A Novel Perturbation Method For Analyzing Singular PDEs

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Background

Nonlinear wave equation

The energy critical focusing wave equation in $\mathbb{R}^3$

$$\partial_{tt}u - \triangle u - u^5 = 0$$

has been the subject of intense investigations in recent years. This equation is known to be locally well-posed, and the solutions have finite energy. Duyckaerts, Kenig, and Merle established a classification of possible blow up dynamics, but the existence of all allowed scenarios calls for further investigation.

Krieger, Schlag, and Tataru (2009) showed existence of blow-up solutions (at $t = 0$) of the form

$$u(t,x) = \lambda(t)^{\frac{1}{2}} W(\lambda(t) r) + \varepsilon(t,x), \quad \lambda(t) \to \infty \text{ as } t \to 0+$$

where $W(r) = (1 + r^2/3)^{-\frac{1}{2}}$ is a special stationary solution, $\lambda(t) = t^{-1-\nu}$, $\nu > 1/2$, and $\varepsilon(t,r)$ is small in a suitable sense. This was accomplished in two steps:

I. Construction of arbitrarily good approximate radial solutions

II. Existence of an exact solution shown by analyzing the spectrum of the Schrödinger operator $-\triangle - 5W^4$

Improvement

Step II above can be accomplished by studying fixed point problem on a Banach space of the form

$$x = F(x) + Ax + x_0$$

where the norm of the linear operator A is not small but A^n has small operator norm provided n is sufficiently large. This new technique, developed by Donninger and Krieger, makes it possible to show existence of an exact solution using a reasonably but not arbitrarily good approximate solution inside the light cone $r \leq t$.

Scenario

The technique of Donninger and Krieger allows us to seek radial solutions of the form

$$u(t,r) = \lambda(t)^{\frac{1}{2}} W(\lambda(t)r) + \varepsilon(t,r), \quad \lambda(t) \to \infty \text{ as } t \to 0+ \tag{1}$$

where $\lambda(t) = t^{-1-\nu(t)}$ and $\nu(t)$ is not a constant. It turns out that one only needs to construct an approximate solution with two steps of corrections, i.e. $u = u_0 + v_1 + v_2$ where

$$u_0(t,r) = \lambda(t)^{\frac{1}{2}} W(r\lambda(t)) = \lambda(t)^{\frac{1}{2}} W(R) \tag{2}$$

$u_0(t,r) = \lambda(t)^{\frac{1}{2}} W(r\lambda(t)) = \lambda(t)^{\frac{1}{2}} W(R)$

Since the radial quintic wave equation is

$$\mathcal{L}_{quintic} u := u_{tt} - u_{rr} - \frac{2}{r} u_r - u^5 = 0 \tag{3}$$

It is straightforward to show that the leading error term $e_0 := \mathcal{L}_{quintic} u_0$ blows up as $t \to 0$ like t^{-2}.
To reduce the error, we plug $u_0 + v_1$ into (3) and discard ∂_t to obtain the linearized operator

$$L_0 := \partial_R^2 + \frac{2}{R} \partial_R + 5W^4(R) \tag{4}$$

and set

$$\lambda^2(t) L_0 v_1 = e_0$$

$L_0 = \partial_R^2 + \frac{2}{R}\partial_R + 5W^4(R)$

The ODE $\lambda^2(t)L_0 v_1 = e_0$ can be solved explicitly and v_1 decays like $(t\lambda(t))^{-2}$. We set $u_1 = u_0 + v_1$ and $e_1 := \mathcal{L}_{quintic}(u_1)$.
It can be shown that the leading behavior of e_1 is

$$t^2\lambda^{-\frac{1}{2}}(t)e_1^0(t,r) := c_1(t)a\mu^{-1}(t) + c_2(t)\mu^{-2}(t) \tag{5}$$

where $a = r/t$, $\mu(t) = t\lambda(t)$, and $c_{1,2}(t)$ are called admissible, meaning they are bounded together with all powers of $t\partial_t$ as $t \to 0+$.
The error is small at the regime $0 < r \ll t$ but not if r is comparable to t. Thus we need to introduce another correction v_2.

v_2

We construct v_2 from the PDE

$$t^2\left(v_{tt} - v_{rr} - \frac{2}{r}v_r\right) = -t^2 e_1^0 \tag{6}$$

Note that the nonlinearity on the left-hand side is of a higher order and thus discarded.
We seek a solution in the form

$$v_2(t,r) = \lambda(t)^{\frac{1}{2}}\left(\mu^{-1}(t)q_1(a,t) + \mu^{-2}(t)q_2(a,t)\right) \tag{7}$$

with boundary conditions $q_1(0,t) = 0, q_1'(0,t) = 0$ and $q_2(0,t) = 0$, $q_2'(0,t) = 0$. These translate into the boundary conditions $v(t,0) = 0$, $\partial_r v(t,0) = 0$ at $r = 0$.

