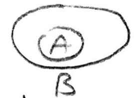
Topics not to miss when you review and practice problems

Operations with sets

$\boxed{1}$ Show that if A, B are two sets such that $A \cap B = A$ and $A \cap B = B$ then $A = B$

Solution

[Intuitively, if $A \cap B = A$  then $A \subseteq B$.]

Rigorously, assume $A \cap B = A$ then, since $A \subseteq A \cap B$

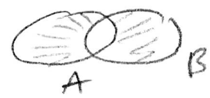
then $x \in A \Rightarrow x \in A \cap B \Rightarrow x \in A$ and $x \in B$ so $x \in B$. Thus $A \subseteq B$. (1)

Same proof, switching A and B shows that $A \cap B = B \Rightarrow B \subseteq A$ (2)

Then (1) and (2) $\Rightarrow A = B$

$\boxed{2}$ Define the symmetric difference of the sets A, B by

$$A \Delta B = (A \setminus B) \cup (B \setminus A)$$



Prove that

$$A \Delta B = (A \cup B) \setminus (A \cap B)$$

Solution $\boxed{\text{Note}}$ (We could do "$\subseteq$" then "$\supseteq$" but "$\supseteq$" is just a walk-back of "$\subseteq$" since all statements are $\Leftrightarrow$. see also $\boxed{8}$)

$x \in A \Delta B \Leftrightarrow x \in A \setminus B$ or $x \in B \setminus A$

$\Leftrightarrow (x \in A$ and $x \notin B)$ or $(x \in B$ and $x \notin A) \Leftrightarrow$

$\Leftrightarrow [x \in A$ or $(x \in B$ and $x \notin A)]$ and $[x \notin B$ or $(x \in B$ and $x \notin A)]$

$\Leftrightarrow [(x \in A$ or $x \in B)$ and $\underbrace{(x \in A \text{ or } x \notin A)}_{\text{True}}]$ and $[\underbrace{(x \notin B \text{ or } x \in B)}_{\text{True}}$ and $(x \notin B$ or $x \notin A)]$

$\Leftrightarrow (x \in A$ or $x \in B)$ and $(x \notin B$ or $x \notin A)$

$\Leftrightarrow (x \in A \cup B)$ and $\neg(x \in B$ and $x \in A)$

$\Leftrightarrow (x \in A \cup B)$ and $\neg(x \in A \cap B)$

$\Leftrightarrow (x \in A \cup B)$ and $(x \notin A \cap B)$.

$\Leftrightarrow x \in (A \cup B) \setminus (A \cap B)$

Q.E.D.

Bijections, Inj, Surj.

[3.] Let $f: A \to B$, $g: B \to C$

a) Show that if f and g are bijective then $g \circ f$ is bijective.

b) Show $(g\circ f)^{-1}(y) = f^{-1}(g^{-1}(y))$.

Sol.

a) $A \xrightarrow{f} B \xrightarrow{g} C$ $(g\circ f): A \to C$

• Show injective: assume $x_1, x_2 \in A$ so that $(g\circ f)(x_1) = (g\circ f)(x_2)$

and show that $x_1 = x_2$.

$(g\circ f)(x_1) = (g\circ f)(x_2) \iff g(f(x_1)) = g(f(x_2))$

Since g is injective $\Rightarrow f(x_1) = f(x_2)$ and since f is inj $\Rightarrow x_1 = x_2$

• Show surjective: for any $y \in C$ we find $x \in A$ so that $(g\circ f)(x) = y$

Indeed, since g is surjective, there is $z \in B$ so that $g(z) = y$

Since f is surjective, there is $x \in A$ so that $f(x) = z$

So $(g\circ f)(x) = g(f(x)) = g(z) = y$.

b) $(g\circ f)^{-1}: C \to A$ so that if $(g\circ f)(x) = y$ then $(g\circ f)^{-1}(y) = x$.

As we saw above, if we denote $f(x) = z$ then $x = f^{-1}(z)$

and if $g(z) = y$ then $z = g^{-1}(y)$

So $x = f^{-1}(z) = f^{-1}(g^{-1}(y))$ Thus $x = f^{-1}(g^{-1}(y)) = (g\circ f)^{-1}(y)$

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[4] Consider $f: (-\infty, 0] \to [1, +\infty)$, $f(x) = x^2 + 1$. Is it bijection? If so, find f^{-1}

Sol.

