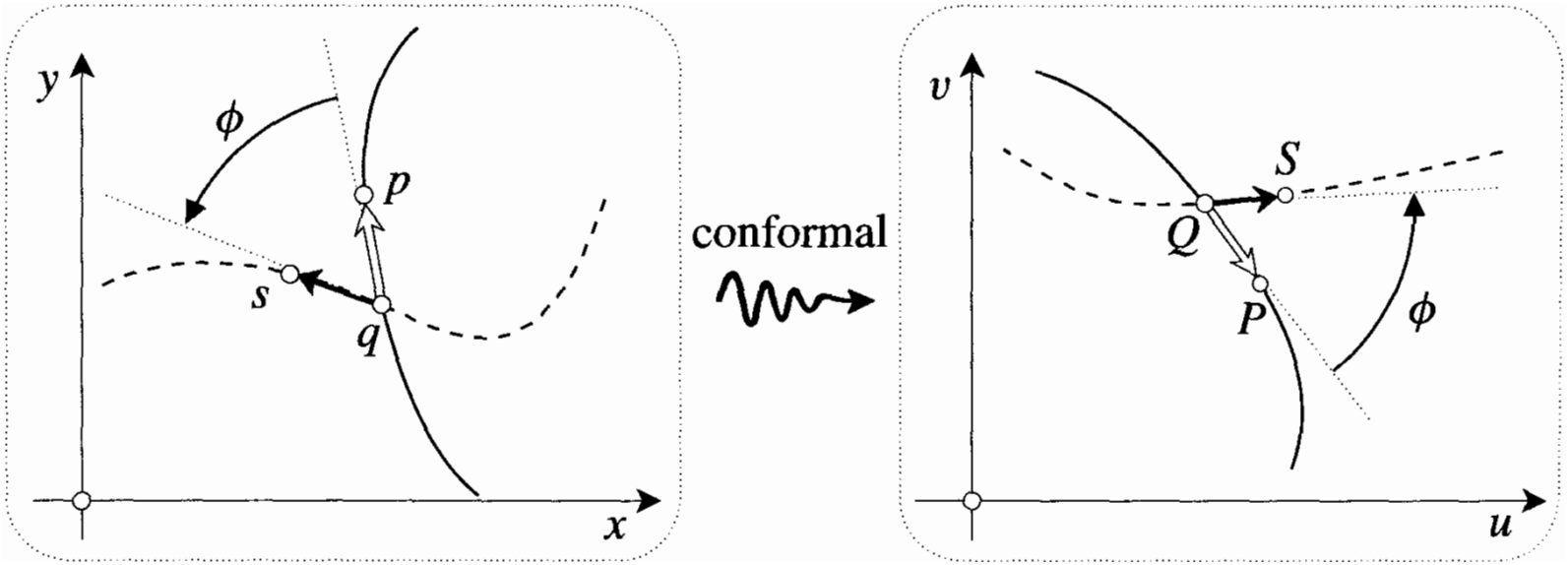
Estimations universelles pour les solutions d'EDP elliptiques non linéaires

Louis Dupaigne
LAMFA, UMR CNRS 6140
Université de Picardie Jules Verne (Amiens)

1. **Singular Yamabe Problem:** can an open set of Euclidean space be conformally transformed into a **complete** manifold of **prescribed curvature**?
2. **Extinction of super Brownian motion:** for which open sets of Euclidean space can we guarantee that super Brownian motion never reaches the boundary ?
3. **Universal estimates:** can we obtain (pointwise) *a priori* estimates for solutions of a nonlinear elliptic PDE, without knowing their boundary values ?

