Estimations universelles pour les solutions d’EDP elliptiques non linéaires

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1. **Singular Yamabe Problem:** can an open set of Euclidean space be conformally transformed into a **complete** manifold of **prescribed** curvature?

2. **Extinction of super Brownian motion:** for which open sets of Euclidean space can we guarantee that super Brownian motion never reaches the boundary?

3. **Universal estimates:** can we obtain (pointwise) *a priori* estimates for solutions of a nonlinear elliptic PDE, without knowing their boundary values?
Definition

A map $\psi : \Omega \subset \mathbb{R}^N \to \mathbb{R}^N$ is **conformal** if it preserves angles.

Given a metric $g$ on a manifold $M$, a metric $g'$ is conformal to $g$ if for some $\rho : M \to (0, +\infty)$,

$$g'_{ij} = \rho \ g_{ij}$$
Example: the TGV metric
Assume $N = 2$. If $g' = \rho g = e^{2u}g$, then the Gaussian curvature equation holds:

$$-\Delta_g u + K_g = K_{g'} e^{2u} \quad \text{in } M$$

Assume $N \geq 3$. If $g' = \rho g = u^{\frac{4}{N-2}}g$, then the scalar curvature or Yamabe equation holds:

$$-\Delta_g u + c_N R_g u = c_N R_{g'} u^{\frac{N+2}{N-2}} \quad \text{in } M,$$

where $c_N = \frac{N-2}{4(N-1)}$.

In particular, if $M = \Omega$ is an open set of Euclidean space,

$$-\Delta u = c_N R_{g'} u^{\frac{N+2}{N-2}} \quad \text{in } \Omega.$$
Recall that we want to find a **complete** metric $g'_{ij} = \rho \delta_{ij}$ on an open set $\Omega$ of Euclidean space. This implies (case $N \geq 3$):

\[
\begin{cases}
-\Delta u = c_N R_{g'} u^{\frac{N+2}{N-2}} & \text{in } \Omega. \\
u = +\infty & \text{on } \partial \Omega.
\end{cases}
\]

A solution to the above equation is called a **BBUS** or **large solution**.

**Remark**

*If $\partial \Omega$ is smooth, there is no solution having positive scalar curvature.*
The Poincaré metric

Let $\Omega = B$ denote the unit ball of $\mathbb{R}^N$ and try to solve the singular Yamabe problem for constant negative curvature $R_{g'}$. Then, if $N = 2$, the problem becomes

$$\begin{cases}
\Delta u = 4e^{2u} & \text{in } B. \\
u = +\infty & \text{on } \partial B,
\end{cases}$$

or, if $N \geq 3$,

$$\begin{cases}
\Delta u = N(N - 2)u^{\frac{N+2}{N-2}} & \text{in } B. \\
u = +\infty & \text{on } \partial B.
\end{cases}$$

Let $R$ be the hyperbolic radius, defined by

$$R := \rho^{-1/2} = \begin{cases}
e^{-u} & \text{if } N = 2 \\
\rho^{-\frac{2}{N-2}} & \text{if } N \geq 3
\end{cases}$$

Then, there exists a unique BBUS explicitly given by

$$R = 1 - |x|^2.$$

$(B, g')$ is the Poincaré disk.

See http://www.poincare.fr for more.
Let \((B_t)_{t \geq 0}\) be Brownian motion in \(\mathbb{R}^N\), i.e.

- \(B_t\) takes values in \(\mathbb{R}^N\). Reducing to the case \(N = 1\),
- \(t \mapsto B_t\) is almost surely continuous
- \(B_t\) has independent increments i.e. for \(0 \leq s < t\),

\[
(B_t - B_s) \perp \| (B_r)_{0 \leq r \leq s}
\]

- \(B_t\) has stationary increments

\[
(B_t - B_s) \overset{\mathcal{L}}{=} B_{t-s} - B_0
\]

\(\sim B_t \sim \mathcal{N}(rt, \sigma^2 t)\). Normalization: \(r = 0, \sigma = 1\).
Exit time and the Dirichlet problem

Let $\Omega$ be a bounded open set of $\mathbb{R}^N$. Consider the exit time of Brownian motion.