$v_2(t,r) = \lambda(t)^{\frac{1}{2}}\big(\mu^{-1}(t)q_1(a,t) + \mu^{-2}(t)q_2(a,t)\big)$

The functions $q_{1,2}$ satisfy the PDEs

$$t^2\Big(-\big(\partial_t + \frac{\beta_1(t)}{t}\big)^2 + \partial_{rr} + \frac{2}{r}\partial_r\Big)q_1(a,t) = c_1(t)a \tag{8}$$

$$t^2\Big(-\big(\partial_t + \frac{\beta_2(t)}{t}\big)^2 + \partial_{rr} + \frac{2}{r}\partial_r\Big)q_2(a,t) = c_2(t) \tag{9}$$

where

$$\beta_j(t) = -(j-1/2)\frac{t\dot{\mu}}{\mu} - \frac{1}{2}, \quad j = 1,2.$$

$t^2\Big(-\big(\partial_t+\frac{\beta_2(t)}{t}\big)^2+\partial_{rr}+\frac{2}{r}\partial_r\Big)q_2(a,t)=c_2(t)$

The key is to show regularity of $q_{1,2}$, i.e.

$$\partial_t^k\partial_a^l q_2(a,t)=O(t^{-k}a^{2-l});\quad \partial_t^k\partial_a^l q_1(a,t)=O(t^{-k}a^{3-l}) \quad (10)$$

from which one could estimate $u_2:=u_0+v_1+v_2$ and $e_2:=\mathcal{L}_{quintic}(u_2)$, which results in

$$t^2\lambda^{-\frac{1}{2}}(t)e_2(t,r)=\mu^{-2}(t)O\left(\frac{\log(R+2)}{R+1}\right),0<r<t \quad (11)$$

This error is sufficiently small to prove existence of an exact solution.

Analysis of $q_{1,2}$

Setting

For convenience we focus on the analysis for $q = q_2$ (q_1 being similar), where $t^2\Big(-\big(\partial_t + \frac{\beta(t)}{t}\big)^2 + \partial_{rr} + \frac{2}{r}\partial_r\Big)q(a,t) = c(t)$ where

$$\beta(t) = -\frac{3t\dot{\mu}}{2\mu} - \frac{1}{2}, \quad j = 1, 2.$$

and $c(t) = c_2(t)$ which comes from the leading order

$$t^2\lambda^{-\frac{1}{2}}(t)e_1(t,r) \sim c_1(t)a\mu^{-1}(t) + c_2(t)\mu^{-2}(t)$$

is a finite sum of $(t\partial_t)^\ell$ applied to $const.\omega_{1,2}(t)$ given below.

$\omega_{1,2}(t)$

Note that

$$\omega_1(t) = t^2\left(\frac{\lambda'}{\lambda}\right)'(t) = (t\partial_t - 1)\left(t\frac{\lambda'}{\lambda}\right)(t)$$

$$\omega_2(t) = \left(t\frac{\lambda'}{\lambda}\right)^2(t)$$

where

$$\left(t\frac{\lambda'}{\lambda}\right)(t) = 1 + \kappa(t) := 1 - \frac{t\dot{\mu}(t)}{\mu(t)}$$

Thus $c(t)$ is a finite sum of $(t\partial_t)^\ell$ applied to a polynomial of $\kappa(t)$, and $\beta(t) = \frac{3}{2}\kappa(t) - \frac{1}{2}$.

$\kappa(t)$

The equation $t^2\Big(-\big(\partial_t+\frac{\beta(t)}{t}\big)^2+\partial_{rr}+\frac{2}{r}\partial_r\Big)q(a,t)=c(t)$ can be rewritten as

$$\begin{aligned}\big((1-a^2)\partial_a^2+(2(\beta(t)-1)a+2a^{-1})\partial_a\\ -\beta^2(t)+\beta(t)-t\dot\beta(t)\big)q(a,t)\\ -(t^2\ddot q(a,t)+2\beta(t)t\dot q(a,t))+2at\partial_a\dot q(a,t)=c(t)\end{aligned}\tag{12}$$

which is obviously singular near $a=1$.
If $\kappa(t)$ is a constant, it has a solution independent of t and it becomes an ODE, which was studied in Krieger, Schlag, and Tataru (2009). However, if $\kappa(t)$ depends on t, one needs to study the PDE and show that there is a regular solution satisfying the boundary conditions $q_2(0,t)=0$, $q_2'(0,t)=0$.

