Moreover, $C_q = 1$ for $1 \leq q \leq 2$. 
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**Theorem.** (Bing-Zheng Li-EB, 2012) Let $\rho$ be a non-negative bounded Borel measure on $S^d$ such that $\text{supp} \rho \setminus \partial S^d \neq \emptyset$, and let $f \in L^q_\rho(S^d)$, $1 \leq q < \infty$. Then

$$\| f - M_{n, \rho} f \|_{L^q_\rho} \leq 2 K_q \left( f, \frac{C_q}{\sqrt{n}} d \left[ \rho(S^d) \right]^{\frac{1}{q}} \right), \quad 1 \leq q < \infty,$$

where $C_q$ is a constant that depends only on $q$. Moreover, $C_q = 1$ for $1 \leq q \leq 2$.

**Remark:** the cases $q = 1$, $q = 2$ are due to Bing-Zheng Li.
Thank you for your attention!