Definition

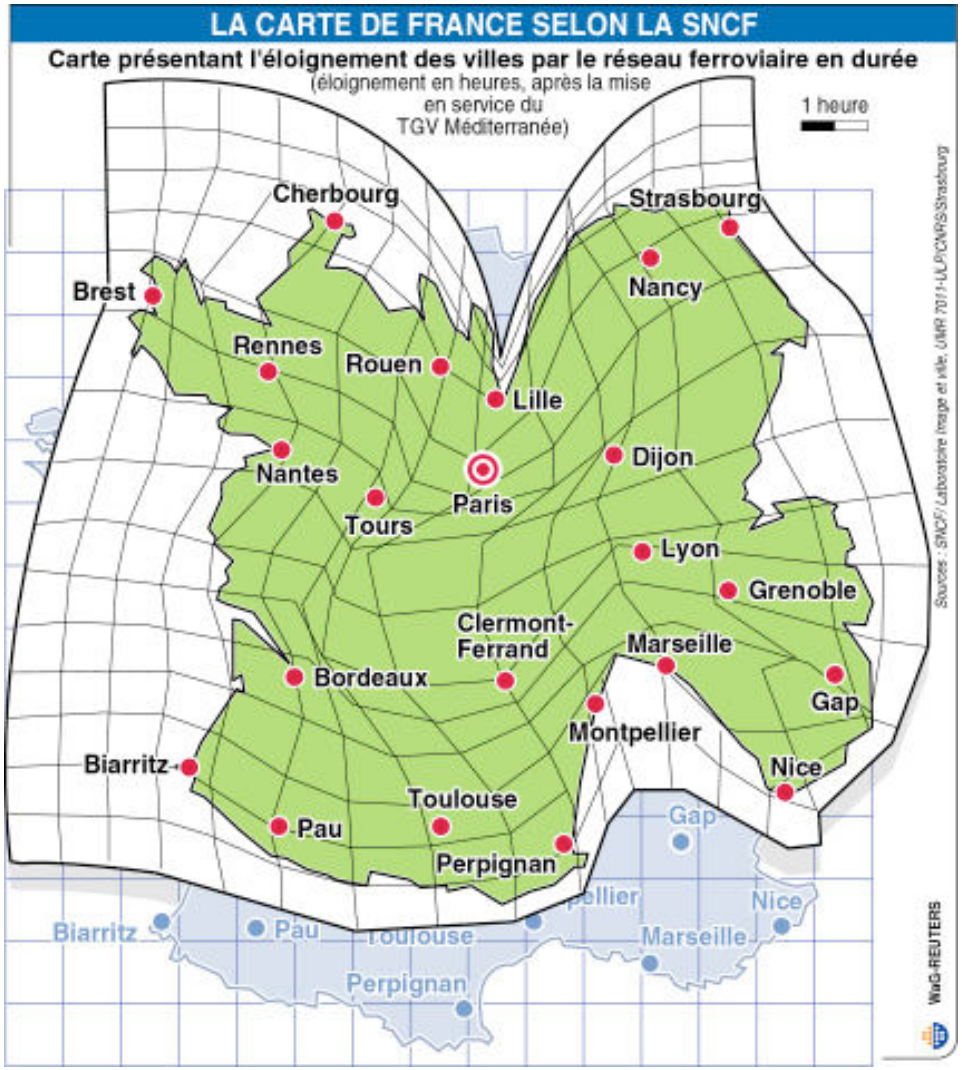
A map $\psi : \Omega \subset \mathbb{R}^N \to \mathbb{R}^N$ is **conformal** if it preserves angles.



Given a metric g on a manifold M, a metric g' is conformal to g if for some $\rho : M \to (0, +\infty)$,

$$g'_{ij} = \rho \, g_{ij}$$

Example: the TGV metric



How is curvature affected by a conformal transformation ?

- Assume $N = 2$. If $g' = \rho\, g = e^{2u} g$, then the Gaussian curvature equation holds:
$$-\Delta_g u + K_g = K_{g'} e^{2u} \qquad \text{in } M$$
- Assume $N \geq 3$. If $g' = \rho\, g = u^{\frac{4}{N-2}} g$, then the scalar curvature or **Yamabe equation** holds:
$$-\Delta_g u + c_N R_g u = c_N R_{g'} u^{\frac{N+2}{N-2}} \qquad \text{in } M,$$
where $c_N = \frac{N-2}{4(N-1)}$.
In particular, if $M = \Omega$ is an open set of Euclidean space,
$$-\Delta u = c_N R_{g'} u^{\frac{N+2}{N-2}} \qquad \text{in } \Omega.$$

The singular Yamabe problem

Recall that we want to find a **complete** metric $g'_{ij} = \rho\delta_{ij}$ on an open set Ω of Euclidean space. This implies (case $N \geq 3$):

$$\begin{cases} -\Delta u = c_N R_{g'} u^{\frac{N+2}{N-2}} & \text{in } \Omega. \\ u = +\infty & \text{on } \partial\Omega. \end{cases}$$

A solution to the above equation is called a **BBUS** or **large solution**.

Remark

If $\partial\Omega$ is smooth, there is no solution having positive scalar curvature.

The Poincaré metric

Let $\Omega = B$ denote the unit ball of $\mathbb{R}^N$ and try to solve the singular Yamabe problem for constant negative curvature $R_{g'}$. Then, if $N = 2$, the problem becomes

$$\begin{cases} \Delta u = 4e^{2u} & \text{in } B. \\ u = +\infty & \text{on } \partial B, \end{cases}$$

or, if $N \geq 3$,

$$\begin{cases} \Delta u = N(N-2)u^{\frac{N+2}{N-2}} & \text{in } B. \\ u = +\infty & \text{on } \partial B. \end{cases}$$

Let R be the **hyperbolic radius**, defined by

$$R := \rho^{-1/2} = \begin{cases} e^{-u} & \text{if } N = 2 \\ u^{-\frac{2}{N-2}} & \text{if } N \geq 3 \end{cases}$$

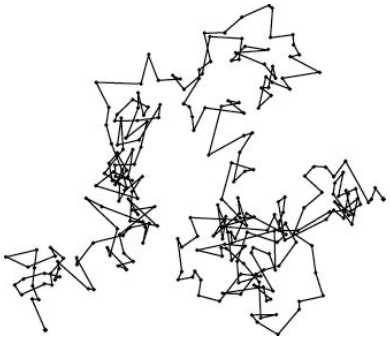
Then, there exists a unique BBUS explicitly given by

$$R = 1 - |x|^2.$$

(B, g') is the **Poincaré disk**.