Toy Model

We first study the toy model

$$\Big((1-a)\partial_a + t\partial_t + \kappa(t)\Big)u(a,t) = 0$$

which can be solved explicitly.
If $\kappa(t) = \nu + \varepsilon t^\gamma$ then the solution is

$$u(a,t) = (1-a)^\nu e^{\frac{\varepsilon\left((1-a)^\gamma - 1\right)t^\gamma}{\gamma}}$$

This suggests that a perturbation expansion in powers of εt^γ is likely to work. In this case the blowup profile $\lambda(t)$ is monotonic.

Oscillations

What's more interesting is the possibility of oscillating behaviors. We consider

$$\lambda(t) = t^{-1-\nu} \exp(-\varepsilon_0 \sin(\log t)), \quad \nu > 3, \ |\varepsilon_0| \ll 1 \quad (13)$$

This is of the form $\lambda(t) = t^{-1-\nu(t)}$ with

$$\nu(t) = \nu + \varepsilon_0 \frac{\sin(\log t)}{\log t} \to \nu \ \text{ as } \ t \to 0+$$

This corresponds to

$$\kappa(t) = \nu + \varepsilon_0 \cos(\log t) = \nu + \frac{\varepsilon_0}{2}(t^i + t^{-i})$$

With this choice of $\lambda(t)$, in the equation

$$\begin{aligned}\big((1-a^2)\partial_a^2 + (2(\beta(t)-1)a + 2a^{-1})\partial_a \\ -\beta^2(t) + \beta(t) - t\dot{\beta}(t)\big)q(a,t) \\ -(t^2\ddot{q}(a,t) + 2\beta(t)t\dot{q}(a,t)) + 2at\partial_a\dot{q}(a,t) = c(t)\end{aligned} \tag{14}$$

we have

$$\beta(t) = \tilde{\nu}_j + \tilde{\epsilon}(t^i + t^{-i}))$$

$$c(t) = \sum_{n=0}^{N}\sum_{m=0}^{n} \tilde{\epsilon}^n \tilde{c}_{n,m}^{(j)} t^{(n-2m)i} \tag{15}$$

where $\tilde{\nu}_j > 1$ and $\tilde{\epsilon} = const.\varepsilon_0$.

Solution

We seek a solution to (14) of the form

$$q(a,t) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \tilde{\varepsilon}^n g_{n,m}(a)\, t^{(n-2m)i} \tag{16}$$

by plugging it into the equation. Collecting powers of $\tilde{\varepsilon}$ and t^i we obtain the ODE

$$\begin{aligned}
&(1-a^2)g''_{n,m}(a) + (2(\tilde{\nu}-1)a + 2a^{-1} + 2ai(n-2m))g'_{n,m}(a) \\
&+ (\tilde{\nu} - \tilde{\nu}^2 + (n-2m)(n-2m+i-2\tilde{\nu}i))g_{n,m}(a) \\
&= -2a(g'_{n-1,m}(a) + g'_{n-1,m-1}(a)) - (1+i-2\tilde{\nu}-2i(n-2m))g_{n} \\
&- (1-i-2\tilde{\nu}-2i(n-2m))g_{n-1,m-1}(a) + 2g_{n-2,m-1}(a) \\
&\qquad + g_{n-2,m}(a) + g_{n-2,m-2}(a) + \tilde{c}_{n,m}(a) =: R_{n,m}(a)
\end{aligned}$$

$g_{n,m}$ can be solved recursively as

$$g_{n,m}(a) = \frac{(1+a)^{\tilde{\nu}+1+(n-2m)i}}{2a(\tilde{\nu}+1+(n-2m)i)} \int_0^a x(1+x)^{-\tilde{\nu}-1-(n-2m)i} R_{n,m}(x)dx - \frac{(1-a)^{\tilde{\nu}+1+(n-2m)i}}{2a(\tilde{\nu}+1+(n-2m)i)} \int_0^a x(1-x)^{-\tilde{\nu}-1-(n-2m)i} R_{n,m}(x)dx \tag{17}$$

Recall that $R_{n,m}(x)$ depends on $g_{n-k,m-j}(x)$ for $k = 1, 2$ and $j = 0, 1, 2$.

To show that $q(a,t)$ given by the expansion is indeed a solution, it is sufficient to show that $g''_{n,m}(a)$ is continuous and for some $C_0 > 0$ we have

$$\|g^{(k)}_{n,m}\|_\infty := \sup_{a\in[0,1]} |g^{(k)}_{n,m}(a)| \leqslant C_0^n$$

Note that the boundary conditions $q(0,t) = \partial_a q(0,t) = 0$ are satisfied since $g_{n,m}(0) = g'_{n,m}(0) = 0$.
This is accomplished by estimating integrals using elementary calculations.

