On Existence of Ball Small Sets With Aronszajn Null Complements in Banach Spaces

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Main results

Theorem

Let X be an infinite dimensional separable Banach space with modulus of convexity of power type $p \geq 2$. Then there exists a Borel set $A \subset X$ which is ball small and whose complement is Aronszajn null.

Corollary

There exist Borel ball small subsets of $A \subset \ell_1$ and $B \subset L_1$ whose complements are Aronszajn null.

Christensen's conjecture

Proposition (Duda, 2004)

Let X be a separable Banach space and $D \subset X$ be a Borel ball small set. Suppose that $X \setminus D$ is Aronszajn null. Then there exists a nonempty closed set A and a Borel set Q which is not Haar null such that the metric projection $P_A(x)$ is empty for each $x \in Q$.

Applying our main Theorem gives the following Corollary, which negatively resolves a conjecture due to J. P. R. Christensen (1973):

Corollary

Let X be an infinite-dimensional separable superreflexive space such that X has modulus of convexity of power type p for some $p \geq 2$. Then there exists a nonempty closed set A and a Borel set Q which is not Haar null such that $P_A(x) = \emptyset$ for all $x \in Q$.

Metric projection

Remark

A similar result holds for "farthest points" as well. See [Duda, 2004] for details.

Let X be a normed linear space, $A \subset X$, and $x \in X$.

Definition

We define the *metric projection* $P_A(x)$ as the set

$$\{y \in A : \|x - y\| = \mathrm{dist}(x, A)\}.$$

Modulus of convexity

Let X be a normed linear space. The *modulus of convexity* δ_X is defined for $\varepsilon \in (0, 2]$ as

$$\delta_X(\epsilon) = \inf\{1 - \|x + y\|/2 : x, y \in S_X, \|x - y\| \geq \varepsilon\}.$$

Definition

We say that δ_X is of *power type* $p \geq 2$ provided there exists $C > 0$ such that $\delta_X(\varepsilon) \geq C\varepsilon^p$ for $\varepsilon \in (0, 2]$.

Remark

Hanner's results imply that spaces L_p and ℓ_p (where $1 < p < \infty$) have modulus of convexity of power type $\max(2, p)$*.*

Ball small sets

Let X be a normed linear space, $A \subset X$ and $r > 0$.

Definition

We say that X is *r-ball porous* provided for each $x \in A$ and $\varepsilon > 0$ there exists $y \in X$ such that $\|x - y\| = r$ and $B(y, r - \varepsilon) \cap A = \emptyset$.

Definition

We say that $B \subset X$ is *ball small* provided $B = \bigcup_{i=1}^{\infty} A_n$ where A_n is r_n-ball porous for some $r_n > 0$.

Aronszajn and Haar null sets

Let X be a Banach space and A, B be Borel subsets of X.

Definition

We say that A is *Haar null* provided there exists a Borel probability measure μ on X such that $\mu(A + x) = 0$ for each $x \in X$.

Definition

We say that B is *Aronszajn null* provided that for each sequence $(x_i)_i$ whose closed linear span is X, there exist Borel sets $B_i \subset X$ such that $B = \bigcup_i B_i$ and intersection of each B_i with any line in the direction of x_i has one-dimensional Lebesgue measure 0.

Aronszajn null sets are Haar null, but not vice versa.

Porous sets with small complements

Definition

A set $E \subset X$ is called *porous* if there is $c > 0$ such that for every $x \in E$ and every $\delta > 0$ there is a $z \in X$ such that $0 < \|z - x\| < \delta$ and $E \cap B(z, c\|z - x\|) = \emptyset$.

A result related to the main theorem was proved earlier:

Theorem (Preiss, Tišer)

Every separable infinite-dimensional separable Banach space X may be decomposed into two sets U and V such that U is of linear measure zero on every line, and V is a countable union of closed porous sets.

Facts about null sets

We will need the following two facts about Aronszajn null sets:

Lemma (Matoušek, Matoušková)

Let X be a separable Banach space, $A \subset X$ Borel and $Y \subset X$ closed subspace of finite codimension. Let $n \in \mathbb{N}$ be such that the intersection of A with any n-dimensional affine subspace parallel to Y is of n-dimensional measure zero. Then A is Aronszajn null.

Lemma (Matoušková)

Let X, Y be separable infinite-dimensional Banach spaces, and $T : X \to Y$ a continuous linear surjective map. Let $A \subset Y$ be Aronszajn null. Then $T^{-1}(A)$ is Aronszajn null.

Ball small sets and quotient maps

Proposition

Let X be a separable Banach space and $Y \subset X$ be a subspace.