$$\tau_\Omega = \inf\{t > 0 : B_t \in \Omega^c\}$$

**Problem** if $B_{t=0} = x \in \partial \Omega$, is it true that $\tau_\Omega = 0$ a.s. ?

Take $g \in C(\partial \Omega)$ and for $x \in \overline{\Omega}$, set

$$u(x) = \mathbb{E}^x(g(B_{\tau_\Omega})).$$

Then,

$$\Delta u = 0, \quad \text{in } \Omega$$

and $\tau_\Omega = 0$ iff for all such $g$,

$$u(x) = g(x), \quad \text{on } \partial \Omega.$$

**Theorem (Wiener criterion)**

$\Omega$ is Dirichlet regular iff

$$\int_0^1 \frac{\text{Cap}(B(x, r) \cap \Omega^c)}{r^{N-2}} \frac{dr}{r} = +\infty, \quad \text{for all } x \in \partial \Omega.$$
Fix $x \in \mathbb{R}^N$ and set

$$\mathcal{W}_x = \{ \text{finite path started at } x \}$$

$$= \{ w \in C([0, \zeta_w], \mathbb{R}^N) : w(0) = x \}.$$

$\zeta_w$ is the **lifetime** of $w$, and $\hat{w} = w(\zeta_w)$ its **tip**.

Heuristically, the **Brownian snake** satisfies

- $W_t$ takes values in $\mathcal{W}_x$
- the lifetime $\zeta_t = \zeta_{W_t}$ is b.m. (reflected at 0)
- when $\zeta_t$ decreases, the path $W_t$ is decreased from its tip
- When $\zeta_t$ increases, the path $W_t$ is extended by adding little pieces of $N$-dim. b.m. at its tip.
Brownian snake

\[ W_s(t) = W_{s'}(t), \quad \text{for } 0 \leq r \leq \min_{p \in [s, s']} \zeta_p =: m \]

\[ (W_{s'}(m + t) - W_{s'}(m))_{0 \leq t \leq \zeta_{s'} - m} \text{ is b.m. in } \mathbb{R}^N \perp W_s \]
Exit set

Given an open set $\Omega \subset \mathbb{R}^N$, and a finite path $w \in \mathcal{W}_x$, set

$$
\tau(w) = \inf\{t \geq 0 : w(t) \in \Omega^c \}.
$$

The set of exit points of b.s. is given by

$$
\mathcal{E}_\Omega = \{W_t(\tau(W_t)) : t \geq 0, \tau(W_t) < \infty \}.
$$

There exists a ($\sigma$–finite) measure (the excursion measure of b.s.) s.t. $u(x) = \mathbb{N}_x(\mathcal{E}_\Omega \neq \emptyset)$ solves

$$
\begin{cases}
\Delta u = 4u^2, & \text{in } \Omega, \\
u = +\infty & \text{on } \partial\Omega
\end{cases}
$$

whenever $\Omega$ is Dirichlet regular.
Assume $u$ solves a nonlinear PDE (no BC's) of the form

$$\Delta u = f(u) \quad \text{in } \Omega.$$ 

Fix a subdomain $\omega \subset \subset \Omega$. Want $C = C(\omega, \Omega, f, N)$ s.t.

$$|u| \leq C \quad \text{in } \omega,$$

More precisely, does there exist a function $g = g_{N,f}$ such that

$$|u(x)| \leq g(\text{dist}(x, \partial \Omega)) \quad \text{for all } x \in \Omega ?$$
The Lane-Emden equation

Fix $p > 1$, $\Omega$ an open set of $\mathbb{R}^N$, and consider

$$-\Delta u = |u|^{p-1}u \quad \text{in } \Omega.$$ 

A solution $u \in C^2(\Omega)$ can be seen as a **critical point of the energy** functional

$$E_\omega(u) = \frac{1}{2} \int_\omega |\nabla u|^2 \, dx - \frac{1}{p+1} \int_\omega |u|^{p+1} \, dx,$$

where $\omega \subset \subset \Omega$.

i.e. for $\varphi \in C^1_c(\Omega)$,

$$D E(u)(\varphi) = \int_\Omega (-\Delta u - |u|^{p-1}u)\varphi \, dx$$

Compute the **second variation of energy**:

$$D^2 E(u)(\varphi) = \int_\Omega \left( |\nabla \varphi|^2 - p|u|^{p-1}\varphi^2 \right) \, dx.$$
Definition

- A solution is said to be stable whenever $D^2 \mathcal{E}(u)(\varphi) \geq 0$ for all $\varphi \in C_c^1(\Omega)$.
- More generally, $u$ has finite Morse index $k \in \mathbb{N}$ if $k$ is the maximal dimension of any subspace $X \subset C_c^1(\Omega)$ s.t. $D^2 \mathcal{E}(u)(\varphi) < 0$ for all $\varphi \in X \setminus \{0\}$.