Since $g_{n,m}$ is represented as an integral of $R_{n,m}$, we have the estimates

$$\begin{aligned} |g_{n,m}(a)| &\leqslant \frac{\sqrt{2}(2^{\tilde{\nu}-1}+2^{-1})a\|R_{n,m}\|_\infty}{(|n-2m|+\tilde{\nu}+1)} \\ |(ag_{n,m}(a))'| &\leqslant \sqrt{2}(2^{\tilde{\nu}-2}+2^{-1})a^2\|R_{n,m}\|_\infty \\ |g'_{n,m}(a)| &\leqslant \sqrt{2}(2^{\tilde{\nu}-1}+2^{-1})(1+(\tilde{\nu}+1)^{-1})\|R_{n,m}\|_\infty \end{aligned} \tag{18}$$

Since $R_{n,m}(x)$ depends on $g_{n-k,m-j}(x)$ and $g'_{n-k,m-j}(x)$ for $k=1,2$ and $j=0,1,2$, we have

$$|R_{n,m}(x)| \leqslant (C_1+\hat{c}_1)^{n+1} \tag{19}$$

Therefore

$$\|g_{n,m}^{(k)}(a)\|_\infty \leqslant C_0^n, \ 0 \leqslant k \leqslant 2 \tag{20}$$

meaning the sum

$$q_2(a,t) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \tilde{\varepsilon}^n g_{n,m}(a)\, t^{(n-2m)i} \tag{21}$$

is convergent for small $\tilde{\varepsilon}$ and $\partial_t^k \partial_a^l q_2(a,t) = O(t^{-k} a^{2-l})$. Similarly $\partial_t^k \partial_a^l q_1(a,t) = O(t^{-k} a^{3-l})$.

Correction

Therefore the second correction

$$v_2(t,r) = \frac{\lambda(t)^{\frac{1}{2}}}{\mu^2(t)}(R\tilde{q}_1(a,t) + q_2(a,t)) \tag{22}$$

satisfies the desired estimate

$$\partial_t^k \partial_a^j v_2(t,at) = O\left(t^{-k}a^{-j}\frac{\lambda(t)^{\frac{1}{2}}a^2(1+R)}{\mu^2(t)}\right) \tag{23}$$

Approximation

Finally the approximate solution $u_2 = u_0 + v_1 + v_2$ satisfies

$$u_2(t,r) = \sqrt{\lambda(t)}(W(R) + \mu^{-2}(t)O(R)),\ 0 < r < t$$

$e_2 := \mathcal{L}_{quintic} u_2$ satisfies

$$t^2 \lambda^{-\frac{1}{2}}(t) e_2(t,r) = \mu^{-2}(t) O\left(\frac{\log(R+2)}{R+1}\right),\ 0 < r < t$$

and their derivatives also satisfy corresponding estimates. Finally the function $u_2(t,r)$ admits a C^2- extension beyond the light cone $r \leq t$ with finite energy, which is shown using standard smooth bump function arguments. Existence of an exact solution follows from the arguments of Donninger and Krieger.

Possible Extensions

Limitation

Recall that our setting is $\lambda(t) = t^{-1-\nu(t)}$ with

$$\nu(t) = \nu + \varepsilon_0 \frac{\sin(\log t)}{\log t} \to \nu \quad \text{as} \quad t \to 0+$$

One question of interest is whether the oscillations could remain visible, i.e. $\nu(t)$ does not approach ν.
One may want to consider $\nu(t) = \nu + \varepsilon_0 \sin(\log t)$ but the oscillation is too large. A better option would be

$$\nu(t) = \nu + \varepsilon_0(\sin(\log(-\log t)) + \cos(\log(-\log t)))$$

which corresponds to

$$\kappa(t) = \nu + \varepsilon_0((-\log t)^i + (-\log t)^{-i})$$

Difficulty

Unlike the previous choice of $t^{\pm i}$, which keeps its shape under the operator $t\partial_t$, we have

$$t\partial_t(-\log t)^{\pm i} = \mp i(-\log t)^{\pm i-1}$$

which complicates the analysis.
Besides, previously we had $((1-a)t)^{\pm i} = (1-a)^{\pm i}t^{\pm i}$ but the expression $\log^{\pm i}((1-a)t)$ cannot be simplified.
Although the existence of a regular solution may hold, and it may have a transseries expansion, the analysis is likely to be very complicated. Thus a better idea is needed.

THE END

Thank you very much!