See http://www.poincare.fr for more.

Brownian motion



Let $(B_t)_{t\geq 0}$ be **Brownian motion** in $\mathbb{R}^N$, i.e.

- B_t takes values in $\mathbb{R}^N$. Reducing to the case $N = 1$,
- $t \mapsto B_t$ is almost surely continuous
- B_t has independent increments i.e. for $0 \leq s < t$,

$$(B_t - B_s) \perp\!\!\!\perp (B_r)_{0\leq r\leq s}$$

- B_t has stationary increments

$$(B_t - B_s) \overset{\mathcal{L}}{=} B_{t-s} - B_0$$

$\rightsquigarrow B_t \sim \mathcal{N}(rt, \sigma^2 t)$. Normalization: $r = 0, \sigma = 1$.

Exit time and the Dirichlet problem

Let Ω be a bounded open set of $\mathbb{R}^N$. Consider the **exit time** of Brownian motion.

$$\tau_\Omega = \inf\{t > 0 \ : \ B_t \in \Omega^c\}$$

Problem if $B_{t=0} = x \in \partial\Omega$, is it true that $\tau_\Omega = 0$ a.s. ?

Take $g \in C(\partial\Omega)$ and for $x \in \overline{\Omega}$, set

$$u(x) = \mathbb{E}^x(g(B_{\tau_\Omega})).$$

Then,

$$\Delta u = 0, \qquad \text{in } \Omega$$

and $\tau_\Omega = 0$ iff for all such g,

$$u(x) = g(x), \qquad \text{on } \partial\Omega.$$

Theorem (Wiener criterion)

Ω is Dirichlet regular iff

$$\int_0^1 \frac{Cap(B(x,r) \cap \Omega^c)}{r^{N-2}} \frac{dr}{r} = +\infty, \qquad \textit{for all } x \in \partial\Omega.$$

