Real Numbers

1. Decimal representation

The following theorems are not hard to prove, but we will not do so here.

Theorem *Decimal representation of natural numbers*

For any $n \in \mathbb{N}$ there are unique $k \in \mathbb{N} \cup \{0\}$ and $a_0, a_1, \ldots, a_k \in \{0, 1, 2, \ldots 9\}$, $a_k \neq 0$ so that

$$n = a_k \cdot 10^k + a_{k-1} \cdot 10^{k-1} + \ldots + a_1 \cdot 10 + a_0$$

The numbers $a_0, a_1, \ldots, a_k$ are called the digits of n and we can write the decimal representation of n as $a_k a_{k-1} \ldots a_1 a_0$. For example by the number $4,527$ we mean

$$4,527 = 4 \cdot 10^3 + 5 \cdot 10^2 + 2 \cdot 10 + 7$$

If we allow for negative powers of 10 we obtain decimal representations of numbers less than 1, for example

$$0.256 = \frac{2}{10} + \frac{5}{10^2} + \frac{6}{10^3}$$

and by doing long division one finds that

$$\frac{1}{4} = 0.2500000... = \frac{2}{10} + \frac{5}{10^2}$$

$$\frac{1}{3} = 0.333333... := 0.\overline{3} = \frac{3}{9}$$

$$\frac{139}{330} = 0.421212121... := 0.4\overline{21} = \frac{4}{10} + \frac{21}{990}$$

Theorem *Decimal representation of rational numbers* Any rational number has a decimal representation so that, *eventually*, the decimals repeat periodically: if $m, n \in \mathbb{N}$, $m < n$ then there are unique k, p natural and unique $a_1, \ldots, a_k$ and $b_1, \ldots, b_p$ in $\{0, 1, 2 \ldots, 9\}$ so that

$$\frac{m}{n} = 0.a_1 \ldots a_k \overline{b_1 \ldots b_p} := \frac{a_1}{10} + \ldots + \frac{a_k}{10^k} + \frac{b_1 \ldots b_p}{10^k(10^p - 1)}$$

A wrinkle: a number with its decimal representation ending with an infinite array of 9, say $x = a_1 a_2 \ldots a_n . b_1 b_2 \ldots b_k 9999\ldots$ with $b_k \neq 9$, is the same number as the one with the infinite string of 9s replaced by 0s and b_k replaced by $b_k + 1$. For example

$$2.34\overline{9} = 2.35 \quad ; \quad 0.499999... = 0.4\overline{9} = 0.5 \quad ; \quad 0.999999... = 0.\overline{9} = 1.0$$

Irrational numbers: these are numbers that cannot be written as the ratio of two integers. Their decimal representation consists of their integral part followed by an infinity of decimals, not ending by a repeating sequence.

For example

$$3.141592653589793238...$$

These are the first digits of a special number we call π. The number π is defined through its properties (π is the ratio between the circumference of any circle and its diameter), and not through its decimals.

Another example: $\sqrt{2} = 1.4142135623730...$ is that number whose square is 2; this number turns out to be irrational.

We can define real numbers by their decimal representation. A real number is an infinite sequence of digits as follows: an integer number followed, after the decimal point, by an infinite sequence of digits:

$$a_1a_2a_3\ldots a_n\,.\,b_1b_2b_3\ldots \qquad \text{where} \;\; a_1, a_2, \ldots, a_n, b_1b_2b_3\ldots \in \{0, 1, 2, \ldots\ 9\}$$

with the agreement that
$a_1a_2\ldots a_n.b_1b_2\ldots b_k9999\ldots = a_1a_2\ldots a_n, b_1b_2\ldots(b_k+1)0000\ldots$

So can we really work with irrational numbers, if we can only know a finite number of their decimals?

When we do calculations for practical applications, we do use decimals, but we stop after a finite number of them: in fact we take an approximation by a rational number. Do we then need all those infinitely many decimals? And how do we work with irrational numbers? Answer: through the properties that those numbers have.

In §2 we take a look at the properties of real numbers.

