Review Problems.

(1) Let x_n be an unbounded sequence of real numbers.

a) Prove that if x_n is increasing then $\lim x_n = +\infty$

b) Prove that if x_n is decreasing then $\lim x_n = -\infty$

[These are theorems that you should know to use and to prove.]

(2) Let x_n, y_n be two sequences of real numbers so that, for some $m \in \mathbb{N}$, $x_n \leq y_n$, $\forall n \geq m$. Then prove that

a) if $x_n \to +\infty$ then $y_n \to +\infty$

b) if $y_n \to -\infty$ then $x_n \to -\infty$

(3) Show that if $x_n \to +\infty$ and $a_n \to a$ then

a) If $a > 0$ then $x_n a_n \to +\infty$

b) If $a < 0$ then $x_n a_n \to -\infty$

c) If $a = 0$, give examples for which the limit is!

(i) $+\infty$ (ii) $-\infty$ (iii) any $c \in \mathbb{R}$.

(4) 3.6 #2

(5) 3.6 #7

(6) 3.6 #9

(7) State the Bolzano-Weierstrass theorem

(8) Consider the sequence $b_n = n[1+(-1)^n]$.

a) Is it bounded?

b) Find its lim inf and lim sup

(9) a) What does it mean that a sequence x_n does not converge to x?

b) What does it mean that a sequence x_n is not Cauchy?

(10) Let x_n, y_n be Cauchy sequences

a) Show $x_n + y_n$ is a Cauchy sequence.

b) Show $x_n y_n$ is a Cauchy sequence

c) Show $|x_n - y_n|$ is Cauchy

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(11) True or false:

a) every subsequence of a Cauchy sequence is Cauchy

b) a Cauchy sequence is convergent

c) a Cauchy sequence is bounded.

d) a Cauchy sequence can be properly divergent

e) If x_n, y_n are Cauchy then $x_n - y_n$ is Cauchy.

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(12) Let a_n be defined recursively by $a_1 = 1$, $a_{n+1} = a_n + (-1)^n n^3$.

Show a_n is not Cauchy.

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(13) Prove that the sequence $a_n = \dfrac{(n^2+20n+35)\sin^3 n}{3n^2+n+1}$

has a convergent subsequence.

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(14) Show that $x_n = 2\cos n - \sin n$ has a convergent subsequence

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(15) True or false: (a) there is a sequence converging to 5, containing a subsequence converging too.

(b) there are unbounded sequences with bounded subseq.

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(16) Show that the sequence $x_{n+1} = \frac{1}{6}(x_n^2 + 8)$, $x_1 = a$ is convergent and find its limit assuming that $0 < a < 2$.

Hint: show $0 \le x_n \le 2$ and that x_n is monotone.

What happens if $a = 2$? What happens if $2 < a < 4$?

(17) True or false? Prove if true, find counterexample if false.

Let $x_n > 0$, $\forall n$

(a) If $x_{n+1} - x_n \to 0$ then x_n converges

(b) If $|x_{n+1} - x_n| < |x_n - x_{n-1}|$ then x_n converges

(c) If x_n is Cauchy, then x_n is contractive

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(18) True or false?

Let x_n be a sequence of integers such that $x_{n+1} \neq x_n$, $\forall n$.

Then x_n has a monotone subsequence.

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never mind!!!

~~(19) Let $1 \le x_1 \le x_2 \le 2$ and $x_{n+2} = \sqrt{x_{n+1} x_n}$~~

~~Show that a) $\frac{x_{n+1}}{x_n} \geq \frac{1}{2}$, $\forall n$,~~

~~b) $|x_{n+1} - x_n| \le \frac{2}{3} |x_n - x_{n-1}|$~~

~~c) x_n converges~~

~~d) OPTIONAL: find $\lim x_n$. Find a formula for x_n~~

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(20) Let $a_1 = 1$, $a_{n+1} = 3 + \frac{a_n}{2}$

Show a_n is convergent and find $\lim a_n$

OPTIONAL: can you find a formula for a_n?

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(21) Find lim sup and lim inf for

a) $a_n = \frac{1}{n}$ b) $b_n = (-1)^n \frac{3n+1}{n+1}$

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(22) Exercises from 3.7 similar to those in homework.