Brownian snake

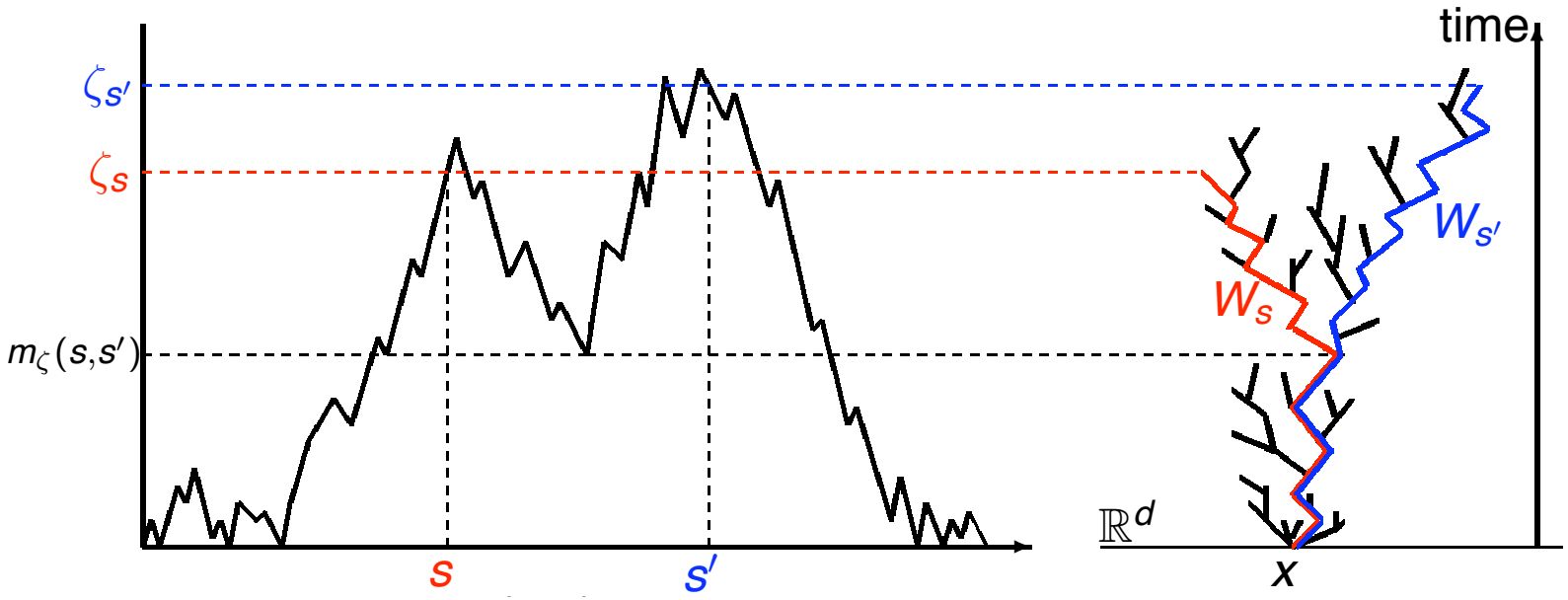
Fix $x \in \mathbb{R}^N$ and set

$$\begin{aligned}\mathcal{W}_x &= \{\text{finite path started at } x\} \\ &= \{w \in C([0,\zeta_w],\mathbb{R}^N) \ : \ w(0) = x\}.\end{aligned}$$

ζ_w is the **lifetime** of w, and $\hat{w} = w(\zeta_w)$ its **tip**.
Heuristicaly, the **Brownian snake** satisfies

- W_t takes values in $\mathcal{W}_x$
- the lifetime $\zeta_t = \zeta_{W_t}$ is b.m. (reflected at 0)
- when ζ_t decreases, the path W_t is decreased from its tip
- When ζ_t increases, the path W_t is extended by adding little pieces of N-dim. b.m. at its tip.

Brownian snake



- W_t takes values in $\mathcal{W}_x$
- the lifetime ζ_t is b.m. (reflected at 0)
- conditionnally on $(\zeta_t)_{t\geq 0}$, if $0 \leq s < s'$,

$$W_s(t) = W_{s'}(t), \qquad \text{for } 0 \leq r \leq \min_{p\in[s,s']} \zeta_p =: m$$

$$(W_{s'}(m+t) - W_{s'}(m))_{0\leq t\leq \zeta_{s'}-m} \text{ is b.m. in } \mathbb{R}^N \perp\!\!\!\perp W_s$$

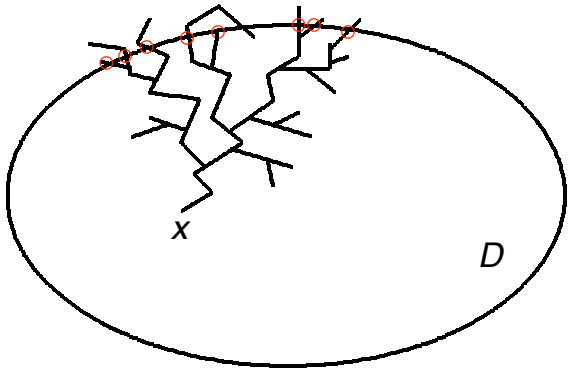
Exit set

Given an open set $\Omega \subset \mathbb{R}^N$, and a finite path $w \in \mathcal{W}_x$, set

$$\tau(w) = \inf\{t \geq 0 \ : \ w(t) \in \Omega^c.\}$$

The set of **exit points** of b.s. is given by

$$\mathcal{E}_\Omega = \{W_t(\tau(W_t)) \ : \ t \geq 0, \tau(W_t) < \infty\}\}$$

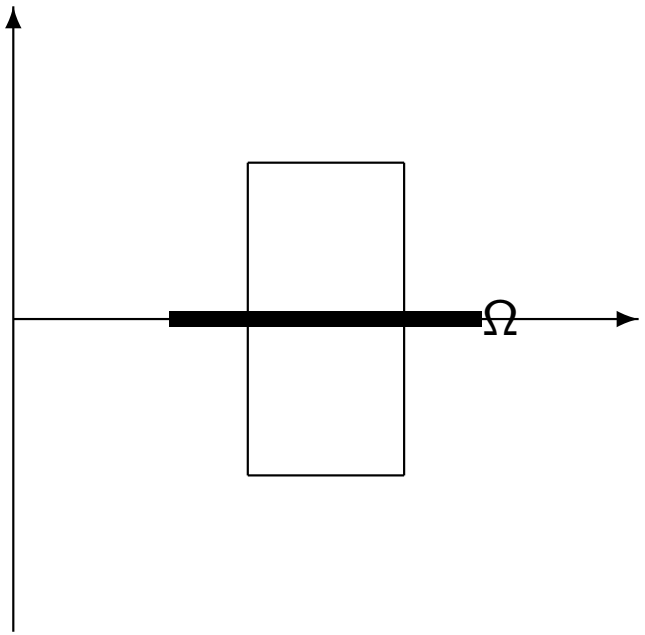


There exists a ($\sigma-$finite) measure (the excursion measure of b.s.) s.t. $u(x) = \mathbb{N}^x(\mathcal{E}_\Omega \neq \emptyset)$ solves

$$\begin{cases} \Delta u = 4u^2, & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega \end{cases}$$

whenever Ω is Dirichlet regular.

