Some review problems

1. Find the maximum of the expression $\dfrac{5x^2 - 2xz + 2y^2 + 5z^2}{x^2 + y^2 + z^2}$ when (x, y, z) take all the real values except for $(0, 0, 0)$.

2. Find the singular value decomposition and the Moore-Penrose pseudoinverse of the matrix

$$M = \begin{bmatrix} a & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The last page of the exam will contain the SVD theorem as showed on the web site.

3. Consider the system $M\mathbf{x} = \mathbf{b}$ where

$$M = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

(a) Find all the vectors $\mathbf{b}$ for which the system is solvable.
(b) Assuming the system $M\mathbf{x} = \mathbf{b}$ is solvable, find its general solution.
(c) Find a basis for the column space of M.
(d) What is the rank of M^T? Justify.

4. Let M be a rectangular $m \times n$ matrix, with complex entries.
(a) What is the dimension of MM^*?
(b) Show that MM^* is self-adjoint.
(c) Show that MM^* is positive semi-definite.
(d) What can we say for sure about the eigenvalues of MM^*?
(e) Is MM^* always diagonalizable?