5101 (AU 2017) **Homework 1** Name(s):

1. Show that the space of all solutions of the linear equation $u'' = u$ is a linear space. You may, or may not, use the fact that

$$\{u = \text{real valued function}\,|\,u'' = u\} = \{C_1 e^t + C_2 e^{-t}\,|\,C_{1,2} \in \mathbb{R}\}$$

2. a) Show that the set of continuous functions over the interval $[a, b]$:

$$C[a, b] = \{f : [a, b] \to \mathbb{R}\,|\,f \text{ continuous on } [a, b]\}$$

is a linear space over the scalar field $\mathbb{R}$.

b) Is it a linear space over the scaler field $\mathbb{C}$? Why?

c) Show that $C_0[a, b] := \{f \in C[a, b]\,|\,f(a) = f(b) = 0\}$ is a subspace of $C[a, b]$.

d) Is $\{f \in C[a, b]\,|\,\int_a^b f(x)\,dx = 0\}$ a subspace of $C[a, b]$? Justify.

3. a) Let X and Y be two vector spaces over F. Let

$$X \oplus Y = \{(\mathbf{x}, \mathbf{y})\,|\,\mathbf{x} \in X,\ \mathbf{y} \in Y\}$$

Show that $X \oplus Y$ is a vector space over F with addition and scalar multiplication defined component-wise, as

$$(\mathbf{x}_1, \mathbf{y}_1) + (\mathbf{x}_2, \mathbf{y}_2) = (\mathbf{x}_1 + \mathbf{x}_2, \mathbf{y}_1 + \mathbf{y}_2), \quad c(\mathbf{x}, \mathbf{y}) = (c\mathbf{x}, c\mathbf{y})$$

$X \oplus Y$ is called the (external) *direct sum* of the vector spaces X and Y.

b) Given $\{\mathbf{v}_i\}_{i=1,\ldots n}$ a basis for X, and $\{\mathbf{w}_j\}_{j=1,\ldots,k}$ a basis for Y, find a basis for $X \oplus Y$. Express the dimension of $X \oplus Y$ in terms of dim$X$ and dim$Y$.

4. In each of the following cases, establish whether or not the given set of vectors is linearly independent or linearly dependent in the given vector/linear space. Explain.

a) $1, \cos t, \cos 2t, \ldots, \cos nt$ in $C[0, 2\pi]$.

b) $p(t) = (t-1)(t-2)(t-3)$, $q(t) = t(t-2)(t-3)$, $r(t) = t(t-1)(t-3)$, $s(t) = t(t-1)(t-2)$ in $\mathcal{P}$.

c) $t^{\sqrt{2}}$, t^e, t^π in $C(0, \infty)$

d) $\cosh x$, $\cosh(x-1)$ in $C(\mathbb{R})$.

e) $(1, 1, 1, 1)$, $(0, 2, 1, -1)$, $(2, -4, -1, 5)$ in $\mathbb{R}^4$.

f) $(1, 1, 1, 1)$, $(0, 2, 1, -1)$, $(2, -1, 1, -1)$ in $\mathbb{R}^4$.