1. *a)* Find a unitary similarity transformation which brings the (lower triangular) matrix $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$ to upper triangular form.

b) Find a unitary similarity transformation which brings

$$B = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

into upper triangular form. *Hint:* can you use *a)*?

c) What are the eigenvalues of B?

2. Let N be a normal matrix, with eigenvalues/vectors $\lambda_k, \mathbf{v}_k,\ k = 1, \ldots, n$.

a) What are the eigenvalues/vectors of its adjoint N^*?

b) If U is a unitary matrix diagonalizing N, what unitary matrix will diagonalize N^*?

3. *a)* Let $\mathbf{u}$ be a unit vector in $\mathbb{C}^n$. Show that $\mathbf{u}\mathbf{u}^*$ is the matrix which represents the orthogonal projection along the direction of $\mathbf{u}$.

b) Let A be an $n \times n$ self-adjoint matrix and $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ its orthonormal eigenvectors. Show the **spectral decomposition** of A:

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^* + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^* + \ldots \lambda_n \mathbf{u}_n \mathbf{u}_n^*$$

The matrices $\mathbf{u}_1\mathbf{u}_1^*,\ \mathbf{u}_2\mathbf{u}_2^*, \ldots, \mathbf{u}_n\mathbf{u}_n^*$ are called **spectral projections.**

4. Let $a > 0$ and k be constants. Calculate

$$\int_{-\infty}^{\infty} e^{-ax^2+kx}\, dx$$

(*Hint:* complete the square and use the fact that $\int_{-\infty}^{\infty} e^{-y^2}\, dy = \sqrt{\pi}$.)

5. Integrals as in problem 4. are generalized to several dimensions as

$$\mathcal{J} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\langle \mathbf{x}, A\mathbf{x}\rangle + \langle \mathbf{k}, \mathbf{x}\rangle} dx_1 \cdots dx_n$$

where A is a positive definite matrix (to ensure that the integrals converge), and $\mathbf{k}$ is a constant vector.

You are asked to calculate $\mathcal{J}$ using the following steps.

I. Change coordinates in $\mathbb{R}^n$ to diagonalize the quadratic form, and show that in these new coordinates

$$\mathcal{J} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\langle \mathbf{y}, \Lambda \mathbf{y} \rangle + \langle B\mathbf{k}, \mathbf{y} \rangle} dy_1 \cdots dy_n$$

where B is a constant matrix, and Λ is a diagonal matrix, which you need to find (in terms of A).

(*Hint:* unitary transformations preserve lengths, angles, hence volumes and the element of volume: if $\mathbf{y} = U\mathbf{x}$ with U unitary then $dx_1 \cdots dx_n = dy_1 \cdots y_n$.)

II. Use an affine change of coordinates $\mathbf{y} = \mathbf{w} + \mathbf{b}$ with $\mathbf{b}$ a constant vector that you need to find so that you complete the squares in the exponential:

$$e^{-\langle \mathbf{y}, \Lambda \mathbf{y} \rangle + \langle B\mathbf{k}, \mathbf{y} \rangle} = e^{-\langle \mathbf{w}, \Lambda \mathbf{w} \rangle} e^{-\frac{1}{4}\langle E\mathbf{k}, \mathbf{k} \rangle}$$

Find the vector $\mathbf{b}$ and the matrix E (in terms of A).

III. Show that

$$\mathcal{J} = C\, \pi^{n/2} e^{-\frac{1}{4}\langle \mathbf{k}, E\mathbf{k} \rangle}$$

and express the constant C in terms of A.

IV. Check that for $n = 1$ your result agrees with the solution to problem 4.