Final Review

1. Tangent plane, linear approx

Ex. 1. Show that the eq

$$z e^{z+2x+3y} + xy = 0$$

defines implicitly a function $z = z(x,y)$ near the point $(1,0,0)$

Sol. $F(x,y,z) = 0$ Need $\frac{\partial F}{\partial z}(1,0,0) \neq 0$

$F(1,0,0) = 0$

$$\frac{\partial F}{\partial z} = e^{z+2x+3y} + z e^{z+2x+3y}$$

$$\left.\frac{\partial F}{\partial z}\right|_{(1,0,0)} = e \neq 0$$

2. Find the eq of the tan plane to this surface at $(1,0,0)$

Find a unit normal at this point

Sol. $\nabla F \perp \{F = c\}$

$$\nabla F = \frac{\partial F}{\partial x} i + \frac{\partial F}{\partial y} j + \frac{\partial F}{\partial z} k$$

$$= \left(2z e^{z+2x+3y} + y\right) i + \left(3z e^{z+2x+3y} + x\right) j + (\ ) k$$

$\nabla F(1,0,0) = j + e k$

$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$ plane

$x - 1 + y + ez = 0$

3. Find the linear approx of $z = z(x,y)$ at $(1,0,0)$

Sol. $z = -\frac{1}{e}(x-1+y)$

2. Extrema

Find local max, min, saddle of

$f(x,y) = x^4 + y^4 - 4xy + 1$

Sol. Critic. pts $\begin{cases} \nabla f \text{ undef} \\ \nabla f = 0 \end{cases}$

$\frac{\partial f}{\partial x} = 4x^3 - 4y = 0$

$\frac{\partial f}{\partial y} = 4y^3 - 4x = 0$

$x^3 = y$

$y^3 = x \Rightarrow x^9 = x$

$x(x^8 - 1) = 0$ $\begin{cases} 0 \to (0,0) \\ 1 \quad (1,1) \\ -1 \quad (-1,-1) \end{cases}$

Second deriv test

$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}$

$D = f_{xx} f_{yy} - f_{xy}^2$

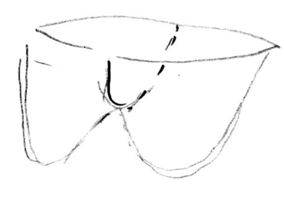
If $D > 0$: $f_{xx} > 0 \Rightarrow$ min
(or $f_{yy} > 0$)

$f_{xx} < 0 \Rightarrow$ max
(or $f_{yy} < 0$)

$D < 0$: Saddle
$D = 0$ inconclusive

	at (0,0)	at (1,1), (-1,-1)
$f_{xx} = 12x^2$	0	12
$f_{xy} = -4$	-4	-4
$f_{yy} = 12y^2$	0	12
	$D < 0$	$D = 12^2 - 16 > 0$
	Saddle pt	$12 > 0$ min

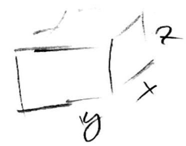
Graph



3. Lagrange multipliers

Ex Find the dimensions of the rectangular box with largest Vol if its total surface area is $6a^2$.

Sol



Maximize $f = xyz$ where $2xy + 2yz + 2zx = 6a^2$

$g = xy + yz + zx = 3a^2$ constraint

Critical pts $\nabla f = \lambda \nabla g$

$g = a^2$

$$f_x = \begin{cases} yz = \lambda (y+z) & x \\ xz = \lambda (x+z) & y \\ xy = \lambda (x+y) & z \\ xy + yz + zx = a^2 \end{cases}$$

Note: $\lambda = 0 \Rightarrow y$ or $z = 0 \Rightarrow$ Vol $= 0$ no

So $\lambda \neq 0$, $x, y, z \neq 0$

$$\left.\begin{aligned} xyz &= \lambda (xy + xz) \\ xyz &= \lambda (xy + zy) \end{aligned}\right] \quad xy + xz = xy + zy \Rightarrow y = x$$

$$xyz = \lambda (xz + yz)$$

$$x = y = z \Rightarrow 3x^2 = a^2 \Rightarrow x = y = z = a$$

2. Extrema, Lag mult

Ex. Find the extreme values of $f(x,y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \le 1$

Sol.
- Domain – bounded, closed $\Rightarrow$ f has abs max & min
- these are found at: critical pts ($\nabla f = 0$; ∇f undefined), on ∂D