Try injective: assume for $x_1, x_2 \in (-\infty, 0]$ $f(x_1) = f(x_2) \Rightarrow x_1^2 + 1 = x_2^2 + 1 \Rightarrow x_1 = \pm x_2$

But $x_1, x_2 \in (-\infty, 0]$ so $x_1, x_2 \le 0$ so $x_1 = x_2$. Injective!

Try surjective. Let $y \in [1, \infty)$. Can we find x, $f(x) = y$? $x^2 + 1 = y$

so $x = \pm\sqrt{y-1}$ well defined since $y \ge 1$. Choose $-$ since $x \le 0$.

So $f^{-1}(y) = -\sqrt{y-1}$ Bijective!

$\boxed{5}$ Let $f:(1,\infty)\to A$ where $A\subseteq\mathbb{R}$

$f(x)=\frac{x-1}{x+1}$

a) Show f is injective

b) Determine A so that f is surjective

c) For this A, find f^{-1}.

Sol.

a) Let $x_{1,2}>1$ so that $f(x_1)=f(x_2)$ so $\frac{x_1-1}{x_1+1}=\frac{x_2-1}{x_2+2}$

Actually we can simplify f (and our work) a bit : note that

$f(x)=\frac{x-1}{x+1}=\frac{x+1-2}{x+1}=1-\frac{2}{x+1}$

So $1-\frac{2}{x_1+1}=1-\frac{2}{x_2+1}\Rightarrow\frac{1}{x_1+1}=\frac{1}{x_2+1}\Rightarrow x_1+1=x_2+1\Rightarrow x_1=x_2$

injective

b) We need $A=R(f)$.

Since $f(x)=1-\frac{2}{x+1}$ is increasing since $\frac{1}{x+1}\searrow$ so $-\frac{2}{x+1}\nearrow$

so for $x>1$ we have $\underset{\underset{0}{\parallel}}{f(1)}<f(x)<\underset{\underset{1}{\parallel}}{\lim_{x\to\infty}f(x)}$

so $0<f(x)<1$ so $R(f)=(0,1)$ Let $A=(0,1)$

c) Solve for x : $y=1-\frac{2}{x+1}$ where $y\in(0,1)$

so $\frac{2}{x+1}=1-y\Rightarrow\frac{x+1}{2}=\frac{1}{1-y}\Rightarrow x=\frac{2}{1-y}-1=f^{-1}(y)$

Direct and inverse images

$\boxed{6}$ Let $f: A \to B$ function, $E \subseteq A$

Show that $E \subseteq f^{-1}(f(E))$

Give an example where E is a proper subset

Solution

Let $x \in E$. We need to show that $x \in f^{-1}(f(E))$, which means that $f(x) \in f(E)$ true by the definition of $f(E)$ since $x \in E$.

Example when $\subsetneq$: Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.

$E = [0,3] \Rightarrow f(E) = [0,9]$

and $f^{-1}([0,9]) = [-3,3]$

$\boxed{7}$ Let $f: A \to B$ and $H \subseteq B$. Show that $f(f^{-1}(H)) \subseteq H$.

Do we always have equality?

Solution

Let $y \in f(f^{-1}(H))$. This means that $y = f(x)$ for some $\underbrace{x \in f^{-1}(H)}_{\text{meaning } f(x) \in H}$

$\Downarrow$

$y \in H$ QED

Attempt to show $H \subseteq f(f^{-1}(H))$

so let $y \in H$. Let $x \in f^{-1}(H)$ meaning $f(x) \in H$ Hmmm...

what if H contains elements outside $R(f)$? then we cannot have $H \subseteq f(\text{Something})$.

Example $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

Let $H = [-3,0]$. Then $f^{-1}(H) = \{0\}$ and $f(f^{-1}(H)) = f(\{0\}) = \{0\}$

So not always.

