(A) Yes! If p is a period then $f(x+p)=f(x)$, $\forall x \in \mathbb{R}$
f restricted to $[0,p]$ is continuous so it is bounded: $\exists M>0$, $|f(x)| \leq M$ $\forall x \in [0,p]$

Now for other $x \in \mathbb{R}$, $\exists n \in \mathbb{Z}$ so that $x-np \in [0,p]$
What is n? We have $np \leq x < (n+1)p$ for $n = [\frac{x}{p}]$
Claim: $f(x) = f(x-np)$ so $|f(x)| \leq M$ too.
Proof is by induction: first show $f(x) = f(x+mp)$, $\forall m \in \mathbb{Z}$
(use induction on m). Then replace x by $x-mp$ and $\Rightarrow f(x-mp)=f(x)$

(B) Let $f:[a,b]\to[a,b]$ continuous
Define $g:[a,b]\to\mathbb{R}$ by $g(x)=f(x)-x$: $\Rightarrow g$ continuous
Then $g(a)=f(a)-a \geq 0$ since $f(a)\in[a,b]$
$g(b) = f(b)-b \leq 0$ since

If $f(a)-a=0$ then $c=a$
If $f(b)-b=0$ then $c=b$
otherwise, by the IVT, $\exists c \in (a,b)$, $g(c)=0$ $\Rightarrow f(c)=c$

(C) Define $h:[0,1]\to\mathbb{R}$ by $h(x)=f(x)-g(x)$
Since f,g are cont on $[0,1]$ $\Rightarrow h$ is cont on $[0,1]$.
$h(0)=f(0)-g(0)\geq 0$
$h(1)=f(1)-g(1)\leq 0$
same explanations as above in B $\Rightarrow \exists c$ $h(c)=0$

(D) Let $x_1, x_2 \in [0,2]$

Chord joining $(x_1, f(x_1))$ and $(x_2, f(x_2))$ has length

$$l(x_1,x_2) = \sqrt{(x_1-x_2)^2 + (f(x_1)-f(x_2))^2}$$

Need to show: $\exists\, x_1, x_2 \in [0,2]$ so that $l(x_1,x_2)=1$

Equivalently, $l^2(x_1,x_2)=1$

$\left.\begin{array}{l} l^2(0,2) = \sqrt{4+0} = 2 \\ l^2(0,0) = 0 \end{array}\right\}$ I guess l^2 is continuous of 2 variables?

Note: we can fix $x_1=0$. So:

Consider $l(0,x) = \sqrt{x^2 + [f(x)-f(0)]^2}$ continuous on $[0,2]$

$l(0,0)=0$

$l(0,2)=2$

By the IVT, $l(0,x)$ takes all values between 0 and 2, in particular the value 1: $\exists\, c \in (0,2)$ st $l(0,x)=1$.

(E) $\left.\begin{array}{l} f: \text{interval} \to \mathbb{R} \\ \text{continuous} \end{array}\right\} \Rightarrow f(\text{interval}) = \text{interval}$ by the preservation of intervals theorem

So $B=\{0,1\}$ cannot be $f(I)$

<u>Remark</u> if $f(x)=c$ then $f(I)=\{c\}=[c,c]$ a "degenerate" interval.

(F) 3 real rooots... $p(x) = x^3 - 3x^2 - x + 2$ continuous.

$\lim_{x \to -\infty} p(x) = -\infty$ so if I take a really small x I get $p<0$

Try $x = -10$: $p(-10) = -1{,}000 + 300 + 10 + 2 < 0$) one root here by IVT perhaps more?

$p(0) = 2 > 0$

$p(1) = \cancel{1} - 3 - \cancel{1} + 2 = -1$) one root here

$p(2) = 8 - 12 - 2 + 2 < 0$

$p(10) = 1000 - 300 - 10 + 2 > 0$) one root here

It has real roots: between $(-10, 0)$, also $(0, 1)$ and $(2, 10)$

(G) Hmm... interesting..

let us try a simplified version $n = 2$, $x_1 = a$, $x_2 = b$

Show $\exists c [a, b]$, $f(c) = \frac{f(a) + f(b)}{2}$

this one is easy $\frac{f(a) + f(b)}{2}$ = midpoint of $[f(a), f(b)]$

so it is between $f(a)$ and $f(b)$. By The IVT, $\exists c$, $f(c) = \frac{f(a) + f(b)}{2}$

Now for $n=2$ but $x_1, x_2 \in [a, b]$ abitrary similarly $\exists c \in (x_1, x_2)$ by IVT

Now for $n = 3$ ...? I would think that the arithmetic average is between the lowest value and the largest one..

I got it! f cont on $[a, b] \Rightarrow f$ has abs max and abs min

$\exists a_0 \in [a, b]$, $f(a_0) = \min\{f(x) \mid x \in [a, b]\}$

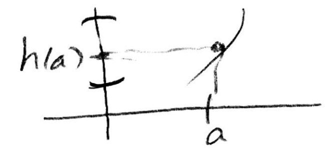
$b_0 \in [a, b)$ $f(b_0) = \max\{f(x) \mid x \in [a, b)\}$

then $f(a_0) \leq f(x_i) \leq f(b_0), \forall i \Rightarrow f(a_0) \leq \frac{f(x_1) + \ldots + f(x_n)}{n} \leq f(b_0)$

By IVT $\exists c \in (a_0, b_0)$, $f(c) = \uparrow$. QED

1. Let $h = g - f$. Since f and g are cont on $\mathbb{R}$, so is h.

We had an exercise: if $h(a) > 0$ then $h(x) > 0$ for all x in a neighborhood of a. Since we can only quote theorems but not exercises, we prove it here.



