Solution

(10) No, e.g. $x_n = n$ Any $x_{n_k} = n_k \to \infty$

(11) No, see above

But: Review also Bolzano Weierstrass theorem
and the Monotone Sequence theorem

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(12) $f:[a,b]\to\mathbb{R}$ cont

No!! eg $f(x)=\begin{cases}1, & 0\le x\le 1\\ 0, & 1<x\le 2\end{cases}$ $f:[0,2]\to\mathbb{R}$ is discont at $x=1$

(13) a, b) Yes

Proof Δ ineq

c) No. Eg. $f=g=x$, $f,g:\mathbb{R}\to\mathbb{R}$ unif cont. Needs proof

But $fg=x^2$ is not unif cont on $\mathbb{R}$

U.C. $\forall\varepsilon>0, \exists\delta>0$ s.t. $\forall x,y\in\mathbb{R}$, $|x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$

Negate. $\exists\varepsilon_0>0$, $\forall\delta>0$, $\exists x_\delta, y_\delta$, $|x_\delta-y_\delta|<\delta$, $|f(x_\delta)-f(y_\delta)|\geq\varepsilon_0$

$\exists\varepsilon_0>0$, $\exists x_n, y_n$, $|x_n-y_n|<\frac{1}{n}$, $|x_n^2-y_n^2|\geq\varepsilon_0$

or any $\delta_n\to 0$

Let $x_n=n+\frac{1}{2n}$, $y_n=n$

$x_n^2-y_n^2=1+\frac{1}{n}>1$. Let $\varepsilon_0=1$

(14) No $f(x)=x$

(15) Yes. Max-Min thm

(16) Yes. Let $c\in\mathbb{R}$. $\varepsilon>0$. Will find $\delta>0$ st $|x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$

f is L so $|f(x)-f(y)|\le L|x-y|<L\delta=L'$

Let $\delta=\frac{\varepsilon}{L}$

(17) No. $f(x)=x^2$

Lip. $\exists L>0, \forall x,y\in\mathbb{R}, |f(x)-f(y)|\le L|x-y|$

Not L: $\forall L>0, \exists x_L, y_L, |f(x_L)-f(y_L)| > L|x_L-y_L|$

$\forall n>0, \exists x_n, y_n, |f(x_n)-f(y_n)| > n|x_n-y_n|$

or any $a_n \to \infty$

$x_n = n+1, \ y_n = n$

$x_n^2 - y_n^2 = 2n+1 > n$

(18) If $L<1$ Yes. otherwise No.

has $L=2$

Ex. $f(x)=2x$, $x_n = 2^{n-1}$

$x_{n+1} - x_n = 2^n - 2^{n-1} = 2^{n-1}$

$x_n - x_{n-1} = 2^{n-2}$

$|x_{n+1}-x_n| > |x_n - x_{n-1}|$

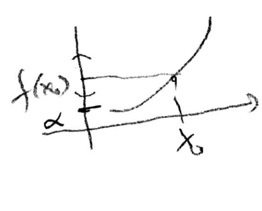
(19) Yes Cauchy $\Rightarrow$ convergent $\Rightarrow$ bounded.

We had a theorem

(20) Yes. Let $\varepsilon=1$. $\exists \delta>0$ s.t. $|x-c|<\delta \Rightarrow |f(x)-f(c)|<1$, $\forall x \in V_\delta(c)$

so $f(c)-1 < f(x) < f(c)+1$

(21) Yes

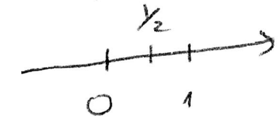


$\exists \delta>0, |x-x_0|<\delta \Rightarrow |f(x)-f(x_0)| < \frac{f(x_0)-\alpha}{2}$

(22) No. Countable Theorem

(101) Let $\varepsilon>0$. Will find $\delta>0$ s.t. $|x-1|<\delta \Rightarrow |f(x)-f(1)|<\varepsilon$

$$|f(x)-f(1)| = \left|\frac{1}{x}-1\right| = \frac{|x-1|}{x}$$



Will choose $\delta \le \frac{1}{2}$ so that $\frac{1}{x} \geq 2$ so

Let $\delta = \min\{\frac{\varepsilon}{2}, \frac{1}{2}\}$ then $2\delta \le \varepsilon$.

$$\frac{|x-1|}{x} \le 2|x-1| < 2\delta$$

(102) $f(x)=x^2$ = polynomial for $x\in(0,+\infty)$ hence continuous
$f(x)=x$ = polynomial for $x\in(-\infty,0)$ hence continuous.

At $x=0$: $\lim\limits_{x\to 0-} f(x) = \lim\limits_{x\to 0-} x^2 = 0$
$\lim\limits_{x\to 0+} f(x) = \lim\limits_{x\to 0+} x = 0$] equal and equal $f(0)$

Continuous!

Cont on $\mathbb{R}$!

(103) As above, g is cont at $x=0$.
Also g has continuous extension to $[-1,1]$: $\tilde{g}(x)=\begin{cases} 2x & \text{for } 0\le x\le 1 \\ x^2 & \text{for } -1\le x<0\end{cases}$

hence unif cont!

(104) Let $\varepsilon>0$. Since $\lim\limits_{x\to c} f(x)=L$, $\forall \varepsilon_1>0$ $\exists\, \delta_1>0$ st $0<|x-c|<\delta_1 \Rightarrow |f(x)-L|<\varepsilon_1$
and $\forall \varepsilon_2>0$ $\exists\, \delta_2>0$, $0<|x-c|<\delta_2 \Rightarrow |g(x)-M|<\varepsilon_2$
and $|g(x)|\le u$ for $x\in V_{\delta_3}(c)$

Will find $\delta>0$, and $\delta\le\min\{\delta_1,\delta_2,\delta_3\}$ such that if $|x-c|<\delta$

$$|fg(x)-LM| = |f(x)g(x) - L\cdot g(x) + L\cdot g(x) - LM| \le$$
$$\le |f(x)-L|\,|g(x)| + |L|\,|g(x)-M| \le u\varepsilon_1 + |L|\varepsilon_2$$

Let $\varepsilon_1 = \frac{\varepsilon}{2u}$, $\varepsilon_2 = \frac{\varepsilon}{2|L|}$ then $u\varepsilon_1 + |L|\varepsilon_2 = \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$

QED