Review questions

What can you say on the connection between

$f^{-1}(C \cap D)$ and $f^{-1}(C), f^{-1}(D)$
$f^{-1}(C \cup D)$

$f(E \cup F)$ and $f(E), f(F)$
$f(E \cap F)$

$\boxed{8}$ Let $A \xrightarrow{f} B \xrightarrow{g} C$ functions. Let $H \subseteq C$
Show that $(g \circ f)^{-1}(H) = f^{-1}(g^{-1}(H))$

Solution

"$\subseteq$" Let $x \in (g \circ f)^{-1}(H)$. This means that $(g \circ f)(x) \in H$
so $g(f(x)) \in H \Rightarrow f(x) \in g^{-1}(H) \Rightarrow x \in f^{-1}(g^{-1}(H))$

"$\supseteq$" Conversely, let $x \in f^{-1}(g^{-1}(H)) \Rightarrow f(x) \in g^{-1}(H) \Rightarrow \underbrace{g(f(x))}_{=(g \circ f)(x)} \in H \Rightarrow$

$\Rightarrow x \in (g \circ f)^{-1}(H)$

This is

Note that we just walked back the proof of "$\subseteq$". which we could have done because instead of "$\Rightarrow$" at each step we can write $\Leftrightarrow$,

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$\boxed{9}$ Let $a \in \mathbb{R}$. Use mathematical induction to show that $(a^m)^n = a^{mn}$ for all $m, n \in \mathbb{N}$.

Solution
Let $P(n)$: $(a^m)^n = a^{mn}$

$P(1)$: $(a^m)^1 = a^{1 \cdot m} \Leftrightarrow a^m = a^m$ true

$P(n) \Rightarrow P(n+1)$. Assume $(a^m)^n = a^{mn}$ and show $(a^m)^{n+1} = a^{m(n+1)}$

$(a^m)^{n+1} = (a^m)^n \cdot a^m \underset{\text{since } P(n) \text{ is true}}{=} a^{mn} \cdot a^m = a^{mn+m} = a^{m(n+1)}$

By the princ of math ind, $P(n)$ is true for all $n \in \mathbb{N}$ QED

[10] Use mathematical induction to show that

$n(n^2-1)$ is divisible by 6. for all $n \in \mathbb{N}$

[Note: this is $(n-1)n(n+1)$ product of 3 consecutive #s, so yes, it is divisible by 3, and by 2 since it is a prod of 2 consecutives]

Solution

Let $P(1)$: 0 is divisible by 6

Assume $6 | n(n^2-1)$ and show that $6 | (n+1)[(n+1)^2-1]$

So we assumed that $n(n^2-1) = 6k$ with $k \in \mathbb{N}$.

then $(n+1)[(n+1)^2-1] = (n+1)(n^2+2n) = n(n+1)(n+2)$ Hmmm

$= n(n+1)(n-1+3) = n(n+1)(n-1) + 3n(n+1) = 6k + 3n(n+1)$ (*)

Since $n(n+1)$ is the product of 2 consecutive numbers, it is even: $n(n+1) = 2m$

thus (*) $= 6k + 6m = 6(k+m)$ so $P(n+1)$ is true.

By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

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[11] Prove or disprove the statement:

"there is no natural number so that $(n+1)! = n! + 1$"

Solution

$(n+1)!$ is $(n+1)$ times greater than $n!$. So eventually it is $>$.

Let us check the first few: for $n=1$ $2! \overset{?}{=} 1! + 1$ True.

So the statement is false.

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[12] Show that $(n+1)! > n! + 1$ for $n \in \mathbb{N}$ large enough.

$\boxed{\text{Sol.}}$ $n=1$ - false.

$n=2$: $3! > 2! + 1$ True

So $P(2)$ is true. Use induction: assume $(n+1)! > n! + 1$ and $(n \geq 2)$ and show $(n+2)! > (n+1)! + 1$

Since: $(n+2)! = (n+2) \cdot (n+1)! > (n+2)[n! + 1] = (n+2) n! + n + 2$

$> (n+1) n! + n + 2 = (n+1)! + (n+2) > (n+1)! + 1$

By the princ. of math induction $P(n)$ is true for all $n \geq 2$. QED

[23] Prove that if the sets A, B are countable, then $A\times B$ is countable

Solution

$$A\times B = \{(a,b) \mid a\in A,\ b\in B\} = \bigcup_{a\in A} \{(a,b) \mid b\in B\} \quad (*)$$

The sets $\{(a,b) \mid b\in B\}$ is countable because there is a bijection

$f: B \to \{(a,b) \mid b\in B\}$

$f(b) = (a,b)$

and B is countable. Since A is countable, the union $(*)$ is countable.