Remark

- If $\Omega$ is bounded and $u \in C^2(\overline{\Omega})$, $k$ is the number of negative eigenvalues of the linearized operator $-\Delta - p|u|^{p-1}$.
- So, every solution $u \in C^2(\Omega)$ has finite Morse index on $\omega \subset\subset \Omega$. 
Theorem (Davila-D-Farina)

Assume $1 \leq N \leq 10$ and take any $p > 1$, $p \neq \frac{N+2}{N-2}$. Let $u \in C^2(\Omega)$ be a solution of finite Morse index $k$. There exists a constant $C = C(N, p, k)$ s.t.

$$|u(x)| \leq C \text{dist}(x, \partial \Omega)^{-\frac{2}{p-1}}, \quad \text{for all } x \in \Omega.$$
if $N \geq 11$, the universal estimate holds iff $p < p_c(N)$. For $p \geq p_c(N)$, consider

$$
\begin{cases}
-\Delta u = \lambda (1 + u)^p & \text{in } B \\
u = 0 & \text{on } \partial B.
\end{cases}
$$

Then ([Joseph-Lundgren]),
the universal estimate implies the Liouville property.

- if \( p = \frac{N+2}{N-2} \), the estimate fails. Take

\[
 u(x)^{-\frac{2}{N-2}} = c_N \left( \lambda + \frac{|x|^2}{\lambda} \right).
\]

Note: the stereographic projection \( \pi : S^N \to \mathbb{R}^N \cup \{\infty\} \) is conformal and \( g_{ij} = u^{\frac{4}{N-2}} \delta_{ij} \) is the pullback of the standard metric on \( S^N \).

- the universal estimate extends to weak solutions \( u \in H^1(\Omega) \cap L^p(\Omega) \). It fails for weak solutions belonging to \( L^p(\Omega) \). Counter-example:

\[
 u(x) = c_{N,p} |x|^{-\frac{2}{p-1}}.
\]
Theorem (Davila-D-Farina)

Let \( u \in H^1(\Omega) \cap L^p(\Omega), \ u > 0, \) have finite Morse index \( k \) and take \( N \geq 11, \ p \geq p_c(N). \) Then, \( u \in C^\infty(\Omega \setminus \Sigma), \) where

\[ \mathcal{H}_{dim}(\Sigma) \leq N - 2 \frac{p + \gamma}{p - 1}, \]

where \( \gamma = 2p + 2\sqrt{p(p - 1)} - 1. \)

Remark

\( p = p_c(N) \) solves

\[ N - 2 \frac{p + \gamma}{p - 1} = 0. \]
If \( p \in \left( \frac{N+2}{N-2}, p_c(N) \right) \), ([Joseph-Lundgren]),
\[
\|u\|_{L^\infty(B)} \leq \lambda \lambda^* 2(N - 2) \quad \frac{1}{1} \leq N \leq \frac{2}{3} \leq N \leq \frac{9}{10}
\]

If \( 1 < p < \frac{N+2}{N-2} \) and \( u > 0 \), then all solutions have bounded Morse index ([Gidas-Spruck, Serrin-Zhou]). Fails again for sign-changing solutions ([Rabinowitz, Ambrosetti-Rabinowitz]).
Assume $f \geq 0$ and consider

$$\Delta u = f(u) \quad \text{in } \Omega$$

**Theorem (Keller, Osserman)**

*Let $\Omega$ open set of $\mathbb{R}^N$, $N \geq 1$. Assume $f \in C(\Omega)$, $f \geq 0$, $f$ increasing.*

- if $f > 0$, there exists $g = g_{N,f}$ s.t.
  $$u(x) \leq g(\text{dist}(x, \partial \Omega)) \quad \text{for all } x \in \Omega, u$$

  iff

  $$(KO) \quad \int_{-\infty}^{+\infty} \frac{dt}{\sqrt{F(t)}} < \infty, \quad \text{where } F' = f.$$  

- if $f(0) = 0$, there exists $\tilde{g} = \tilde{g}_{N,f}$ s.t.
  $$0 \leq u(x) \leq \tilde{g}(\text{dist}(x, \partial \Omega)) \quad \text{for all } x \in \Omega, u \geq 0$$

  iff (KO) holds.
Discussion: Liouville, regularity of $f$, BBUS

- **Liouville property:**
  \[
  \lim_{d \to 0} g = +\infty \quad \lim_{d \to 0} \tilde{g} = +\infty \\
  \lim_{d \to +\infty} g = -\infty \quad \lim_{d \to +\infty} \tilde{g} = 0
  \]

- **need not assume** $f$ continuous ([Olofsson])
- **existence of BBUS:** take $\Omega$ bounded, $f(0) = 0$.
  If $\Omega$ is Dirichlet regular, we can easily solve
  \[
  \begin{cases}
  \Delta u_k = f(u_k) & \text{in } \Omega, \\
  u_k = k & \text{on } \partial \Omega.
  \end{cases}
  \]
  Then $u_k \rightarrow U$, where $U$ is a BBUS.