Universal estimates



Assume u solves a nonlinear PDE (**no BC's**) of the form

$$\Delta u = f(u) \qquad \text{in } \Omega.$$

Fix a subdomain $\omega \subset\subset \Omega$. Want $C = C(\omega, \Omega, f, N)$ s.t.

$$|u| \leq C \qquad \text{in } \omega,$$

More precisely, does there exist a function $g = g_{N,f}$ such that

$$|u(x)| \leq g(dist(x, \partial\Omega)) \qquad \text{for all } x \in \Omega \text{ ?}$$

The Lane-Emden equation

Fix $p > 1$, Ω an open set of $\mathbb{R}^N$, and consider

$$-\Delta u = |u|^{p-1} u \qquad \text{in } \Omega.$$

A solution $u \in C^2(\Omega)$ can be seen as a **critical point of the energy** functional

$$\mathcal{E}_\omega(u) = \frac{1}{2} \int_\omega |\nabla u|^2 \, dx - \frac{1}{p+1} \int_\omega |u|^{p+1} \, dx, \qquad \text{where } \omega \subset\subset \Omega.$$

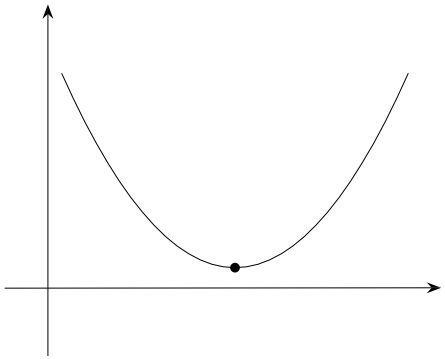
i.e. for $\varphi \in C_c^1(\Omega)$,

$$D\mathcal{E}(u)(\varphi) = \int_\Omega (-\Delta u - |u|^{p-1} u)\varphi \, dx$$

Compute the **second variation of energy**:

$$D^2\mathcal{E}(u)(\varphi) = \int_\Omega \left(|\nabla \varphi|^2 - p|u|^{p-1}\varphi^2 \right) \, dx.$$

Stability



Definition

- A solution is said to be **stable** whenever $D^2\mathcal{E}(u)(\varphi) \geq 0$ for all $\varphi \in C_c^1(\Omega)$.
- More generally, u has finite **Morse index** $k \in \mathbb{N}$ if k is the maximal dimension of any subspace $X \subset C_c^1(\Omega)$ s.t. $D^2\mathcal{E}(u)(\varphi) < 0$ for all $\varphi \in X \setminus \{0\}$.

Remark

- *If Ω is bounded and $u \in C^2(\overline{\Omega})$, k is the number of negative eigenvalues of the linearized operator $-\Delta - p|u|^{p-1}$.*
- *So, every solution $u \in C^2(\Omega)$ has finite Morse index on $\omega \subset\subset \Omega$.*

Universal estimate for the Lane-Emden equation

Theorem (Davila-D-Farina)

Assume $1 \leq N \leq 10$ *and take any* $p > 1$, $p \neq \frac{N+2}{N-2}$. *Let* $u \in C^2(\Omega)$ *be a solution of finite Morse index* k. *There exists a constant* $C = C(N, p, k)$ *s.t.*

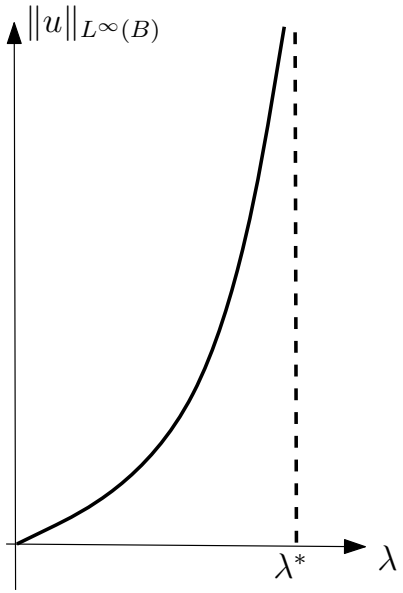
$$|u(x)| \leq C\, dist(x, \partial\Omega)^{-\frac{2}{p-1}}, \qquad \text{for all } x \in \Omega.$$

Discussion: critical dimension, critical exponent

- if $N \geq 11$, the universal estimate holds iff $p < p_c(N)$. For $p \geq p_c(N)$, consider

$$\begin{cases} -\Delta u = \lambda(1+u)^p & \text{in } B \\ u = 0 & \text{on } \partial B. \end{cases}$$