1. *Let $q : X \to X/Y$ be the canonical quotient map. Then for each $x \in X$ and $r > 0$ we have*

$$q^{-1}(B_{X/Y}(q(x), r)) = B_X(x, r) + Y.$$

   Thus, if $A \subset X/Y$ is a ball small subset of X/Y, then $q^{-1}(A)$ is a ball small subset of X.

2. *If X has modulus of convexity of power type p, then so does X/Y.*

Proof of the main theorem

Reduction to the case with a basis:

Choose $Y \subset X^*$ with a basis. Then $W = X/Y^*$ has basis and modulus of convexity of power type p. Let $D \subset W$ be an Aronszajn null set with a ball small complement. Then $q^{-1}(D)$ is the desired set (using Lemmas and Proposition from previous slides). Thus, we can assume that X has a basis.

Compatible measures on affine finite-dimensional subspaces

Let X be a Banach space. For every n-dimensional subspace $Y \subset X$ fix an isomorphism $M_Y : Y \to \mathbb{R}^n$ with $\|M_Y\| \leq 1$. Let λ_Y be the induced measure (by M_Y). For any affine subspace $W \subset X$ parallel to Y, define $\lambda_W(C) := \lambda_Y(C - o_W)$ for any Borel $C \subset W$ where $o_W \in W$ is fixed.

Lemma

$A \subset W$ Borel. Then:

1. $\lambda_W(A) = \lambda_W(A + y)$ *for any* $y \in Y$,
2. $\lambda_W(A) = \lambda_Z(A - o_W + o_Z)$ *for any affine subspace Z parallel to W of same dimension,*
3. $\eta^n \lambda_W(A) = \lambda_{W_\eta}(\eta A)$ *for any* $\eta > 0$,
4. $\lambda_W(B(s,t) \cap W) \leq vt^n$ *for any* $s \in W$ *and* $t > 0$, *where* $v := \sup_n \mathrm{vol}_n(B_{\mathbb{R}^n}(0,1)) < \infty$.

A geometric lemma

Lemma

Suppose that a Banach space X has modulus of convexity of power type p. Then for each $N \in \mathbb{N}$ there exists $\beta = \beta(p, N, X) > 0$ such that whenever Z is an N-dimensional affine subspace of X, and $x \in X$ satisfies

$$\mathrm{dist}(x, Z) \geq 1 - \rho$$

for some $\rho \in (0, 1)$, then

$$\lambda_Z(B(x, 1) \cap Z) \leq \beta \rho^{N/p}.$$

Test cubes

Definition

An N-dimensional *test cube* U will be any set of the form

$$U = \left\{ x + \sum_{i=1}^{N} \alpha_i u_i : \alpha_i \in [0,1] \right\},$$

where $u_i \in \overline{B}_X(0,1)$, and u_i's are linearly independent.

Proposition

Let X be an infinite dimensional separable Banach space with a basis and with modulus of convexity of power type p. Then there exists an $N \in \mathbb{N}$ such that for each $\varepsilon > 0$ we can find $r > 0$ and a countable $C \subset X$, such that for each $x \in X$ and $\psi, \xi > 0$ there exist infinitely many $c_n, \tilde{c}_n \in C$ with

1. $\|\tilde{c}_n - (2x - c_n)\| < \xi$,
2. $\|c_n - c_m\| \geq r - \psi$, $\|\tilde{c}_n - c_m\| \geq r - \psi$ *for* $n \neq m$,
3. $\|\tilde{c}_n - c_n\| = 2r$,
4. $\|x - c_n\| \leq r + \xi$, $\|x - \tilde{c}_n\| \leq r + \xi$ *(and thus* $B(C, r + \delta) = X$ *for all* $\delta > 0$*)*,
5. $\lambda_{\mathrm{aff}(U)}(U \cap B(C, r)) < \varepsilon$ *for any N-dimensional test cube U.*

Proof of the proposition

Let $(e_k)_k$ be the basis of X and let $(f_k)_k$ be the dual basis with $\|f_k\| = 1$. Let $(x_k)_k$ be a dense finitely-supported (relative to $(e_k)_k$) sequence in X with each point repeated infinitely many times. Find $n_k \in \mathbb{N}$ and $s_k \in S_X$ so that

- $\max\{\max(\operatorname{supp} x_k), n_{k-1}\} < n_k$,
- $\|s_k\| = \langle f_{n_k}, s_k \rangle = 1$,
- $|f_{n_k}(s_l)| \leq 1/k$ for $l < k$.

