3.12. **Determinant of a product.** In this section we show that

$$\det(AB) = \det A \det B \tag{20}$$

and other cool facts.

The matrices E_{ij}, $G_j(t)$ (for $t \neq 0$) and $F_{ij}(\alpha)$ are called **elementary matrices**.

I. We will first remark that all the elementary matrices are invertible, and formula (20) holds when A is an elementary matrix. Indeed:

• E_{ij} is obtained from the identity matrix by exchanging rows i and j. (Indeed, note that $\mathbf{e}_i M$=the i'th row of M.) Also, E_{ij} is invertible (it is its own inverse) and $\det E_{ij} = -1$. Therefore formula (20) holds when $A = E_{ij}$.

• $G_j(t)$ is obtained from the identity matrix by replacing the 1 in the diagonal in position j by t. $G_j(t)$ is invertible (what is its inverse?) and, since $\det G_j(t) = t$, then formula (20) holds when $A = G_j(t)$.

• $F_{ij}(\alpha)$ is obtained from the identity matrix by adding to its i'th row α times its j'th row (it is a matrix having one on the diagonal, α in position (i, j) and zeroes everywhere else. It is invertible, with inverse $F_{ij}(-\alpha)$, and had determinant 1, so formula (20) holds when $A = F_{ij}(\alpha)$.

II. Note that as a consequence of the following Proposition and I, formula (20) holds when A and B are invertible.

Proposition 14. *Any invertible matrix is a product of elementary matrices.*

You may know that, using elementary transformation, any square matrix can be brought to upper triangular form. But if the matrix is invertible, then it can be brought to identity (because, after it is brought to an upper triangular form, no elements on the diagonal are zero). Instead of a rigorous proof, we illustrate its idea on an example:

Let $A = \begin{bmatrix} 2 & 6 \\ 3 & 4 \end{bmatrix}$. Note $G_1(1/2)A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} := A^{(1)}$

then

$$F_{21}(-3)A^{(1)} = \begin{bmatrix} \mathbf{e}_1^T \\ \mathbf{e}_2 - 3\mathbf{e}_1^T \end{bmatrix} A^{(1)} = \begin{bmatrix} 1 & 3 \\ 0 & -5 \end{bmatrix} := A^{(2)}$$

Next

$$G_2(-1/5)A^{(2)} = \begin{bmatrix} 1 & 0 \\ 0 & -1/5 \end{bmatrix} A^{(2)} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} := A^{(3)}$$

and finally

$$F_{12}(-3)A^{(3)} = \begin{bmatrix} \mathbf{e}_1^T - 3\mathbf{e}_2^T \\ \mathbf{e}_2 \end{bmatrix} A^{(3)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

This shows that

$$F_{12}(-3)G_2(-1/5)F_{21}(-3)G_1(1/2)A = I$$

therefore

$$A = [F_{12}(-3)G_2(-1/5)F_{21}(-3)G_1(1/2)]^{-1} \tag{21}$$
$$= G_1(1/2)^{-1}F_{21}(-3)^{-1}G_2(-1/5)^{-1}F_{12}(-3)^{-1}$$
$$= G_1(2)F_{21}(3)G_2(-5)F_{12}(3)$$

III. If B is not invertible, then $\det B = 0$ and B has a nonzero null space, but then so does AB hence $\det AB = 0$.

Finally, if A is not invertible, then $\det A = 0$ and its column space (the range of $\mathbf{x} \to A\mathbf{x}$) is not the full space, so A is not onto; then neither is AB hence $\det AB = 0$.

Formula (20) has been established in all cases.