Solution 2 $\varphi: \mathbb{N}\to A$ bij. $\psi: \mathbb{N}\to B$ bij. Let $f: \mathbb{N}\times\mathbb{N}\to A\times B$, $f(m,n) = (\varphi(n), \varphi(m))$ bijection

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[24] Let U be an uncountable set, and $C\subseteq U$, countable. Show that $U\setminus C$ is uncountable.

Solution

Assume that $U\setminus C$ is countable then U is countable, contradiction

Since $U = (U\setminus C)\cup C$

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[25] Here is another proof that $\mathbb{Z}$ is countable.

Let $\mathbb{Z}_+ = \mathbb{N}$ the set of positive integers, and $\mathbb{Z}_-$ the set of negative integers, which is countable since

$f: \mathbb{N}\to\mathbb{Z}_-$
$f(n) = -n$ is a bijection.

Then $\mathbb{Z} = \mathbb{N}\cup\{0\}\cup\mathbb{Z}_-$ a countable union of countable sets, thus countable.

13 True or False?

a) $\mathbb{Q}$ is countable
b) $\mathbb{Q}$ is finite
c) $\mathbb{Q}$ is uncountable
d) $\mathbb{Q}$ is denumerable

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14 True or false? If true, prove it. If false, give a counterexample

a) Any subset of $\mathbb{N}$ is finite
b) Any subset of $\mathbb{N}$ is denumerable
c) Any subset of $\mathbb{N}$ is countable

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15 True or False? If true, prove it, if false, give a counterexample

a) If A is a finite set, $A \neq \phi$, then there is a $k, n \in \mathbb{N}$ and a bijection $f: \{k+1, k+2, \dots, k+n\} \longrightarrow A$.

b) Any function $f: \{1, 2, \dots n\} \to \mathbb{N}$ is not injective
c) Any function $f: \{1, 2, \dots n\} \to \mathbb{N}$ is not surjective
d) Any function $f: \{1, 2, \dots n\} \to \mathbb{N}$ is not bijective

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16 True or false? Justify

a) $\mathbb{Z} \times \mathbb{Z}$ is finite
b) $\mathbb{Z} \times \mathbb{Z}$ is countable
c) $\mathbb{Z} \times \mathbb{Z}$ is uncountable
d) $\mathbb{Z} \times \mathbb{Z}$ is denumerable

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17 True or false? If true prove it, if false give a counterexample

"If there is $f: A \to B$ bijection then A countable $\Leftrightarrow$ B is countable"

$\boxed{18}$ Find all the values of x that satisfy

$$|x+2| \geq x+5$$

$$|x+2| = \begin{cases} x+2 & \text{if } x+2 \geq 0 \text{ i.e. } x \geq -2 \\ -(x+2) & \text{if } x+2 < 0 \text{ i.e. } x < -2 \end{cases}$$

Sol I

Case I: If $x \geq -2$ the ineq is $x+2 \geq x+5 \iff 2 \geq 5$ false!
No $x \geq -2$ satisfies

Case II If $x < -2$ the ineq is $-x-2 \geq x+5 \iff 2x \leq -7 \iff x \leq -\frac{7}{2}$

so $x \in (-\infty, -2) \cap (-\infty, -\frac{7}{2}) = (-\infty, -\frac{7}{2})$ final answer.

Sol II We can get rid of $| \ |$ by squaring both sides. They must be positive!

Note: if $x+5 < 0$, that is, $x < -5$ the ineq is true since

$$\underbrace{|x+2|}_{+} \geq \underbrace{x+5}_{-}$$

Now if $x \geq -5$ I can square both sides and get

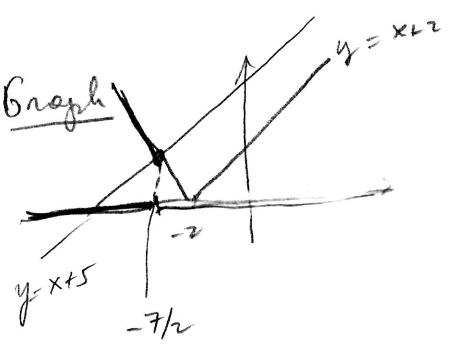
$$(x+2)^2 \geq (x+5)^2 \iff \cancel{x^2} + 4x + 4 \geq \cancel{x^2} + 10x + 25$$

