Singular Value Decomposition Theorem

Let M be an $m \times n$ matrix. Then

$$M = U\Sigma V^*$$

where:

• U is a unitary matrix whose columns are eigenvectors of MM^*

• V is a unitary matrix whose columns are eigenvectors of M^*M

• Σ is an $m \times n$ diagonal matrix

More precisely:

∘ if $U = [\mathbf{u}_1, \ldots, \mathbf{u}_r, \mathbf{u}_{r+1}, \ldots, \mathbf{u}_m]$ and $V = [\mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}, \ldots, \mathbf{v}_n]$ then for $j = 1, \ldots r$ the vectors $\mathbf{u}_j$ and $\mathbf{v}_j$ correspond to the eigenvalue $\lambda_j \neq 0$ while all the others correspond to the eigenvalue 0.

∘ The diagonal matrix Σ has $\Sigma_{jj} = \sigma_j = \sqrt{\lambda_j}$ for $j = 1, \ldots, r$, and all other elements are 0.

∘ Also, $\mathbf{u}_j = \frac{1}{\sigma_j} M\mathbf{v}_j$ for $j = 1, \ldots, r$.