Since h is continuous at a,

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|x-a| < \delta \Rightarrow |h(x) - h(a)| < \varepsilon$

Let $\varepsilon = \frac{h(a)}{2}$. Then $\exists \delta > 0$ s.t. $|x-a| < \delta \Rightarrow |h(x) - h(a)| < \frac{h(a)}{2}$

So $h(a) - \frac{h(a)}{2} < h(x) < h(x) + \frac{h(a)}{2} \Rightarrow h(x) > \frac{h(a)}{2} > 0$.

for all $x \in V_\delta(a)$.

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2. If $f(a) < g(a) \Rightarrow$ (see problem 1) $\exists \delta$, $f(x) < g(x)$, $\forall x \in V_\delta(a)$ so continuous at a.

so $\max\{f(x), g(x)\} = g(x)$, $\forall x \in V_\delta(a)$ so cont at a.

If $f(a) > g(a)$, similarly, $\max = f(x)$, $\forall x \in V_\delta(a)$

If $f(a) = g(a)$: Let $\varepsilon > 0$. Since f is cont at a and so is g:

$\exists \delta_1 > 0$ s.t. $|x-a| < \delta_1 \Rightarrow |f(x) - f(a)| < \varepsilon$

$\exists \delta_2 > 0$ s.t. $|x-a| < \delta_2 \Rightarrow |g(x) - \underbrace{f(a)}_{=g(a)}| < \varepsilon$

Let $\delta = \min\{\delta_1, \delta_2\}$. Since

$$f(a) - \varepsilon < f(x) < f(a) + \varepsilon$$

$$f(a) - \varepsilon < g(x) < f(a) + \varepsilon \quad \forall x \text{ s.t. } |x-a| < \delta$$

$$\Rightarrow f(a) - \varepsilon < \max\{f(x), g(x)\} < f(a) + \varepsilon$$

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3 Yes. First note the domain is $[0,1) \cup (1,\infty)$. Then $\frac{\sqrt{x}-1}{x-1} = \frac{\sqrt{x}-1}{(\sqrt{x}-1)(\sqrt{x}+1)} = \frac{1}{\sqrt{x}+1}$ limit $= \frac{1}{2}$

4. $\frac{1}{x}$ continuous on $\mathbb{R}\setminus\{0\}$, $\sin x$ cont on $\mathbb{R}$

so $\sin\frac{1}{x}$ cont on $\mathbb{R}\setminus\{0\}$.

At $x=0$: $\lim\limits_{x\to 0} \sin\frac{1}{x}$ does not exist, so ~~f~~ not cont at 0.

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5. As above, g is cont at all points in $\mathbb{R}\setminus\{0\}$.

$\lim\limits_{x\to 0} g(x) = 0$ (squeeze theorem, write details)

and $0 = g(0)$ so g is also cont. at 0. $\underline{\underline{A}}$ cont on $\mathbb{R}$

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6. As in 4., h is cont on $\mathbb{R}\setminus\{0\}$.

Now at 0: $\lim\limits_{x\to 0} e^{-\frac{1}{x^2}} = ?$ $\quad \lim\limits_{x\to 0} \frac{1}{x^2} = +\infty$ so $\lim e^{-\frac{1}{x^2}} = 0 = h(0)$

so h is cont on $\mathbb{R}$

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7. Recall $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$. So $\lim\limits_{x\to 0} \frac{\sin 3x}{\sin 5x} = \lim\limits_{x\to 0} \frac{\sin 3x}{3x} \cdot \frac{5x}{\sin 5x} \quad \frac{3}{5} = \frac{3}{5}$

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8 $\sqrt{x+\sqrt{x+\sqrt{x}}} - \sqrt{x} = \dfrac{\cancel{x}+\sqrt{x+\sqrt{x}} \cancel{-x}}{\sqrt{x+\sqrt{x+\sqrt{x}}} + \sqrt{x}}$

$= \dfrac{\cancel{\sqrt{x}}\sqrt{1+\frac{1}{\sqrt{x}}}}{\cancel{\sqrt{x}}\left(\sqrt{1+\frac{\sqrt{x+\sqrt{x}}}{x}} + 1\right)} = \frac{1}{2}$ since $\frac{\sqrt{x+\sqrt{x}}}{x} = \frac{\sqrt{x}\sqrt{1+\frac{1}{\sqrt{x}}}}{x} \to 0$

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9. $-1 \le \sin x^{10} \le 1$ so $\underset{\downarrow\atop 0}{-\frac{1}{x}} \le \frac{\sin x^{10}}{x} \le \underset{\downarrow\atop 0}{\frac{1}{x}}$ $\quad$ $\lim = 0$ by squeeze theorem.

10 $\lim \left(\frac{x}{\sqrt{x^2+x+1}} + \frac{\cos x}{\sqrt{x^2+x+1}} \right)$

$= \lim \left(\frac{\cancel{x}}{\cancel{x}\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}} + \frac{\cos x}{\sqrt{x^2+x+1}} \right)$

$\searrow 1$

and $\quad -\frac{1}{\sqrt{x^2+x+1}} \leq \frac{\cos x}{\sqrt{x^2+x+1}} \leq \frac{1}{\sqrt{x^2+x+1}}$

$\downarrow 0 \qquad\qquad \downarrow 0$

squeeze theorem $\Rightarrow$ $\downarrow 0$ This is a sketch, You need to write nicer...