1.1. **Representation of real numbers in other bases.** Real numbers can be represented by digits in other bases too. If instead of base 10 use use base $r \in \mathbb{N}$, then any natural number n can be *uniquely* written as

$$n = b_kr^k + b_{k-1}r^{k-1} + \ldots + b_1r + b_0, \quad \text{with } b_0, \ldots, b_k \in \{0, 1, \ldots, r-1\},\ b_k \neq 0$$

The base $r = 2$ is often used.

2. Defining real numbers by properties

The set $\mathbb{R}$ of all real numbers defined above by their decimal representation has the following properties **I-V**:

I. Addition

There is an operation between any two real numbers a, b, called addition, denoted $a + b$ so that:
A1. $a + b = b + a$ (commutativity)
A2. $(a + b) + c = a + (b + c)$ (associativity)
A3. $a + 0 = 0 + a = a$ (existence of the zero element)
A4. Any number a has an opposite, denoted $-a$, so that $a + (-a) = 0$.

II. Multiplication

There is an operation between any two real numbers a, b, called multiplication, denoted ab (or $a \cdot b$) so that:
M1. $ab = ba$ (commutativity)
M2. $(ab)c = a(bc)$ (associativity)
M3. $a \cdot 1 = 1 \cdot a = a$ (1 is a neutral element)
M4. Any number $a \neq 0$ has an inverse, denoted $\frac{1}{a}$, so that $a \cdot \frac{1}{a} = 1$.
D. $a(b + c) = ab + ac$ (distributivity of multiplication over addition)

III. Order properties

[There is a total order relation on $\mathbb{R}$, compatible with operations; equivalently:]
There is a nonempty subset $\mathbb{P} \subset \mathbb{R}$, called the set of positive real numbers, so that
i. If $a, b \in \mathbb{P}$ then $a + b \in \mathbb{P}$.
ii. If $a, b \in \mathbb{P}$ then $ab \in \mathbb{P}$.
iii. If $a \in \mathbb{R}$ then exactly one of the following hold:

$$a \in \mathbb{P}, \qquad a = 0, \qquad -a \in \mathbb{P}$$

IV. Completeness (existence of the least upper bound)

Any nonempty set of real numbers which is bounded above has a supremum.

Note that the set of rational numbers have all the properties **I-III**.

Property **IV.** will be elucidated in a short while. It is this property that brings completeness to the real numbers, that makes possible having lots and lots of limits.

Definition *A set $\mathbb{R}$ with properties* **I-IV** *is called the set of real numbers.*

That is, $\mathbb{R}$ is any totally ordered field which is complete.

This is an axiomatic definition: properties **I-IV** are taken to be axioms, that is, statements considered to be true. All other properties of real numbers are deduced from these, using logic.

When an axiom system is established there are two major questions:

1) Are there enough axioms to match our intuition on the concept we want to define?
The first three are not enough, since they are also satisfied by $\mathbb{Q}$.

2) Are they consistent? That is, is there a set for which the specified axioms are true?
Yes, there is, since we do have our model with decimal representation of real numbers. They satisfy **I-III** by the way operations are defined, and **IV** needs a proof.
Other models for axioms **I-IV**: the number line, many physical quantities: temperature, velocity (on a straight line), time, etc.

Remarks (easy to see when one thinks in decimal representations):
1. between any two rational numbers there is another rational number
(can you imagine how the set of all rationals looks like when plotted on the number line?);
2. between any two rational numbers there also are irrational numbers;
3. similarly, between any two irrationals there are rationals, and irrationals.

Exercises

2.1 Convert to a fraction the following numbers

$$\mathbf{a} = 3.45\overline{123} \qquad \mathbf{b} = 0.\overline{1} \qquad \mathbf{c} = 0.\overline{9}$$

Explain the result you got in **c**.

2.2 Which of the properties **I-IV** are *not satisfied* by the set of integer numbers?

2.3 The properties listed at §2 are enough to deduce all the other properties of real numbers that you know. Here is an example.
Use *only the properties listed in §2* to show that if $a < b$ then $-a > -b$.