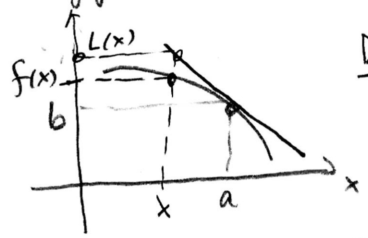
# Linear approximation

Basically, approximate the function by its tangent:




Def. $L(x) = f(a) + f'(a)(x-a)$ is the linear approximation of $f$ at $a$

(Recall, tan line is $y - f(a) = f'(a)(x-a)$)

For functions of 2 variables we approximate the function by its tangent plane:

Def. The linear approximation of $f(x,y)$ at $(a,b)$ is

$$L(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

(recall eq of the tan plane is $z - f(a,b) = \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$ )

More variables:

Def The linear approximation of $f(x,y,z)$ at $(a,b,c)$ is

$$L(x,y,z) = f(a,b,c) + \frac{\partial f}{\partial x}(a,b,c)(x-a) + \frac{\partial f}{\partial y}(a,b,c)(y-b) + \frac{\partial f}{\partial z}(a,b,c)(z-c)$$

Def. Analogue for any number of variables.

As in 1-var, the closer $(x,y)$ is to $(a,b)$, the better the approximation

Example $f(x,y) = \sqrt{xy^3}$. Linearize near $(5,5)$:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{x}\, y^{3/2}\right) = \frac{1}{2\sqrt{x}}\, y^{3/2} \quad \text{so } \frac{\partial f}{\partial x}(5,5) = \frac{1}{2}\frac{1}{\sqrt{5}} 5^{3/2} = \frac{5}{2} = 2.5$$

$\frac{\partial f}{\partial y} = \sqrt{x}\,\frac{3}{2}\sqrt{y}$ so $\frac{\partial f}{\partial y}(5,5) = \frac{3}{2}\sqrt{5}\sqrt{5} = 7.5$

$f(5,5) = \sqrt{5 \cdot 5^3} = 25$

$L(x,y) = 25 + 2.5(x-5) + 7.5(y-5)$

$L(4.9, 5.1) = 25.5$ while $f(4.9, 5.1) = \sqrt{4.9 \cdot 5.1^3} = 25.494\ldots$

Not Bad!

Homework

1. Consider the function $f(x,y) = \sqrt{10 - x^2 - 5y^2}$
   a) What is the domain of this function?
   b) plot its graph (it is part of a known shape, identify it!)
   c) find its linear approximation at $(2,1)$ and use it to estimate $f(1.95, 1.04)$

2. If $z = x^2 - xy + 3y^2$ and $(x,y)$ changes from $\overset{(x_0,y_0)}{\overset{\shortparallel}{(3,-1)}}$ to $(2.96, -0.95)$ Compare the values of $f(x,y) - f(x_0,y_0)$ and $L(x,y) - L(x_0,y_0)$.

## Taylor polynomial

Better approximations can be obtained by retaining more terms in the Taylor polynomial of a function.

1-var recall $f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots$

$$\cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n + \underbrace{R_n(x)}_{\text{remainder, (small)}}$$

Taylor polyn of

degree 0 : $T_0 f = f(a)$ = approx by a constant

1 : $T_1 f = f(a) + f'(a)(x-a)$ = linear approx

2 : $T_2 f = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$ = approx by parabolas

n $\quad T_n f = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(x-a)^k$

In two variables : for $f(x,y)$

$$T_n f(x,y) = \sum_{k=0}^{n}\sum_{j=0}^{n} \frac{1}{k!}\frac{1}{j!}\frac{\partial^{k+j} f}{\partial x^k \partial y^j}(a,b)(x-a)^k (y-b)^j$$

So: $T_0 f(x,y) = f(a,b)$

$T_1 f(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) \equiv$ Lin approx

$T_2 f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$+ \frac{1}{2!} f_{xx}(a,b)(x-a)^2 + \frac{1}{2!} f_{yy}(a,b)(y-b)^2 + f_{xy}(a,b)(x-a)(y-b)$