Crit. pts: ∇f defined everywhere

$\nabla f = 0$: $\begin{cases} 2x = 0 \\ 4y = 0 \end{cases}$ $(0,0)$ $\boxed{f(0,0) = 0}$

On ∂D: Use Lagrange Mult

$4x^3 = \lambda \cdot 4x^3$
$8y^3 = \lambda \cdot 4y^3$
$x^4 + y^4 = 1$

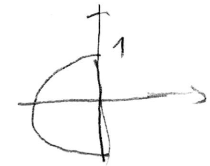
$4x^3(\lambda - 1) = 0$
$4y^3(\lambda - 2) = 0$
$x^4 + y^4 = 1$

I: $\lambda = 1 \Rightarrow y = 0 \Rightarrow x = \pm 1$ $\quad (1,0), (-1,0)$

II: If $\lambda \neq 1 \Rightarrow x = 0 \Rightarrow y = \pm 1$ $\quad (0,1), (0,-1)$

$f(0,0) = 0$ min
$f(1,0) = f(-1,0) = 1$
$f(0,1) = f(0,-1) = 2$ Max

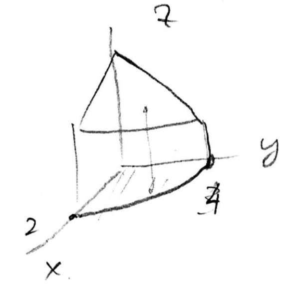
Ex. Find the mass of a semicircle $x^2+y^2 \le 1,\ x \le 0$
if the density function is $\delta(x,y) = (x^2+y^2)^{3/2}$



$$M = \iint_D \delta(x,y)\, dA$$

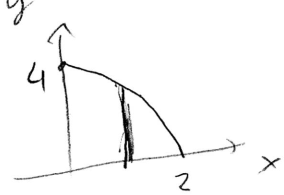
$$\iint_D \sqrt{x^2+y^2}\, dA = \int_{\pi/2}^{3\pi/2} \int_0^1 r^3\, r\, dr\, d\theta$$

Ex Calculate $\iiint_E y\, dV$
where $E =$ the solid in the first octant bd by
the cylinder $y = 4 - x^2$ and the plane $2x+3y+z=6$

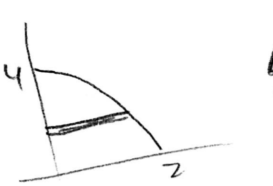


Sol.

$$\iint_D \int_0^{6-2x-3y} y\, dz = \iint_D y(6-2x-3y)\, dA$$



$$= \int_0^2 \int_0^{4-x^2} (6y - 2xy - 3y^2)\, dy\, dx$$



$y = 4 - x^2$
$x^2 = 4 - y$

or $$\int_0^4 \int_0^{\sqrt{4-y}} (6y - 2xy - 3y^2)\, dx\, dy$$

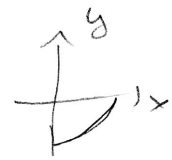
Ex Use spherical coords to calculate the mass of the solid that lies between the spheres $x^2+y^2+z^2=a^2$

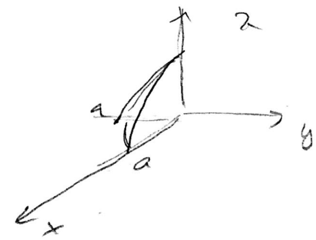
$x^2+y^2+z^2=4a^2$

in the octant with $y<0,\ x>0,\ z>0$, $\delta(x,y,z)=\sqrt{x^2+y^2+z^2}$

Sol

$0 \le \varphi \le \frac{\pi}{2}$





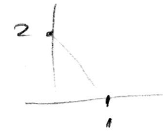
$\frac{3\pi}{2} \le \theta \le 2\pi$

$a \le \rho \le 2a$

$\iiint \sigma\, dV = \iiint \rho \cdot \rho^2 \sin\varphi \, d\varphi\, d\rho\, d\theta$

Line integrals

Calc. $\int_C (3+2xy)dx + (x^2-3y^2)\,dy$



1. $C=$ line-segm from $(1,0)$ to $(0,2)$

Sol. $R(t)=(1,0)(1-t)$

2. On another path

Sol. $\frac{\partial Q}{\partial x} = 2x = \frac{\partial P}{\partial y} \rightarrow$ path indep

3. find the potential

Sol $f_x = 3+2xy$

$f_y = x^2-3y^2$

$f = 3x + x^2y - y^3 + C$

4. Find $\int_C$ on a path from $(0,0)$ to $(1,1)$