Example done well

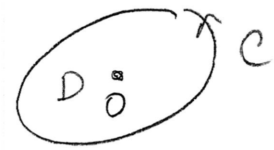
$$\vec F(x,y) = \frac{-y\vec i + x\vec j}{x^2+y^2} = P\vec i + Q\vec j \quad \text{so } P(x,y) = \frac{-y}{x^2+y^2}$$

$$Q(x,y) = \frac{x}{x^2+y^2}$$

satisfies $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Indeed $\frac{\partial P}{\partial y} = \frac{-1}{x^2+y^2} + \frac{y \cdot 2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$$\frac{\partial Q}{\partial x} = \frac{1}{x^2+y^2} - \frac{x \cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

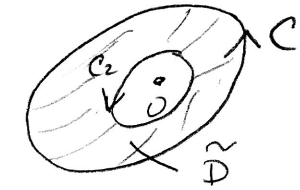


Let C a simple closed curve around O

Then $C = \partial D$. But we cannot use Green's theorem on D

because $\vec F$ is not defined at $(0,0)$!

However, take another curve inside C



we can do Green's on $\tilde D$:

$$\iint_{\tilde D} \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{=0} dA = \int_{\partial \tilde D} P\,dx + Q\,dy = \oint_C + \oint_{C_2} \quad \text{so } \oint_C = \oint_{C_2} \;!$$

Now for C_2 take $x^2+y^2=\varepsilon^2$ $\quad x = \varepsilon\cos t,\ y = \varepsilon \sin t$

$$\oint_{x^2+y^2=\varepsilon^2} \left(\frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy\right) = \int_0^{2\pi} \frac{-\varepsilon\sin t}{\varepsilon^2}(-\varepsilon\sin t) + \frac{\varepsilon\cos t}{\varepsilon^2}\varepsilon\cos t$$

$$= \int_0^{2\pi} dt = 2\pi$$