**Theorem (Dhersin-Legal)**

Take $f(u) = u^2$. \(\exists \text{BBUS} \geq 0 \iff N \leq 3\) or

\[
N \geq 4 \text{ and } \int_0^1 \frac{C_{2,2}(\Omega^c \cap B(x, r))}{r^{N-2}} \frac{dr}{r} = +\infty, \quad \text{for all } x \in \partial \Omega
\]
**Discussion: f not increasing, Liouville again**

- The previous result can be extended to \( f(u) = u^p \), \( p > 1 \) ([Labutin]), with critical exponent \( p = \frac{N}{N-2} \) and capacity \( C_{2,p'} \).

- If \( f \) not increasing, the universal estimate fails. Simply take \( f(u) = u^2 \sin^2(u) \). However,

**Theorem (Dumont-D-Goubet-Radulescu)**

*If \( f \geq 0, \text{loc. Lip., } f(0) = 0, \Omega \text{ Dirichlet regular bounded} \)

\[
\exists \text{BBUS} \geq 0 \iff (KO).
\]

**Theorem (D)**

\[
\text{Liouville property } \iff (KO).
\]

\( \sim \text{Universal Harnack inequality} \)?
Conjectures on BBUS

- ([McKenna]): if $f$ increasing, then the BBUS is unique
- ([Brezis]): if $f$ loc. Lip., $\Omega = B$, then every BBUS is radial
- ([Lazer-McKenna]): characterize all $f$'s for which universal blow-up occurs i.e. any BBUS satisfies

\[ u(x) = g(\text{dist}(x, \partial \Omega)) + o(1), \]

where

\[
\begin{cases}
  g'' = f(g) & \text{in } (0, 1) \\
  g(0^+) = +\infty
\end{cases}
\]

Theorem (Costin-D)
Assume $f$ loc. Lip., $\Omega = B$. Let $U_1, U_2$ be two BBUS. Then,

\[ |U_1 - U_2| \leq C \int_{U_1}^{+\infty} \frac{dt}{F(t)}. \]

If $\exists K$ s.t. $t \mapsto f(t) + Kt$ increasing, then

\[ |\nabla(U_1 - U_2)| \leq C. \]
Uniqueness Let $u_1, u_2$ denote two solutions. It suffices to prove that $u_1 \leq u_2$. If not, let $\omega = \{x \in B : u_1(x) > u_2(x)\}$. Then,

$$\begin{cases}
\Delta(u_2 - u_1) = f(u_2) - f(u_1) \leq 0 & \text{in } \omega, \\
u_2 - u_1 = 0 & \text{on } \partial \omega.
\end{cases}$$

By the maximum principle, $u_2 - u_1 \geq 0$ in $\omega$, a contradiction.

Symmetry The gradient estimate implies $\partial_r u >> \partial_\theta u$. Then, apply the moving plane method ([Porretta-Véron]).

Universal blow-up Set $g = \sqrt{F}$. Then, universal blow-up always holds when

$$\lim_{u \to +\infty} g(u) \int_u^{+\infty} \frac{G(t)}{g(t)^3} \, dt = 0$$

and always fails when $\lim \inf > 0$. 

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Universal estimates
To solve an ODE (or a PDE or a difference equation), say

$$y' - y = 0$$

one can look for a solution in the form of a power series

$$\tilde{y} = \sum a_k t^k.$$ 

That is, provided the solution is analytic, we replace an analytic problem (solving the ODE) with an algebraic problem that can be solved algorithmically. Here:

$$ka_k - a_{k-1} = 0.$$ 

In so doing, we use fundamentally the isomorphism between convergent power series and analytic functions $y = \tilde{y}$.

**Problem**

*For most equations, formal asymptotic expansions thus obtained are (factorially) divergent. How can we make sense of such expressions?*
Euler’s example

What is the meaning for large $t$ of

$$\tilde{y} = \sum_{k=0}^{+\infty} \frac{k!}{(-t)^{k+1}}$$

Euler argues for the preservation of the usual properties of summation: formally, $\tilde{y}$ solves the ODE $\tilde{y}' - \tilde{y} = 1/t$. Thus, denoting $y$ the sum of $\tilde{y}$, we should have

$$y' - y = \frac{1}{t}$$

- all solutions of this ODE are given by $y = e^t Ei(-t) + Ce^t$, where $Ei(x) = \int_{-\infty}^{x} s^{-1} e^s \, ds$.
- formally, $\tilde{y} \to 0$ as $t \to +\infty$. The only solution $y$ having this property is $y = e^t Ei(-t)$. Thus, $e^t Ei(-t)$ is the sum of $\tilde{y}$. 

