Review Solutions

① See the proof in the textbook, Theorem 3.6.3

② See proof to theorem 3.6.4 in the textbook.

③ (a) Let $u>0$. We need to show $\exists K\in\mathbb{N}$ so that $x_n a_n > u,\ \forall n\geq K$

- Since $a_n \to a > 0$ then $\exists N\in\mathbb{N}$ so that $a_n > \frac{a}{2} > 0,\ \forall n\geq N$

because $|a_n - a| < \frac{a}{2}$ for $n\geq$ some N.
[note: we can just as well let $\varepsilon_0 < a$, then $a_n > \varepsilon_0,\ \forall n\geq$ some N]
$\varepsilon_0 > 0$

- Since $x_n \to +\infty$ then $\exists M\in\mathbb{N}$, $x_n > \frac{u}{a/2}$, $\forall n\geq M$

- So for $n\geq \max\{N, M\} \equiv K$ we have $x_n a_n > \frac{u}{a/2}\cdot\frac{a}{2} = u$

(b) Is similar, only we have $a_n < \frac{a}{2} < 0$.

(c) $\lim_{n\to\infty} n\cdot\frac{1}{\sqrt{n}} = +\infty \quad \lim_{n\to\infty} n\cdot\frac{-1}{\sqrt{n}} = -\infty$

$\lim n\cdot\frac{c}{n+1} = c$

④ 3.6 #2 a) $x_n = n,\ y_n = n$
b) $x_n = n^2,\ y_n = n$

⑤ 3.6 #7 Assume $x_n, y_n > 0$, $\lim \frac{x_n}{y_n} = 0$

a) Assume $x_n \to +\infty$. Show $\lim y_n = +\infty$

Intuition: $\frac{x_n}{y_n} \to 0$ means $x_n \ll y_n$, in particular $x_n < y_n$.

Proof: $\frac{x_n}{y_n} \to 0$, let $\varepsilon = 1 \Rightarrow \exists K$, $0 \leq \frac{x_n}{y_n} < 1$ for $n\geq K\in\mathbb{N}$. So $x_n < y_n$, $n\geq K$
Since $x_n \to \infty$ then $y_n \to \infty$ (Theorem)

b) If $0 < y_n < u$, let $\varepsilon > 0$. Since $\frac{x_n}{y_n} \to 0$ $\exists K$, $\frac{x_n}{y_n} < \frac{\varepsilon}{u}$, $\forall n\geq K$
Then $0 < x_n < \frac{\varepsilon}{u} y_n < \varepsilon$.

(6) (3.6 #9) Let $x_n, y_n > 0$, $\frac{x_n}{y_n} \to \infty$

a) If $y_n \to \infty$ then $x_n \to \infty$. Intuition $\frac{x_n}{y_n} \to \infty$ means $x_n >> y_n$

in particular $x_n > y_n$.

Proof: Let $u = 1$, $\exists k \in \mathbb{N}$, $\frac{x_n}{y_n} > 1$, $\forall n \geq k$. So $x_n > y_n$, $\forall n \geq k$

$$\downarrow$$
$$\infty$$

so $x_n \to \infty$

(we had a theorem, sort of squeeze theorem)

b) If $x_n < u$ then $y_n \to 0$

Intuition $\frac{x_n}{y_n} < \frac{u}{y_n}$ cannot $\to \infty$ unless $y_n \to 0$.

Proof Let $\varepsilon > 0$. $0 < y_n = \frac{y_n}{u} \cdot u < \frac{y_n}{x_n} \cdot u$

Since $\frac{x_n}{y_n} \to \infty$ then $\frac{x_n}{y_n} > \frac{u}{\varepsilon}$, $\forall n \geq$ some k

so $\frac{y_n}{x_n} \cdot u < \frac{\varepsilon}{u} \cdot u = \varepsilon$, $\forall n \geq k$.

---

(7) See textbook

---

(8) $b_{2n} = 2n$, $b_{2n-1} = 0$ not bounded above so not bounded.

limsup does not exist (or we could say it is $+\infty$)

liminf $b_n = 0$

---

(9) a) $\exists \varepsilon > 0$ so that $\forall k \in \mathbb{N}$, $\exists n \in \mathbb{N}$, $n \geq k$ so that $|x_n - x| \geq \varepsilon$

b) $\exists \varepsilon > 0$ so that $\forall k \in \mathbb{N}$, $\exists m, n \in \mathbb{N}$, $m, n \geq k$ so that $|x_n - x_m| \geq \varepsilon$

(10), (11) in class

(12) $|a_{n+1}-a_n| = n^3 > 1$

(13) a_n is bounded: $-1 \le \sin^3 n \le 1$,

and $y_n = \dfrac{n^2+20n+35}{3n^2+n+1} = \dfrac{1+\frac{20}{n}+\frac{35}{n^2}}{3+\frac{1}{n}+\frac{1}{n^2}}$ converges to $\frac{1}{3}$, hence bounded.