Then ([Joseph-Lundgren]),



Discussion: Liouville, Yamabe, weak solutions

- the universal estimate implies the Liouville property.
- if $p = \frac{N+2}{N-2}$, the estimate fails. Take

$$u(x)^{-\frac{2}{N-2}} = c_N \left(\lambda + \frac{|x|^2}{\lambda} \right).$$

  Note: the stereographic projection $\pi : \mathbb{S}^N \to \mathbb{R}^N \cup \{\infty\}$ is conformal and $g_{ij} = u^{\frac{4}{N-2}} \delta_{ij}$ is the pullback of the standard metric on $\mathbb{S}^N$.
- the universal estimate extends to weak solutions $u \in H^1(\Omega) \cap L^p(\Omega)$. It fails for weak solutions belonging to $L^p(\Omega)$. Counter-example: $u(x) = c_{N,p}|x|^{-\frac{2}{p-1}}$.

Discussion: $p \geq p_c(N)$?

Theorem (Davila-D-Farina)

Let $u \in H^1(\Omega) \cap L^p(\Omega)$, $u > 0$, have finite Morse index k and take $N \geq 11$, $p \geq p_c(N)$. Then, $u \in C^\infty(\Omega \setminus \Sigma)$, where

$$\mathcal{H}_{dim}(\Sigma) \leq N - 2\frac{p+\gamma}{p-1},$$

where $\gamma = 2p + 2\sqrt{p(p-1)} - 1$.

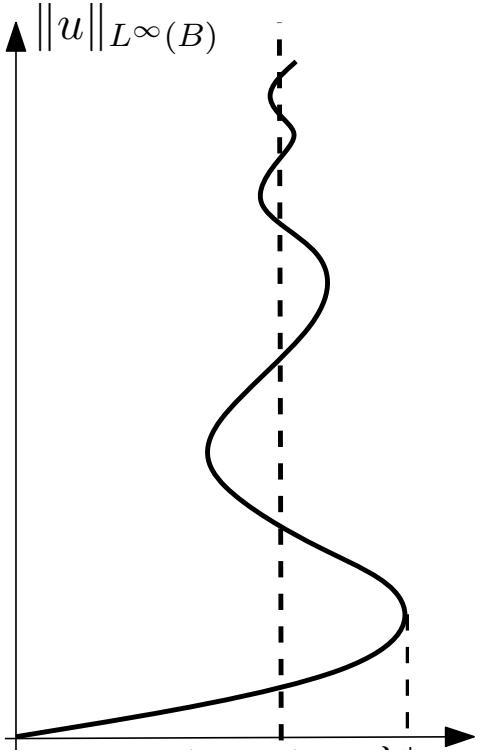
Remark

$p = p_c(N)$ solves

$$N - 2\frac{p+\gamma}{p-1} = 0.$$

Discussion: without Morse index ?

- If $p \in (\frac{N+2}{N-2}, p_c(N))$, ([Joseph-Lundgren]),



- If $1 < p < \frac{N+2}{N-2}$ and $u > 0$, then *all* solutions have bounded Morse index ([Gidas-Spruck, Serrin-Zhou]). Fails again for sign-changing solutions ([Rabinowitz, Ambrosetti-Rabinowitz]).

Assume $\mathbf{f} \geq \mathbf{0}$ and consider

$$\Delta u = f(u) \qquad \text{in } \Omega$$

Theorem (Keller, Osserman)

Let Ω open set of $\mathbb{R}^N$, $N \geq 1$. Assume $f \in C(\Omega)$, $f \geq 0$, f increasing.

- *if $f > 0$, there exists $g = g_{N,f}$ s.t.*

$$u(x) \leq g(dist(x, \partial\Omega)) \qquad \text{for all } x \in \Omega,\ u$$

  iff

$$(KO) \qquad \int^{+\infty} \frac{dt}{\sqrt{F(t)}} < \infty, \qquad \text{where } F' = f.$$

- *if $f(0) = 0$, there exists $\tilde{g} = \tilde{g}_{N,f}$ s.t.*

$$0 \leq u(x) \leq \tilde{g}(dist(x, \partial\Omega)) \qquad \text{for all } x \in \Omega,\ u \geq 0$$

  iff (KO) holds.

Discussion: Liouville, regularity of f, BBUS

- Liouville property:

$$\lim_{d\to 0} g = +\infty \qquad \lim_{d\to 0} \tilde{g} = +\infty$$
$$\lim_{d\to +\infty} g = -\infty \qquad \lim_{d\to +\infty} \tilde{g} = 0$$

- need not assume f continuous ([Olofsson])
- existence of BBUS: take Ω bounded, $f(0) = 0$.
  If Ω is Dirichlet regular, we can easily solve

$$\begin{cases} \Delta u_k = f(u_k) & \text{in } \Omega, \\ u_k = k & \text{on } \partial\Omega. \end{cases}$$

  Then $u_k \nearrow U$, where U is a BBUS.

