5101 (AU 2017) **Homework 2** Name(s):

1. Quasi-triangular matrices: let

$$A = \begin{bmatrix} a_{11} & \dots & a_{1k} & 0 & \dots & 0 \\ \vdots & & \vdots & & & \\ a_{k1} & \dots & a_{kk} & 0 & \dots & 0 \\ a_{k+1,1} & \dots & a_{k+1,k} & a_{k+1,k+1} & \dots & a_{k+1,n} \\ \vdots & & & & & \\ a_{n1} & \dots & a_{nk} & a_{n,k+1} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} B & | & 0 \\ - & - & - \\ C & | & D \end{bmatrix}$$

where B is a $k \times k$ matrix and D is $(n-k) \times (n-k)$. Show the nice formula

$$\det A = \det B \, \det D$$

2. Let $\mathcal{B}_N$ = the set of all linear combinations of $e^{ikt}, k = -N, \dots, 0, \dots N$, with complex coefficients. (Such linear combinations form a set of *band limited* functions.)

It is easy to see that $\mathcal{B}_N$ is a linear space of functions over $\mathbb{C}$, and it can be shown that $e^{ikt}, k = -N, \dots, 0, \dots N$ are linearly independent. Assume these are true.

a) What is $\dim \mathcal{B}_N$?

b) Show that $\{1, \cos t, \dots, \cos Nt, \sin t, \dots, \sin Nt\}$ is a basis for $\mathcal{B}_N$.

3. *Chebyshev polynomials* $T_n(x)$ are defined by the recurrence relation

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \text{ for } n \geq 1$$

a) Find $T_2(x)$, $T_3(x)$, $T_4(x)$.

b) Show that $T_k(\cos t) = \cos kt$ for $k = 0, 1, 2, 3, 4$.

c) Show that $T_0(x), \dots, T_4(x)$ form a basis for $\mathcal{P}_4$.