A product of bounded sequences is bounded, since if $|x_n| \le u$ and $|y_n| \le v$ then $|x_n y_n| = |x_n||y_n| \le uv$

So a_n is bounded so by the Bolzano-Weierstrass theorem it has a convergent subsequence.

(14), (15) in class

(16) • Prove $0 \le x_n \le 2$ $\forall n$ by induction

$n=1$: $0 < a < 2$ true.

Assume true for n and show for $n+1$: $x_{n+1} \ge 0$ is clear

$x_{n+1} \le \dfrac{x_n^2+8}{6} \le \dfrac{2^2+8}{6} = 2$ ✓

• $x_1 = a$, $x_2 = \dfrac{a^2+8}{6}$ / Show $x_n \le x_{n+1}$ by induction.

and $x_1 \le x_2$ since $6a < a^2+8$ since $a^2-6a+8>0$ true for $a \in (-\infty, 2) \cup (4, \infty)$

$n=1$ true ↑. Assume $x_{n-1} \le x_n$ and show $x_n \le x_{n+1}$

But $x_n \le x_{n+1} \Longleftrightarrow \dfrac{x_{n-1}^2+8}{6} \le \dfrac{x_n^2+8}{6} \Longleftrightarrow x_{n-1}^2 \le x_n^2 \Longleftrightarrow |x_{n-1}| \le |x_n|$

(Since all $x_n > 0$) $\Longleftrightarrow x_{n-1} \le x_n$ True!

• So x_n is increasing and bounded $\Rightarrow$ convergent $\Rightarrow$ Cauchy

• let $x = \lim x_n$. Then $x = \frac{1}{6}(x^2+8) \Rightarrow x^2 - 6x + 8 = 0$ $x = \begin{cases} 2 \\ 4 \end{cases}$

Since $x_n \le 2$ then $\lim x_n \le 2$ so it can't be $4 \Rightarrow \lim x_n = 2$

• If $a=2$ then $x_2 = x_1 = 2$ so all $x_n = 2$. • If $2 < a < 4$ then $2 < x_n < 4$ (prove!) x_n increases and $\lim = 4$

(17) a) (F) The harmonic series $x_n = \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n}$ diverges

but $x_{n+1} - x_n = \frac{1}{n+1} \to 0$

b) (F) For the harmonic series $x_{n+1} - x_n = \frac{1}{n+1} < x_n - x_{n-1} = \frac{1}{n}$

c) (F) Problem (16) is such an example.

Another example is $x_n = \frac{1}{n}$:

$$|x_{n+1} - x_n| = \frac{1}{n(n+1)} < |x_n - x_{n-1}| = \frac{1}{n(n-1)}$$

$x_n \to 0$, so it is Cauchy. But it is not contractive.

Assume it was: $\frac{1}{n(n+1)} < \frac{r}{n(n-1)}$, $\forall n \geq 2$ for some $r \in (0,1)$

Then $r > \frac{n-1}{n+1} \Rightarrow r \geq \lim \frac{n-1}{n+1} = 1$ Contradiction.

---

(18) Yes, any sequence has one.

---

(20) Try contractive: $|a_{n+1} - a_n| = \frac{1}{2} |a_n - a_{n-1}|$ Yes!

$\Rightarrow$ Cauchy $\Rightarrow$ convergent

Let $a = \lim a_n \Rightarrow a = 3 + \frac{a}{2} \Rightarrow a = 6$

[OPTIONAL]: the homogeneous part, $x_{n+1} = \frac{1}{2} x_n$ has sol $x_n = \frac{c}{2^n}$

Try $a_n = \frac{c}{2^n} + d$ for some c, d

$$\frac{c}{2^{n+1}} + d = 3 + \frac{1}{2}\left(\frac{c}{2^n} + d\right) \Rightarrow \frac{c}{2^{n+1}} + d = \frac{c}{2^{n+1}} + 3 + \frac{d}{2} \Rightarrow d = 3 + \frac{d}{2}$$

$d = 6$

$a_n = \frac{c}{2^n} + 6$. Now $a_1 = 1$ so $\frac{c}{2} + 6 = 1$ so $c = -10$.

$$a_n = 6 - \frac{10}{2^n}$$

R2 Sol

5

(21) a) a_n converges to

lim sup = lim inf = lim = 0

b) $r_n = \begin{cases} \frac{3n+1}{n+1} \to 3 & \text{for } n \text{ even} \\ -\frac{3n+1}{n+1} \to -3 & \text{for } n \text{ odd} \end{cases}$

lim sup = 3

lim inf = −3