Theorem (Dhersin-Legal)

Take $f(u) = u^2$. $\qquad \exists BBUS \geq 0 \iff N \leq 3$ *or*

$$N \geq 4 \text{ and } \int_0^1 \frac{C_{2,2}(\Omega^c \cap B(x,r))}{r^{N-2}} \frac{dr}{r} = +\infty, \qquad \text{for all } x \in \partial\Omega$$

Discussion: f not increasing, Liouville again

- The previous result can be extended to $f(u) = u^p$, $p > 1$ ([Labutin]), with critical exponent $p = \frac{N}{N-2}$ and capacity $C_{2,p'}$.
- If f not increasing, the universal estimate fails. Simply take $f(u) = u^2 \sin^2(u)$. However,

Theorem (Dumont-D-Goubet-Radulescu)

If $f \geq 0$, loc. Lip., $f(0) = 0$, Ω Dirichlet regular bounded

$$\exists BBUS \geq 0 \iff (KO).$$

Theorem (D)

$$\textit{Liouville property} \iff (KO).$$

$\rightsquigarrow$ Universal Harnack inequality ?

Conjectures on BBUS

- ([McKenna]): if f increasing, then the BBUS is unique
- ([Brezis]): if f loc. Lip., $\Omega = B$, then every BBUS is radial
- ([Lazer-McKenna]) : characterize all f's for which universal blow-up occurs i.e. any BBUS satisfies

$$u(x) = g(dist(x, \partial\Omega)) + o(1),$$

where

$$\begin{cases} g'' = f(g) & \text{in } (0,1) \\ g(0^+) = +\infty \end{cases}$$

Theorem (Costin-D)

Assume f loc. Lip., $\Omega = B$. Let U_1, U_2 be two BBUS. Then,

$$|U_1 - U_2| \leq C \int_{U_1}^{+\infty} \frac{dt}{F(t)}.$$

If $\exists K$ s.t. $t \mapsto f(t) + Kt$ increasing, then

$$|\nabla(U_1 - U_2)| \leq C.$$

Partial solutions to conjectures

Uniqueness Let u_1, u_2 denote two solutions. It suffices to prove that $u_1 \leq u_2$. If not, let $\omega = \{x \in B \ : \ u_1(x) > u_2(x)\}$. Then,

$$\begin{cases} \Delta(u_2 - u_1) = f(u_2) - f(u_1) \leq 0 & \text{in } \omega, \\ u_2 - u_1 = 0 & \text{on } \partial\omega. \end{cases}$$

By the maximum principle, $u_2 - u_1 \geq 0$ in ω, a contradiction.

Symmetry The gradient estimate implies $\partial_r u >> \partial_\theta u$. Then, apply the moving plane method ([Porretta-Véron]).

Universal blow-up Set $g = \sqrt{F}$. Then, universal blow-up always holds when

$$\lim_{u \to +\infty} g(u) \int_u^{+\infty} \frac{G(t)}{g(t)^3} \, dt = 0$$

and always fails when $\liminf > 0$.

From power series to Borel summation

To solve an ODE (or a PDE or a difference equation), say

$$y' - y = 0$$

one can look for a solution in the form of a power series

$$\tilde{y} = \sum a_k t^k.$$

That is, *provided the solution is analytic*, we replace an analytic problem (solving the ODE) with an algebraic problem that can be solved algorithmically. Here:

$$ka_k - a_{k-1} = 0.$$

In so doing, we use fundamentally the isomorphism between convergent power series and analytic functions $y = \tilde{y}$.

Problem

For most equations, formal asymptotic expansions thus obtained are (factorially) divergent. How can we make sense of such expressions ?

Euler's example

What is the meaning *for large t* of

$$\tilde{y} = \sum_{k=0}^{+\infty} \frac{k!}{(-t)^{k+1}}$$

Euler argues for the *preservation of the usual properties of summation*: formally, $\tilde{y}$ solves the ODE $\tilde{y}' - \tilde{y} = 1/t$. Thus, denoting y the sum of $\tilde{y}$, we should have

$$y' - y = \frac{1}{t}$$

- all solutions of this ODE are given by $y = e^t Ei(-t) + Ce^t$, where $Ei(x) = \int_{-\infty}^{x} s^{-1} e^s \, ds$.
- formally, $\tilde{y} \to 0$ as $t \to +\infty$. The only solution y having this property is $y = e^t Ei(-t)$. Thus, $e^t Ei(-t)$ is the sum of $\tilde{y}$.

