Math 5101 HOMEWORK 9

1. Verify that *the polarization identity*

$$\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \, \| i^k \mathbf{x} + \mathbf{y} \|^2$$

holds in any inner product space $(V, \langle , \rangle)$ over the complex numbers $F = \mathbb{C}$.

2. Consider $\mathbb{R}^3$ equipped with the Euclidian inner product: $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y}$. Let $\mathbf{v} = (1, 2, 3) \in \mathbb{R}^3$.

Describe geometrically, and give algebraic formulas, for the set of all vectors $\mathbf{x} \in \mathbb{R}^3$ so that $\langle \mathbf{x}, \mathbf{v} \rangle = 0$.

3. Consider the $C[0, \pi]$, the space of real valued, continuous functions on the interval $[0, \pi]$ equipped with the inner product $\langle f, g \rangle = \displaystyle\int_0^\pi f(t) g(t) dt$.

a) Show that $\mathcal{S} = \{1, \cos t, \cos(2t), \dots, \cos(nt), \dots\}$ is an orthogonal set.

b) Suppose a function f has the form $f(t) = \displaystyle\sum_{k=0}^{n} c_k \cos(kt)$ where c_k are constants. Express each c_k in terms of the function f and functions in $\mathcal{S}$.

4. Consider the $C[-\pi, \pi]$, the space of complex valued, continuous function on the interval $[-\pi, \pi]$ equipped with the inner product $\langle f, g \rangle = \displaystyle\int_{-\pi}^{\pi} \overline{f(t)} g(t) dt$.

a) Show that $\mathcal{F} = \{e^{int} |\ n \in \mathbb{Z}\}$ is an orthogonal set.

b) Suppose a function f has the form $f(t) = \displaystyle\sum_{k=-N}^{N} c_k e^{ikt}$ where c_k are constants. Express each c_k in terms of the function f and functions in $\mathcal{F}$.

5. Consider the linear space of all real valued polynomials $\mathcal{P}$ equipped with the inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t) \, g(t) \, e^{-t^2} \, dt$$

The standard basis of $\mathcal{P}$ consists of all monomials $1, t, t^2, \dots, t^n \dots$.

Use a Gram-Schmidt process on $1, t, t^2$ to obtain a set of orthonormal polynomials with respect to this inner product.

Note: These polynomials $p_0, p_1, \dots, p_n \dots$ obtained by the Gram-Schmidt process are called *Hermite polynomials*. They are one family of orthogonal polynomials; other families are obtained using inner products with different weights.