5101 (AU 2017) **Homework 4** Name(s):

1. Let V be a vector space over F. A linear functional $B : V \oplus V \to F$ is called *bilinear* if it is linear in each argument, that is, if

$$B(c\mathbf{u}_1 + d\mathbf{u}_2, \mathbf{v}) = cB(\mathbf{u}_1, \mathbf{v}) + dB(\mathbf{u}_2, \mathbf{v}), \quad B(\mathbf{u}, c\mathbf{v}_1 + d\mathbf{v}_2) = cB(\mathbf{u}, \mathbf{v}_1) + dB(\mathbf{u}, \mathbf{v}_2)$$

for all scalars c, d and all vectors $\mathbf{u}_{1,2}, \mathbf{v}_{1,2} \in V$.

a) Find all the bilinear functionals $B : \mathbb{R}^2 \oplus \mathbb{R}^2 \to \mathbb{R}$.
(*Hint:* expand vectors in $\mathbb{R}^2$ in a basis.)

b) Consider a bilinear functional $B : \mathbb{R}^2 \oplus \mathbb{R}^2 \to \mathbb{R}$ which also satisfies: $B(\mathbf{u}, \mathbf{u}) = 0$ for all $\mathbf{u} \in \mathbb{C}^2$ and $B(\mathbf{e}_1, \mathbf{e}_2) = 1$. Show that $B(\mathbf{u}, \mathbf{v}) = \pm \det [\mathbf{u} \,|\, \mathbf{v}]$.

c) Let $\mathbf{u}, \mathbf{v}$ be two vectors in $\mathbb{R}^2$. When regarded geometrically as arrows staring at O, they determine a parallelogram. Show that *the area of this parallelogram equals* $\pm \det [\mathbf{u} \,|\, \mathbf{v}]$. (*Hint:* Show that this area is bilinear in $(\mathbf{u}, \mathbf{v})$ which also satisfies the conditions in b) above.)

d) Similarly, three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ determine a parallelepiped, whose volume equals $\pm \det[\mathbf{u} \,|\mathbf{v}\,|\mathbf{w}]$. How would you prove this similarly to the argument above in $\mathbb{R}^2$?

2. a) Let A be an $n \times n$ matrix. Show that $\det(cA) = c^n \det A$.

b) When is it true that $\det(-A) = \det(A)$?

3. Consider the transformation of $\mathbb{R}^3$ given by the matrix multiplication $\mathbf{x} \to M\mathbf{x}$ where

$$M = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

a) Find the range and the null space of this transformation, describing these subspaces by giving a basis and as geometrical objects in $\mathbb{R}^3$.
b) Find all the vectors $\mathbf{b} \in \mathbb{R}^3$ for which the system $M\mathbf{x} = \mathbf{b}$ is soluble.
c) Find the general solution to $M\mathbf{x} = \mathbf{0}$.

4. You showed that $\mathcal{M}_{2,2}(\mathbb{R})$, the set of 2×2 matrices with the real entries, is a linear space of all 2×2 of dimension 4.

a) Show that the trace

$$Tr : \mathcal{M}_{2,2}(\mathbb{R}) \to \mathbb{R}, \quad Tr(A) = A_{11} + A_{22}$$

is a linear functional and use this to show that the set of matrices of zero trace form a subspace in $\mathcal{M}_{2,2}(\mathbb{R})$.

b) What is the dimension of the subspace of matrices of zero trace?