Exercise:

1. Show that any n'th root of 1 (that is, a number z so that $z^n = 1$) has the form $\exp(2\pi ki/n)$ for k integer.

2. Denote $a = \exp(2\pi i/n)$. Show that $1, a, a^2, ..., a^{n-1}$ are all the roots of 1 (that is, there are n of them, and no more).

3. Show that the number a above satisfies $1 + a + a^2 + ... + a^{n-1} = 0$

4. Plot, on separate planes, the roots of 1 for $n = 2$. Then for $n = 3$, then $n = 4$. Then $n = 5$. What do you see?

Solution:

There probably are many ways to solve this, here is my favorite (with some good review items, too).

Recall the polar form of complex numbers: $z = |z|e^{i\theta}$.

Recall: $e^{x+iy} = e^x(\cos y + i \sin y)$. Therefore $e^{2k\pi i} = 1$ for all $k \in \mathbb{Z}$, and, conversely, if $e^{x+iy} = 1$ then $x = 0,\ y = 2k\pi$ for some $k \in \mathbb{Z}$.

1. Let $z \in \mathbb{C}$ so that $z^n = 1$. Take the magnitude on both sides, it follows that $|z|^n = 1$ so $|z| = 1$. Therefore $z = e^{i\theta}$ for some real number θ. Then $z^n = e^{in\theta} = 1$ means that $n\theta = 2k\pi$ for some $k \in \mathbb{Z}$, hence $\theta = 2k\pi/n$.

2. $1, a, a^2, ..., a^{n-1}$ are the numbers $e^{2k\pi/n}$ for $k = 0, 1, \ldots n-1$.

We have $a^n = \exp(2\pi i) = 1$. The other powers: they satisfy $(a^k)^n = (a^n)^k = 1^k = 1$ so they are all roots of 1.

For all other k the numbers repeat themselves: $a^n = 1,\ a^{n+1} = a^n a = a$ etc. More formally:

– for higher k: say $k = Nn + j$ for some integer $j \geqslant 0$. Then $e^{2k\pi/n} = e^{2(Nn+j)\pi/n} = e^{2j\pi/n}$, the numbers start repeating.

- for negative k: add N multiples of n until $Nn + k \in \{0, 1, \ldots, n-1\}$ and repeat the argument above. (What is this magic number N? Divide $k/n = N + (\text{remainder})/n$.)

3. *Recall* the factorization formula $1 - x^n = (1-x)(1 + x + x^2 + \ldots + x^{n-1})$.

We have $a^n = 1$. So $a^n - 1 = (a-1)(1 + a + a^2 + \ldots + a^{n-1}) = 0$. Since $a \neq 1$ then $1 + a + a^2 + \ldots + a^{n-1} = 0$.

4. $n = 2$: two points, $1, -1$, symmetric about O.

$n = 3$: equilateral triangle, with one vertex at 1.

$n = 4$: $1, i, -1, -i$ square with one vertex at 1.

$n = 5$: a regular (why?) pentagon with one vertex at 1.