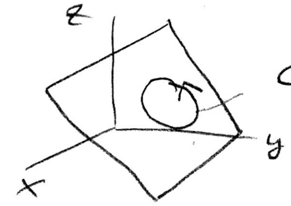
Review problems – Vector calculus

① Let C be a closed curve in the plane $ax+by+cz=d$ and $\vec{F} = mz\,\vec{i} + nx\,\vec{j} + py\,\vec{k}$

Show that $\oint_C \vec{F}\cdot d\vec{R}$ depends only on the area enclosed by C and not on other details about C.



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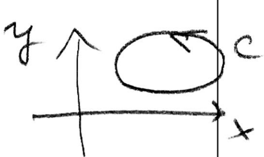
② Let C be a closed curve in the xy plane and let $\vec{F} = -\frac{1}{2}y\,\vec{i} + \frac{1}{2}x\,\vec{j}$. Show that $\oint_C \vec{F}\cdot d\vec{R}$ equals the area of the domain enclosed by C.



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③ Find the outward flux of $\vec{F} = xy\,\vec{i} - \frac{1}{2}y^2\,\vec{j} + z\,\vec{k}$ through S = surface bounded by $z = 4-3x^2-3y^2$ with $1 \le z \le 4$ on the top, on the sides by $x^2+y^2=1$, $0 \le z \le 1$ and $z=0$ on the bottom.

Answer = $\frac{5\pi}{2}$

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④ Find the area of the dome cut from $z = 4-3x^2-3y^2$ by the plane $z=1$

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⑤ $\vec{F} = xy\sin z\,\vec{i} + \cos(xz)\,\vec{j} + y\cos z\,\vec{k}$. Find its outward flux through the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

(6) Find the outward flux of

$$\vec{F} = z^2 x\,\vec{i} + \left(\tfrac{1}{3}y^3 + \tan z\right)\vec{j} + (x^2 z + y^2)\,\vec{k}$$

through S = top half of the sphere $x^2+y^2+z^2=1$.

Hint

Use div theorem

$$\iint_S \vec{F}\cdot d\vec{S} + \underbrace{\iint_D \vec{F}\cdot d\vec{S}}_{\text{evaluate}} = \underbrace{\iiint_E \operatorname{div}\vec{F}\, dV}_{\text{evaluate}}$$

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(7) a) Show that the vector field

$$\vec{F} = (2xy + 2z + 1)\vec{i} + x^2\vec{j} + 2x\vec{k}$$

is conservative and find f so that $\vec{F} = \nabla f$

b) Calculate $\oint_C \vec{F}\cdot d\vec{R}$ where C is an ellipse in the plane $y=1$

c) Calculate $\int_C \vec{F}\cdot d\vec{R}$ where C is a half circle from $(1,1,1)$ to $(10,1,2)$.

Hint Use path independence and calculate on a line segment

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(8) It is know that the gravitational force has the direction opposite to $\vec{R}$ and inverse proportional to $|\vec{R}|^2$: $\vec{G} = -c\dfrac{\vec{R}}{|\vec{R}|^2}$

a) Show that $\vec{G}$ is conservative and find the potential.

b) Show that the work done by $\vec{G}$ from P_0 to P_1 does not depend on the path taken.

More review problems

(9) Consider $f(x, y) = x^2 + 2y^2 - x^2 y + 4$

a) Find its critical points and use the second derivative test to decide their nature.

b) Find its absolute max and min on the circle $x^2 + y^2 = 8$

c) Find its extrema inside the disk $x^2 + y^2 \le 8$. and Explain why a) and b) are relevant for your answer.

A a) $(0,0)$ min

$(2,1)$ saddle

$(-2,1)$ also saddle, f is even in x

b) use Lagrange mult

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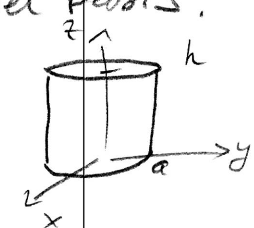
(10) Find the mass of a thin plate shaped as a cylinder of radius a and height h if its density function at each point equals its distance to the lower basis!

$\delta(x,y,z) = z$



Hint parametrize the cylinder using cylindrical coordinates

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(11) Find the mass of a circular wire with linear density $\delta(P)$ proportional to the square of the arc length from P to a fixed point P_0 on the wire.

(12) Find the circulation $\oint_C \vec{F} \cdot d\vec{R}$ of $\vec{F} = y\vec{i} - x\vec{j}$ along the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ going counterclockwise. Do it directly and check your answer using Green's thm.

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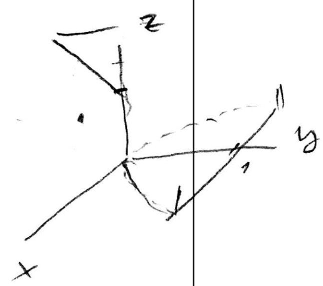
(13) Find $\iiint_E xy \, dx\, dy\, dz$ where E is bounded by the cylinder $y = x^2$ and the planes $y = 1$, $z = x + 1$, $z = 0$



Sol

$0 \le z \le x+1$

$x^2 \le y \le 1$

so $\int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{x+1} xy \, dz\, dy\, dx$