Borel summation: heuristics

- In practice, divergent series have coefficients $a_k = O(k^n)$.
- A change of variable allows to choose $n = 1$. E.g. $k!^2 t^{-k} \sim (2k)! (\sqrt{t})^{-2k} = j!/y^j$.
- Laplace transform $\mathcal{L} Y(t) = \int_0^{+\infty} e^{-pt} Y(p) \, dp$. Note that
  $$\mathcal{L}^{-1} \frac{k!}{t^{k+1}} = p^k$$
  Thus formally,
  $$\mathcal{L}^{-1} \sum_{k=0}^{+\infty} \frac{k!}{(-t)^{k+1}} = \sum_{k=0}^{+\infty} (-p)^k = \frac{1}{1+p}.$$
  Still formally, $\mathcal{L} \mathcal{L}^{-1} = Id$, so
  $$\sum_{k=0}^{+\infty} \frac{k!}{(-t)^{k+1}} = \mathcal{L} \mathcal{L}^{-1} \sum_{k=0}^{+\infty} \frac{k!}{(-t)^{k+1}} = \mathcal{L} \frac{1}{1+p} = \int_0^{+\infty} \frac{e^{-pt}}{1+p} \, dp = e^t Ei(-t).$$
The Borel transform $B$ acts on formal power series:

$$\tilde{y} = \sum \frac{y_k}{t^{k+1}} \rightarrow B\tilde{y} = \sum \frac{y_k}{k!} p^k$$

The Borel summation $LB$ is given by

$$y = LB\tilde{y} = \int_{0}^{+\infty} e^{-pt} B\tilde{y}(p) \, dp$$

This requires that

- $B\tilde{y}$ has nonzero radius of convergence (true after our change of variables)
- $B\tilde{y}$ continues analytically in $p \in \mathbb{R}^+$: serious restriction. Take e.g. $\tilde{y} = \sum \frac{k!}{t^{k+1}}$. Then, $B\tilde{y} = \frac{1}{1-p}$.
- $B\tilde{y}$ is exponentially bounded on $\mathbb{R}^+$. 

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Universal estimates
Borel summation is: \( g = \mathcal{L}(AC)\mathcal{B}\tilde{g} \)

**Generalized Borel summation.**

(1) AC is replaced in Écalle’s theory by **medianization**: averages with universal weights\(^1\) of analytic continuations above and below \( \mathbb{R}^+ \). Medianization is very widely applicable, unlike mere analytic continuation.


**Generalized Borel summation:**

Borel transform \( \circ \) Medianization \( \circ \) Acceleration \( \circ \) Laplace transform.

\(^1\)Independent of series.
Let $d(x) = 1 - r$, $p > 1$ and $k_0 = [2/(p - 1)] + 1$. Consider

$$\Delta u = u^p.$$ 

Then, the blow-up solution satisfies

$$u = d^{-\frac{2}{p-1}} \sum_{k=0}^{k_0} a_k d^k + o(1)$$

Now, for

$$\Delta u = u(\ln u)^4,$$

get

$$u = e^{\frac{1}{d}} \sum_{k=0}^{+\infty} a_k d^k + \exp. small$$
The general case: starting point

Let \( v = du/dr \) and multiply the equation by \( v \). We get

\[
\frac{d}{dr} \left( \frac{v^2}{2} \right) + \frac{N - 1}{r} v^2 = \frac{d}{dr} (F(u)).
\]

We define the resulting error term by

\[
g := -\frac{v^2}{2} + F(u),
\]

which satisfies the D.E. \( \frac{dg}{dr} = \frac{N - 1}{r} v^2 \) Change independent variable \( u = u(r) \).

Thinking of \( g \) as a function of the variable \( u \), we have

\[
\frac{dg}{du} = \frac{N - 1}{r} v.
\]

Solving for \( v \), we finally obtain

\[
\begin{cases}
\frac{dg}{du} = \frac{N - 1}{r} v = \frac{N - 1}{r} \sqrt{2(F(u) - g)}, \\
\frac{dr}{du} = 1/v = 1/\sqrt{2(F(u) - g)}.
\end{cases}
\]