$$\iff 6x \leq -21 \iff x \leq -\frac{21}{6} = -\frac{7}{2}$$

Sol $x \in [-5, +\infty) \cap (-\infty, -\frac{7}{2}] = [-5, -\frac{7}{2})$

Final solution $(-\infty, -5] \cup [-5, -\frac{7}{2}) = (-\infty, -\frac{7}{2})$

Graph



V is larger for $x < -\frac{7}{2}$

$\boxed{19}$ Find all values of x s.t. $|x+1|+|x-1|<3$

Sol. $|x+1| = \begin{cases} x+1, & x \geq -1 \\ -x-1, & x<-1 \end{cases}$, $|x-1| = \begin{cases} x-1, & x\geq 1 \\ -x+1, & x<1 \end{cases}$

I II III

-1 1

Case I: $x<-1$: $-x-1-x+1<3$

$-2x<3 \iff x>-\frac{3}{2}$

so $x\in(-\infty,-1)\cap(-\frac{3}{2},\infty) = (-\frac{3}{2},-1)$

Case II $-1\leq x<1$: $x+1-x+1<3$

$2<3$ true $x\in[-1,1)$

Case III $x\geq 1$: $x+1+x-1<3$

$2x<3$

$x<\frac{3}{2}$ $x\in[1,\infty)\cap(-\infty,\frac{3}{2}) = [-1,\frac{3}{2})$

So $x\in(-\frac{3}{2},-1)\cup[-1,1)\cup[-1,\frac{3}{2}) = (-\frac{3}{2},\frac{3}{2})$

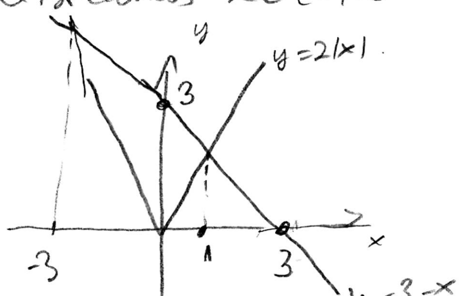
$\boxed{20}$ Find all x which satisfies $2|x|\leq 3-x$. Sketch graph

Solution

If $x\geq 0$: $2x\leq 3-x \iff 3x\leq 3 \iff x\leq 1$ So $x\in[0,+\infty)\cap(-\infty,1] = [0,1]$

If $x<0$: $-2x\leq 3-x \iff -x\leq 3 \iff x\geq -3$ So $x\in(-\infty,0)\cap[-3,\infty) = [-3,0)$

All solutions $x\in[0,1]\cup[-3,0) = [-3,1]$



Graphs intersect
for $x>0$ when $2x = 3-x$ so $x=1$
for $x<0$ when $-2x = 3-x$ so $x=-3$

We see that the line is above the V for $x\in[-3,1]$ indeed.

21 Find all $x \in \mathbb{R}$ that satisfy both $|2-x|<4$ and $|x+2|>1$ simultaneously

Solution

First solve $|2-x|<4$. For $x<2$: $2-x<4 \iff x>-2$ so $x \in (-2,2)$

For $x \geq 2$: $x-2<4 \iff x<6$ so $x \in [2,6)$

All in all $x \in (2,6) \cup (-2,2] = (-2,6) \equiv A$

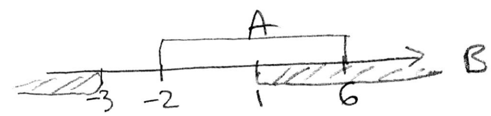
Next solve $|x+2|>1$.

For $x \geq -2$: $x+2>1$ so $x>-1$ so $x \in [-2,+\infty) \cap (-1,+\infty) = (-1,+\infty)$

For $x<-2$: $-x-2>1$ so $x<-3$ so $x \in (-\infty,-2) \cap (-\infty,-3) = (-\infty,-3)$

All in all $x \in (-\infty,-3) \cup (1,+\infty) \equiv B$

Simultaneously if $x \in A \cap B$
$= (1,6)$



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22 Show that $|2x+3y-4z| \leq 2|x|+3|y|+4|z|$ for all $x,y,z \in \mathbb{R}$.

Solution

Use Δ inequality twice:

$$|2x+3y-4z| \leq |2x+3y|+|-4z| = |2x+3y|+4|z| \leq |2x|+|3y|+4|z| = 2|x|+3|y|+4|z|$$