Now define $c_k = x_k + s_k$ and $\tilde{c}_k = x_k - s_k$. Let $C_1 = \{c_k\}$, $\tilde{C}_1 = \{\tilde{c}_k\}$ and $C = C_1 \cup \tilde{C}_1$.

Choice of N

Theorem (Gurarii-Gurarii)

Let E be a superreflexive space. Then there are $1 < t < q < \infty$ and $\gamma > 0$ such that every normalized basic sequence $\{x_n\}$ satisfies

$$(*) \quad \gamma^{-1} \left(\sum |a_n|^q \right)^{1/q} \leq \left\| \sum a_n x_n \right\| \leq \gamma \left(\sum |a_n|^t \right)^{1/t}$$

for every choice of scalars $\{a_n\}$ for which $\sum a_n x_n$ converges.

Let

$$N > pq_1.$$

The basis (e_n) is not normalized but since $1/\gamma \leq \|e_n\| \leq \gamma$ (γ from previous theorem for (f_n)), $(*)$ also holds for (e_n) with t_1, q_1, γ_1.

Proof of proposition (continued)

It is easy to see that (1)–(4) hold for $r = 1$. We will prove that for large enough $K > 0$ we have

$$(**) \quad K^{-N}\lambda_{\mathrm{aff}(T)}(T \cap B(C,1)) < \varepsilon$$

for any T which is a K-times enlarged test cube U. (Once we have this, the proposition follows by taking $r = 1/K$.)
To see that $(**)$ holds, let

$$T = KU = \left\{ Kx + \sum_{i=1}^{N} \alpha_i u_i : \alpha_i \in [0,K] \right\}$$

be our enlarged test cube. Then (take $W = \mathrm{aff}(T)$)

$$K^{-N}\lambda_W(T \cap B(C,1)) = \lambda_V(U \cap B(C/K, 1/K)).$$

Proof of the proposition (continued)

Define

$$I_j = \{k \in \mathbb{N} : 1 - 2^{-j} \leq \mathrm{dist}(T, c_k) < 1 - 2^{-j-1}\}.$$

We have

$$\lambda_{\mathrm{aff}(T)}(T \cap B(c_k, 1)) \leq \beta 2^{-Nj/p}.$$

Thus

$$K^{-N}\lambda_{\mathrm{aff}(T)}(B(C_1, 1) \cap T) \leq \frac{\beta}{K^N}\sum_j 2^{-Nj/p}|I_j|.$$

Take $\eta := 1/(NK2^{j+2})$ and define

$$I'_j = \{k \in I_j : |\langle u_i, f_{n_k}\rangle| < \eta \text{ for all } i = 1, \ldots, N\}.$$

By Gurarii-Gurarii theorem, we have

$$|I_j \setminus I'_j| \leq N\gamma^q\eta^{-q} = \gamma^q N^{q+1}K^q 2^{q(j+2)}.$$

Proof of the proposition (continued)

To bound $|I_j'|$ let $k \in I_j'$. We obtain that $\langle f_{n_k}, x\rangle \geq 2^{-j-2}$ and thus

$$\gamma^{-1}\left(\sum_{i=n_k}^{\infty} |\langle f_i, x\rangle|^q\right)^{1/q} < 4NK.$$

Let $l \in \mathbb{N}$ be the first so that $(\sum_{i=l}^{\infty} |\langle f_i, x\rangle|^q)^{1/q} \leq 4\gamma NK$ and then

$$(4\gamma NK)^q \geq (|I_j'| - 1)2^{-q(j+2)}.$$

The final estimate is

$$K^{-N}\lambda_{\mathrm{aff}(T)}(B(C_1, 1) \cap T) \leq \frac{C(N, p, q, X)}{K^{N-q}}.$$

A similar argument works for $\tilde{C}_1$ (choosing $-f_{n_k}$). Thus for large K,

$$K^{-N}\lambda_{\mathrm{aff}(T)}(B(C, 1) \cap T) < \varepsilon.$$

Proof of the main theorem (continued)

For $m \in \mathbb{N}$ apply previous proposition with $\varepsilon = 1/m$ obtaining $r_m > 0$ and sets C_m. Define

$$E := \bigcap_m B(C_m, r_m).$$

For any N-dimensional test cube U we have that $\lambda_{\mathrm{aff}(U)}(U \cap B(C_m, r_m)) \leq 1/m$ and thus E intersected with any affine N-dimensional subspace of X has measure 0. Thus E is Aronszajn null by an earlier lemma.

Now

$$A = X \setminus E = \bigcup_m (X \setminus B(C_m, r_m))$$

is ball small since previous proposition implies that $X \setminus B(C_m, r_m)$ is r_m-ball porous.

References

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