Borel summation: heuristics

- In practice, divergent series have coefficients $a_k = O(k!^n)$.
- A change of variable allows to choose $n = 1$. E.g. $k!^2 t^{-k} \sim (2k)!(\sqrt{t})^{-2k} = j!/y^j$.
- Laplace transform $\mathcal{L}Y(t) = \int_0^{+\infty} e^{-pt} Y(p)\, dp$. Note that

$$\mathcal{L}^{-1} \frac{k!}{t^{k+1}} = p^k$$

Thus formally,

$$\mathcal{L}^{-1} \sum_{k=0}^{+\infty} \frac{k!}{(-t)^{k+1}} = \sum_{k=0}^{+\infty} (-p)^k = \frac{1}{1+p}.$$

Still formally, $\mathcal{L}\mathcal{L}^{-1} = Id$, so $\sum_{k=0}^{+\infty} \frac{k!}{(-t)^{k+1}} =$

$$= \mathcal{L}\mathcal{L}^{-1} \sum_{k=0}^{+\infty} \frac{k!}{(-t)^{k+1}} = \mathcal{L}\frac{1}{1+p} = \int_0^{+\infty} \frac{e^{-pt}}{1+p}\, dp = e^t Ei(-t)$$

Borel summation: rigorous definition

The Borel transform $\mathcal{B}$ acts on formal power series:

$$\tilde{y} = \sum \frac{y_k}{t^{k+1}} \to \mathcal{B}\tilde{y} = \sum \frac{y_k}{k!} p^k$$

The Borel summation $\mathcal{LB}$ is given by

$$y = \mathcal{LB}\tilde{y} = \int_0^{+\infty} e^{-pt} \mathcal{B}\tilde{y}(p)\, dp$$

This requires that

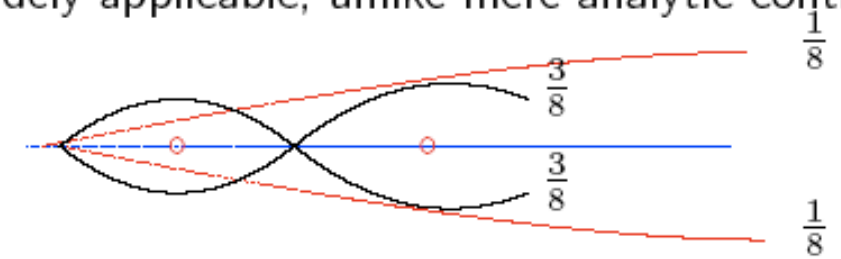
- $\mathcal{B}\tilde{y}$ has nonzero radius of convergence (true after our change of variables)
- $\mathcal{B}\tilde{y}$ continues analytically in $p \in \mathbb{R}^+$: serious restriction. Take e.g. $\tilde{y} = \sum \frac{k!}{t^{k+1}}$. Then, $\mathcal{B}\tilde{y} = \frac{1}{1-p}$.
- $\mathcal{B}\tilde{y}$ is exponentially bounded on $\mathbb{R}^+$.

Generalized Borel summation

Borel summation is: $g = \mathcal{L}(AC)\mathcal{B}\tilde{g}$

Generalized Borel summation.

(1) AC is replaced in Écalle's theory by medianization: averages with universal weights[1] of analytic continuations above and below $\mathbb{R}^+$. Medianization is very widely applicable, unlike mere analytic continuation.



(2) Superexponential growth: multisummability (Ramis, Ecalle). Ecalle: acceleration operators.

Generalized Borel summation:

Borel transform $\circ$ Medianization $\circ$ Acceleration $\circ$ Laplace transform.

[1] Independent of series.

Application to BBUS: examples

Let $d(x) = 1 - r$, $p > 1$ and $k_0 = [2/(p-1)] + 1$. Consider

$$\Delta u = u^p.$$

Then, the blow-up solution satisfies

$$u = d^{-\frac{2}{p-1}} \sum_{k=0}^{k_0} a_k d^k + o(1)$$

Now, for

$$\Delta u = u(\ln u)^4,$$

get

$$u = e^{\frac{1}{d}} \sum_{k=0}^{+\infty} a_k d^k + \text{exp. small}$$

The general case: starting point

Let $v = du/dr$ and multiply the equation by v. We get

$$\frac{d}{dr}\left(\frac{v^2}{2}\right) + \frac{N-1}{r}v^2 = \frac{d}{dr}\left(F(u)\right).$$

We define the resulting error term by

$$g := -\frac{v^2}{2} + F(u),$$

which satisfies the D.E. $\frac{dg}{dr} = \frac{N-1}{r}v^2$ Change independent variable $u = u(r)$. Thinking of g as a function of the variable u, we have

$$\frac{dg}{du} = \frac{N-1}{r}v.$$

Solving for v, we finally obtain

$$\begin{cases} dg/du = \dfrac{N-1}{r}v = \dfrac{N-1}{r}\sqrt{2(F(u)-g)}, \\ dr/du = 1/v = 1/\sqrt{2(F(u)-g)}. \end{cases}$$